Understanding how precalculus teachers develop mathematic knowledge for teaching the idea of proportionality

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Abstract: The purpose of this study was to better understand how a precalculus teacher develops mathematical knowledge for teaching the idea of proportionality. We were also interested in understanding what instructional supports might foster shifts in a teacher's thinking, and then how shifts in her thinking affect her classroom decisions.

The teacher was using a research-based precalculus curriculum designed to help students acquire improved problem solving abilities, and deeper understandings and connections among the courses central ideas. The findings revealed that the research-based professional development and support tools for teaching precalculus can lead to improvements in the teacher's mathematical content knowledge and aspects of her teaching practice. We also observed that the teacher was still making new mathematical connections during her second year of using the materials. We also observed that shifts in a teacher's content knowledge do not always improve the teacher's ability to leverage her student's thinking during teaching.

Key Words: precalculus, proportional reasoning, mathematical knowledge for teaching, linearity

<u>Research questions</u>

We are seeking more detailed understanding of how a teacher develops mathematical knowledge for teaching (MKT), and what mechanisms may support or prevent the teacher in making shifts to more effective teaching. To investigate this, we are focusing on a case study of a teacher teaching about proportional relationships. This teacher is using a research-based curriculum, which was designed as part of an NSF grant. We based our investigation on the framework developed by Silverman and Thompson (2008). The framework characterizes the process by which a teacher's MKT may develop. This provides a hypothetical trajectory for teacher learning that allows us to evaluate the development of a teacher's MKT and identify when barriers are preventing further development of her MKT. Developing a more detailed description of how one teacher's MKT about proportionality emerges over time may help to expand our understanding of Silverman and Thompson's general framework.

With that in mind, the questions we are trying to address follow:

1. How does a teacher's mathematical thinking about proportional relationships shift as she teaches with these activities?
    a. What new mathematical understandings if any emerge as she teaches?
    b. What new mathematical connections are made between ideas?
2. How does a teacher's understanding of student thinking shift as she teaches and reflects on her teaching?
    a. What insights into how her students think does she develop?
    b. How does the teacher perceive the understandings she wants her students to develop as supporting their future learning?
    c. What strategies does she develop to support students in developing new ways of understanding?
3. How do these shifts in thinking about the mathematics and understanding of student thinking change the choices the teacher makes in the classroom?

Relationship of this research to current literature

There seems to be general agreement that there is specialized mathematical knowledge required to teach mathematics effectively. Much work has been done to understand MKT in elementary mathematics (Ball, 1991; Ball, Hill, & Bass, 2005; Hill & Ball, 2004; Ma, 1999)). This research has focused on observing what actions a teacher performs in the classroom, and then on what knowledge a teacher needs to be able to do those actions effectively. Ma (1999) describes a deep connected mathematical knowledge which is necessary for effective elementary mathematics teaching which she calls profound understanding of mathematics.

We are interested in the MKT of teachers who teach more advanced mathematics, specifically precalculus. We want to better understand how teachers develop new meaningful ways of thinking about the mathematics connected with student thinking that allow them to develop activites and conversations that support student learning. In The Teaching Gap (Stigler & Hiebert, 2009) states that when teachers make changes, they are often superficial, because their underlying beliefs and thinking do not change. The framework developed by Silverman & Thompson (2008) describes a process by which MKT may develop for teachers of more advanced mathematics. According to this framework, the teacher first develops a significant mathematical understanding through reflection, and realizes its value to her students. Through further reflection and attention to student thinking the teacher develops ideas of how to support her students in developing a new way of thinking. It is through on-going reflection that a teacher is able to develop new ways of thinking and new approaches to teaching.

Theoretical perspective and/or conceptual framework

We are basing our study on the framework developed by Silverman & Thompson (2008) to describe the process of developing MKT. This process includes a number of stages that are not necessarily linear, but iterative and interrelated. The stages are characterized by ways of thinking that emerge as a teacher develops MKT. We can look for these ways of thinking as evidence of developing MKT. This framework also provides a hypothetical trajectory for developing MKT, which may allow us to provide interventions that encourage a teacher to develop MKT.

