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Decentering: A construct to analyze and explain teacher actions as they relate to student thinking

Dawn Teuscher[1] · Kevin C. Moore[2] · Marilyn P. Carlson[3]

Abstract Mathematics educators and writers of mathematics education policy documents continue to emphasize the importance of teachers focusing on and using student thinking to inform their instructional decisions and interactions with students. In this paper, we characterize the interactions between a teacher and student(s) that exhibit this focus. Specifically, we extend previous work in this area by utilizing Piaget's construct of decentering (The language and thought of the child. Meridian Books, Cleveland, 1955) to explain teachers' actions relative to both their thinking and their students' thinking. In characterizing decentering with respect to a teacher's focus on student thinking, we use two illustrations that highlight the importance of decentering in making in-the-moment decisions that are based on student thinking. We also discuss the influence of teacher decentering actions on the quality of student–teacher interactions and their influence on student learning. We close by discussing various implications of decentering, including how decentering is related to other research constructs including teachers' development and enactment of mathematical knowledge for teaching.

Keywords Decentering · Student thinking · Secondary mathematics teachers · Mathematical knowledge for teaching · Pedagogical actions

✉ Dawn Teuscher
dawn.teuscher@byu.edu

Kevin C. Moore
kvcmoore@uga.edu

Marilyn P. Carlson
marilyn.carlson@asu.edu

[1] Department of Mathematics Education, Brigham Young University, TMCB 191, Provo, UT 84602-6563, USA

[2] University of Georgia, Athens, GA, USA

[3] Arizona State University, Tempe, AZ, USA

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> I found myself actually spending time talking to my students about how they interpreted or solved a problem. I became intrigued with the way that my students looked at concepts in so many different ways. I dared to let my students become the experts in the field, rather than me.—Claudia

The above quote is from an experienced high school mathematics teacher who shifted her instruction to have a greater focus on her students' thinking. Her quest to understand the meanings of her students' thinking led to her advancing her mathematical knowledge and appreciating the diversity in her students' mathematical thinking. Over the past few decades, mathematics educators and writers of several policy documents have emphasized the importance of teachers focusing on and using student thinking to inform their teaching (Clarke 2008; Franke and Kazemi 2001; Levin et al. 2009; NCTM 1991, 2000). As a notable example, Cognitively Guided Instruction emerged as a professional development program with the goal of helping elementary teachers shift their instruction to have a greater focus on students' mathematical thinking (Carpenter et al. 2000). Our meaning of *effectively focusing on student thinking* is the teacher discerning and acting on student thinking in ways that support student learning.

Just as it is important that teachers maintain a focus on student thinking during teaching, Hiebert and Grouws (2007) called for researchers to focus on both the teacher and students when investigating teaching, and particularly when characterizing student–teacher interactions in the classroom. The authors argued that documenting shifts in teaching and instruction is more meaningful when done in the context of characterizing student thinking and learning. Members of the *Learning Mathematics for Teaching Project* (2011) described teaching and instruction as including, "…not only what teachers say and do, but also what students say and do…" (p. 30). Echoing Hiebert and Grouws' sentiments, we add that characterizations of teaching and instruction should include not only what teachers say and do, but also how teachers think, how students think, and, especially, the ways in which teachers incorporate student thinking into their practice.

The purpose of this article is to extend the literature on what is entailed in teachers effectively focusing on and leveraging student thinking during teaching, from the elementary level (Carpenter et al. 2000; Franke et al. 2011; Franke et al. 2009; Jacobs et al. 2010) to the secondary level. In doing so, we apply Piaget's construct of *decentering* (1955; Steffe and Thompson 2000) to make inferences about a teacher's ability to effectively focus on student thinking, and illustrate its usefulness in making instructional decisions that are informed by and advance student thinking. By drawing on the construct of decentering to interpret a teacher's actions and thinking *in the moment*[1] of teaching, our work has a different focus than prior related research at the elementary level. Although the construct of decentering has not been widely used to analyze student–teacher interactions, we suggest that other constructs such as *professional noticing* (Jacobs et al. 2010) and *teacher follow-up* (Franke et al. 2011) tacitly presume decentering, but have not applied decentering in a way that is sensitive to the teacher reflecting on and building an in-the-moment model of student thinking. To illustrate these in-the-moment actions, we share specific instances of a secondary mathematics teacher attempting to make sense of and act on student thinking while teaching. These illustrations illuminate the effect of decentering on student–teacher interactions and a teacher's ability to effectively understand and draw on student thinking to inform her instruction.

[1] By *in the moment* of teaching (or in-the-moment), we mean situations in which teachers are in their classroom interacting with or teaching their students, as opposed to situations in which teachers are reflecting on a classroom session that has concluded or watching classroom video.

Review of literature

Mathematics educators continue to encourage teachers to shift their classroom practices to align with reform- and research-based instruction. Yet this goal remains difficult to accomplish, as interpretations of "reform- and research-based instruction" vary from educator to educator.[2] One such interpretation of research-based instruction is that teachers should move from a focus on computational skills toward a focus on deeper understandings of mathematical ideas, relations, processes, and concepts (Hiebert and Carpenter 1992; Lampert 1990; NCTM 1989, 2000; National Governors Associations Center for Best Practices and Council of Chief State School Officers (NGA Center and CCSSO) 2010). To achieve such a focus, researchers have recommended that students interact in ways that create opportunities to think, discuss, agree, and disagree about mathematics, so their thinking can be understood by their teacher and peers (Ball 1993; Bauersfeld 1995; Cobb et al. 1992; Goos 2004; Lampert 1990; Mercer 2000; Nathan and Knuth 2003; O'Connor 1998).

It seems intuitive that increasing student talk about mathematics should lead to improved learning, as increasing student talk provides students the opportunity to share and reflect on their thinking, and teachers the opportunity to better understand their students' thinking (Woodward and Irwin 2005). However, there is evidence that increasing student talk is not sufficient in and of itself to improve student learning (e.g., Kazemi and Stipek 2001; Nathan and Knuth 2003; Truxaw and DeFranco 2007). For example, Nathan and Knuth (2003) found that increasing the *quantity* of student talk in a middle grades classroom did not necessarily improve students' mathematical understandings. The authors' findings underscore that although it is important that students have substantial opportunities to discuss mathematics, the *quality and nature* of these discussions may be a more pivotal factor for influencing student learning.

A natural approach to improving the quality and nature of classroom discussions is providing students opportunities to share their solutions. By sharing solutions, a student has an opportunity to reflect on the thinking that led to her or his mathematical products. All students in the class also have an opportunity to consider the presented solutions and how they relate to their own approach and thinking. Yet, teachers often have difficulties engaging students in productive conversations about students' solutions and thinking. Several researchers have found that although teachers may ask questions that elicit student solutions, these same teachers struggle to pose questions that are attentive to or extend student thinking and learning as the students share their solutions (Boaler and Brodie 2004; Brodie 2011; Franke et al. 1998; Franke et al. 2009; Walshaw and Anthony 2008). Teachers are often told to 'listen to your students,' but how teachers listen to students and leverage the knowledge acquired from listening is not as obvious to teachers; nor is it well articulated by researchers. Should teachers listen for correct or incorrect answers? Should they listen for student misconceptions? And how should teachers respond to their interpretations of what they have heard? These fundamental questions have, in part, led mathematics educators to analyze classroom discussions using discourse analysis.

Discourse analysis and a lens on student thinking

When analyzing the nature of classroom discussions, researchers have focused on various aspects of discourse. For instance, researchers have examined discourse patterns to

[2] We use the word educator to refer to teachers, teacher educators, mathematics educator-researchers, etc.

describe ways that teachers interact with their students (e.g., univocal or dialogic; Mehan 1979; Sinclair and Coulthard 1975; Wertsch and Toma 1995). Other researchers have analyzed discourse patterns to describe teacher talk and questioning (e.g., revoicing, pressing, and follow-up; Boaler and Brodie 2004; Franke et al. 2011). Researchers have also characterized discourse patterns by modeling desirable and undesirable behaviors of teachers (e.g., focusing or funneling; Rittenhouse 1998; Wood 1998). These different characterizations of discourse patterns in mathematics classrooms provide important information about student–teacher interactions. At the same time, and countering Hiebert and Grouws' (2007) call for more attention to student thinking, these discourse patterns foreground teacher actions in ways that do not attend to the depth and content of *student* responses and thinking, including what a teacher might understand about student thinking. For example, revoicing is a teacher-centered activity only requiring a student to put forth some statement to be revoiced; the mathematical nature of the student's statement is mostly irrelevant to the revoicing construct. Although funneling and focusing provide more emphasis on the students' participation and contributions, these characterizations foreground teachers' actions in ways that do not provide explanatory constructs for teachers' decisions related to student thinking. In other words, these uses of discourse patterns explain what teachers *do,* but they do not provide ways to characterize a teacher's complex mental processes that might lead to and inform what they do.

Maintaining a focus on teachers' actions without giving attention to the mental processes of teachers and students in relation to these actions is not likely to explain the effectiveness of a teacher focusing on and using student thinking. Our research group's past work (Carlson et al. 2007), which examined the degree to which professional learning community facilitators and teachers made sense of and leveraged their peers' thinking, has led us to believe that a similar lens may be useful for characterizing the quality of classroom discussions. Our intention is not to imply that researchers have ignored these aspects of classroom interactions. Rather, prior researchers have not provided explanations for the ways of thinking that teachers might engage in when attempting to promote quality discourse in their classrooms; nor has prior research set such explanations against the backdrop of students' mathematical thinking. As an example, discourse analysis research (Boaler and Brodie 2004; Brodie 2004, 2011; Franke et al. 2011, 2009) has highlighted the importance of teachers evaluating and following up on student responses. With regard to this discourse practice, a focal question becomes: What is the nature of a teacher's mental actions that enables her or him to productively evaluate and follow up on student responses *in the moment* of teaching? It could be the case that a teacher's decisions rely on the correctness of a student's solution. Or, it could be the case that a teacher considers all student solutions as stemming from viable[3] (to the student) ways of thinking that the teacher must discern through continued interaction. In the first case, a teacher's questions are likely to be for the purpose of determining whether students have the correct answer or not, and then correcting the answer if the students do not have the correct answer. In the latter case, a teacher's questions are likely for the purpose of building a model of students' thinking in order to account for how students reached their solution, a model that may or may not inform the teacher's subsequent interactions with her or his students.