According to the Silverman and Thompson framework, the stages in developing MKT include[1]:

- *The teacher develops or becomes aware of her own personal Key Developmental Understanding (KDU).* A KDU (Simon, 2006) is an understanding that is a leap in understanding, is connected to many other concepts and provides a powerful foundation for learning. Courtney (2010) in his dissertation notes that teachers are resistant to the type of reflection required for them to develop a KDU. In this study we attempeted to provide activities and tools that encourage this type of reflection. We hypothesize that the teacher is challenged to reflect on her own understandings as she prepares to teach, and works through problems in the curriculum. Ma (1999) notes that Chinese teachers develop their deep mathematical understandings primarily through studying the curriculum materials with teaching students as the primary focus.
- *The teacher develops models of how students may understand through de-centering[2].* We are interested in how the teacher attends to student thinking and what models she develops of

[1] The description provided here in italics is an abbreviated summary of our interpretation of the Silverman and Thompson framework. We have added notes that describe how the framework relates to our study.

student thinking. The "Teacher Notes" provided as part of the research-based curriculum provide descriptions of student thinking that support the teacher in developing her own model.

- *The teacher develops images of how someone else could develop the KDU they have developed.* We are interested in how the teacher imagines someone else developing the same understandings they have developed. This means developing "personal images that are capable of conveying what one has in mind to someone who does not already understand what one intends to convey."(Thompson, Carlson, & Silverman, 2007) Thompson, et al suggest that this is difficult because teachers have learned to focus on procedures they intend for students to learn, not on ideas they intend for students to develop.
- *The teacher develops a mini-learning theory, including ideas for activities and conversations to develop this way of thinking.* We are interested in how the teacher's ways of thinking affect decisions the teacher makes in the classroom.
- *The teacher sees how this new way of thinking empowers other learning.*

We developed activities and asked questions that attempted to understand how the teacher was developing MKT. The teacher was using a curriculum, which begins with a module that focuses on developing the ideas of quantity, variable, proportionality, rate of change and linearity. The module also emphasizes the network of connections among these ideas. The curriculum presents proportional relationships as relationships between two quantities whose values change together or co-vary in a way such that as the quantities change together they remain in a constant ratio to each other. This emphasis on co-varying quantities focuses attention on the fact that proportionality is describing a relationship between varying quantities, and supports the approach the curriculum uses to support students in solving problems by developing meaningful models of quantities and how they are related.

The understanding of proportional relationships is developed in ways to support future learning and reasoning about average rate of change, exponential functions, and angle measurements in trigonometry.

The curriculum intends to support teachers in developing coherent mathematical thinking with their students through challenging problems set in realistic situations. The curriculum materials provide support for the teacher in the form of workbooks with detailed activities and teacher's notes about how student thinking develops. Teachers also have access to Power Point slides and an on-line forum to exchange ideas with their peers and get support from a curriculum expert. Teachers also attend regular bi-weekly Professional Learning Community (PLC) meetings with other teachers using the materials.

Research methodology

This case study follows a precalculus teacher, Elizabeth who has taught for 7 years, as she teaches a class of 40 high schools students. Elizabeth has taught with the research-based curriculum for one full year and is now in her second year of implementing the materials. The data we report here focuses on her classroom practices during the first two months of the second year of using the curriculum.

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[2] We used the idea of de-centering to describe a behavior in which one attempts to understand the mathematical thinking and/or perspective of someone else for the purpose of adjusting her behavior in order to influence another in specific ways. (Piaget, 1995; Steffe & Thompson, 2000)

Elizabeth recorded her daily goals for student learning before each class and videotaped all lessons. Immediately following class Elizabeth recorded in a journal, instances she recalled in which student thinking was revealed. The intention was that Elizabeth would notice and become more aware of what her students actually said or did in class without feeling the pressure to make an interpretation or judgment. She was also asked to make comments on her mathematical understandings as they related to the student thinking she observed. This form of data collection was made to promote reflection and was based on Mason's (2002) suggestion that humans have a tendency to interpret before observing and noticing, thus limiting the quality of an individuals observations.

The interviews were performed to investigate the teacher's thinking about the mathematical ideas, her attention to her students' thinking and how their thinking had influenced her classroom decisions. The questions were based on teacher reflections, student work collected by the teacher and on notes from classroom observations and videos. The questions posed during the clinical interviews included prompts as deemed useful for advancing both her MKT and classroom practices. The clinical interviews were conducted after the completion of each unit of connected ideas.

Results of research

During the first year of using the curriculum Elizabeth developed new understandings of teaching proportionality including the value of students understanding the three ways of expressing that two changing quantities are related proportionally (i.e., if a and b are related proportionally then as they change together: i) the ratio of a to b remains constant; ii) a and b are related by a constant multiple; iii) if a is scaled by a constant, b is scaled by the same constant). She was also able to see how the idea of proportionality is related to the ideas of constant rate of change, slope and linearity. We also documented that while her understandings are consistent with the curriculum, she is still developing ways of leveraging student thinking and helping students develop meaning. Finally, we give an example that demonstrates Elizabeth identifying an important mathematical understanding in a clinical interview.

First, we discuss how Elizabeth's mathematical understanding had shifted after one year of using the research-based curriculum, while also noting that some connections between the idea of proportionality and other ideas were missing in her conception of proportionality. Before using the curriculum, when Elizabeth was probed to discuss the idea of constant rate of change, she did not spontaneously see how the idea of proportionality could be used to relate ideas of constant rate of change and slope of a line.

I: What connections do you expect students to make between constant rate of change and the slope of the line?
E: Yes, so constant rate of change. First off I would expect them to know that graphically it's a straight line and then second I would expect them to be able to dictate to me that for, uh, over equal amounts of changes of the independent variable the dependent variable is also changing by a same amount
I: What about if you change the input by half the original amount?
E: They should know proportionally that the output should change. You should change it by the half amount as well
I: So, would that be part of your expectations for their understanding?
E: Oh, I'm not sure I've ever really gotten into that not necessarily in a trig pre-calculus class to be honest. ... I can't say that I did in a trig pre-calculus this year.

We can contrast this with a recent classroom interaction, which was observed in her second year of teaching with the materials. The problem is shown in figure 1. Elizabeth guided the class through a conversation that included first identifying that a linear relationship has a constant rate of change, and that in this problem the constant rate of change is -2.5. The class then discussed that the change in y is proportional to the change in x. The students worked this problem using these understandings. This approach to thinking about rate of change and proportionality is quite a contrast to her earlier statement of what she wanted students to understand. In her teaching here, there is a clear connection between the ideas of slope, constant rate of change and the idea that the change in x and the change in y are proportional. We noticed that she did not use the constant multiple representation of a proportional relationship with her students. In an interview we asked her if she could imagine using the constant multiple representation of proportionality in this problem. Elizabeth noted that she is not comfortable with this representation of proportionality, and then noted the connection to the fact that her students also did not seem comfortable with this representation Elizabeth identified this as an area for her to further develop her own personal understanding.

According to her journal entries and informal conversation, Elizabeth finds it challenging to understand student thinking and then find ways to support students in developing their own understanding. Figure 2 shows student work on a problem, which was given on a quiz. The student's answer contains correct ideas, but the student applies the ideas incorrectly. Elizabeth expressed that the student probably understood the ideas, but just confused himself with the words. Another possibility for this confusion is that the student did not have a well-developed meaning of the idea of quantity and changes in the quantities. When this possibility was suggested to Elizabeth she was able to describe questions and activities that could possibly promote student understanding of these ideas.

Figure 3 contains another quiz problem. As Elizabeth was reviewing the quiz, one student asked how you decide which number goes on top of the ratio. Elizabeth guided them to use unit analysis. Later in an interview the researcher asked what understanding that student was missing that caused them to ask that question. After thinking for a while, Elizabeth said "*We could look at what we are trying to model. We are trying to model the cost with respect to distance... So, we want to look at how does the cost change as the miles change? Therefore their constant rate should be the change in cost with respect to miles. That sounds like a tough conversation.*" Elizabeth acknowledged that this understanding of the meaning of the rate of change would help students in solving this problem, and then went on to say she would further support the students' discussing the corresponding changes in cost and miles on a graph so they understood the meaning of the rate. This is a different approach from using unit analysis, and could be an insight that supports her future teaching decisions.

<u>Applications to/implications for teaching practices or further research</u>

This study supports that the research-based curriculum used by this teacher, and the accompanying professional development tools, may be useful for supporting teachers in advancing their MKT relative to specific content. Our findings support those of Silverman and Thompson in revealing that teachers are more effective in advancing their students' knowledge if they have deep personal understanding of ideas and rich connections among ideas. A major challenge in advancing a teacher's MKT is to support them in developing models of student thinking, and ideas about how to support students in developing meaningful mathematical thinking.

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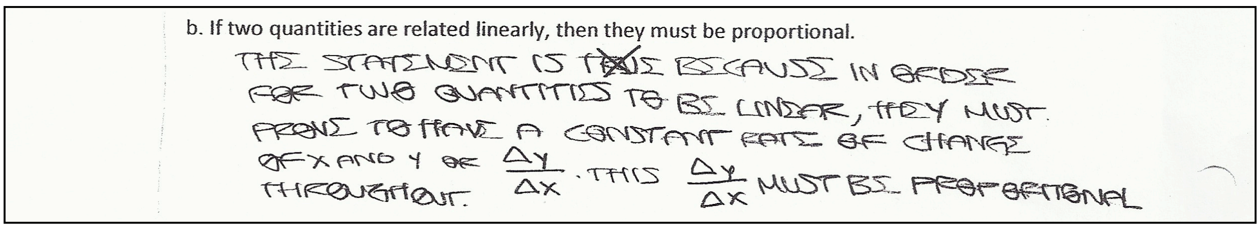
Figures

A linear relationship has a slope of –2.5 and passes through the point (–5, 8). Use the given point and the slope to complete the table of values for this relationship.

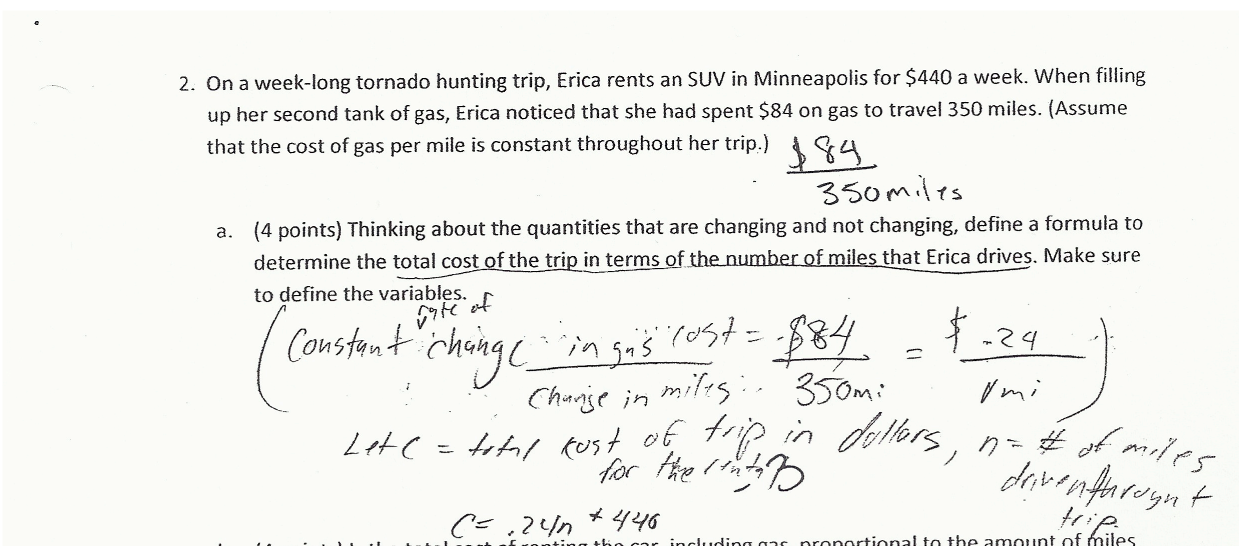
x	*y*
–6	
–3	
–0.8	
0	
2.3	
5	
8	

Figure 1


b. If two quantities are related linearly, then they must be proportional.

THE STATEMENT IS FALSE BECAUSE IN ORDER FOR TWO QUANTITIES TO BE LINEAR, THEY MUST PROVE TO HAVE A CONSTANT RATE OF CHANGE OF X AND Y OR $\frac{\Delta y}{\Delta x}$ THROUGHOUT. THIS $\frac{\Delta y}{\Delta x}$ MUST BE PROPORTIONAL

Figure 2


2. On a week-long tornado hunting trip, Erica rents an SUV in Minneapolis for $440 a week. When filling up her second tank of gas, Erica noticed that she had spent $84 on gas to travel 350 miles. (Assume that the cost of gas per mile is constant throughout her trip.) $\frac{\$84}{350\text{ miles}}$

a. (4 points) Thinking about the quantities that are changing and not changing, define a formula to determine the total cost of the trip in terms of the number of miles that Erica drives. Make sure to define the variables.

(Constant rate of change in gas cost / Change in miles = $84 / 350mi = $.24 / 1mi)

Let C = total cost of trip in dollars for the rental, n = # of miles driven throughout trip

C = .24n + 440

b. (4 points) Is the total cost of renting the car, including gas, proportional to the amount of miles

Figure 3