[3] By claiming a student's strategy or solution stems from a viable way of thinking, we do not mean to imply that a teacher should not evaluate student thinking as correct or incorrect. Instead, we intend to emphasize that, regardless of correctness, a teacher should assume that the student has an understanding that makes sense in that moment to the student.

Decentering

The construct of decentering provides a powerful lens for examining student–teacher interactions because it moves beyond discourse analysis by including a focus on the teacher's interpretations of students' verbal and written explanations to make decisions in the moment of teaching. While the products of discourse analysis provide information about interaction patterns, including characterizations of the types of questions a teacher poses in a lesson, these products do not provide information about the thinking a teacher engages in when deciding what questions to pose, nor do these products provide information set against the backdrop of student thinking. We seek to extend discourse analysis by drawing inferences about how a teacher's understanding of student thinking relates to the teacher's actions in the moment of teaching.

Smith and Stein (2011) alluded to the importance of being attentive to student thinking when describing teachers' uses of probing questions—those questions that require students to expand on their response or explain their thinking in a different way. The authors explained that, "most teachers find probing questions easy to use and typically will get the hang of this style of questioning quickly" (p. 68). Stressing the importance of considering teacher thinking, the authors continued, "*…teachers need to learn how to discern* when this question type can be expected to pay dividends and when it cannot" (p. 68, emphasis added). We conjecture that making such a judgment requires knowledge of the students' thinking and an understanding of the mathematical ideas relevant to the discussion. We approach the purpose and expectations of probing questions in terms of the teacher's ability to build a model of student thinking and effectively advance student thinking and understanding.[4] We believe the construct of *decentering* is useful for placing greater emphasis on how student thinking (or a teacher's interpretation of student thinking) informs the teacher's instructional moves and thinking. We hypothesize that posing probing questions to a student is more effective when the questions are informed by the teacher's understanding of student thinking and mathematical ideas being discussed.

Piaget (1955) introduced the idea of *decentering* (or *decentration*) to characterize the actions of an observer attempting to understand how an individual's perspective differs from her or his own. Piaget was mainly concerned with decentering in relation to a child's development, particularly in explaining a child's transition from being egocentric and unable to consider perspectives, thoughts, and feelings other than her or his immediate own to when a child has the capacity to consider perspectives, thoughts, and feelings independent of her or his immediate own (Chapman 1988; Piaget 1955; Piaget and Inhelder 1967). Piaget explained decentering both in terms of a child's mathematical thoughts (Chapman 1988; Piaget and Inhelder 1967) and in terms of a child's interactions with others (Piaget 1955). As a mathematical example, Piaget and Inhelder (1967) illustrated that a child's spatial reasoning transitions from being constrained to her or his own visual perspective to being able to imagine visual perspectives from positions other than the one he or she was occupying. With respect to interactions, Piaget (1955) discussed the depth at which those involved in interactions consider those understandings and perspectives that are not their own. When speaking of children, he explained:

[4] Adopting a radical constructivist (von Glasersfeld 1995) standpoint, we consider an individual's thinking to be fundamentally unknowable to any other individual. We use "build a model of student's thinking" to refer to when an individual attempts to develop a model of another individual's mental actions that may explain that individual's observable and audible actions.

> If children fail to understand one another, it is because they *think they do* understand one another. The explainer believes from the start that the reproducer will grasp everything, will almost know beforehand all that should be known, and will interpret every subtlety. (p. 116, emphasis added)

Steffe and Thompson (2000) extended the construct of decentering to classify hypothetical interactions between teacher and student, specifically discussing decentering in terms of the manner by which an individual adjusts (or does not adjust) his or her actions in order to understand another individual's (or a group of individuals') thinking. Incorporating a focus on both social interactions and mathematical thinking, Steffe and Thompson (2000) and Thompson (2000) distinguished between two primary ways in which individuals interact with each other. They related these distinctions to the type of models that an individual (implicitly or explicitly) creates of another's thinking.

Thompson (2000) differentiated individuals participating in an interaction unreflectively from those individuals participating in an interaction reflectively with respect to other individuals' contributions. In the context of teaching, Thompson conveyed that, "If teachers do not reflect on aspects of an interaction that are contributed by students, if they are fused with the situation as they have constituted it" (2000, p. 423), then the teacher is constrained to (consciously or sub-consciously) creating first-order models of the interaction. First-order models are, "the models an individual constructs to organize, comprehend and control *his or her experience*, i.e., their own mathematical knowledge" (Steffe and Olive 2010, p. 16, emphasis added), which differ from those models that stem from an individual interacting reflectively in a conscious attempt to set aside one's own thinking and understand what other individuals understand (e.g., decentering). A focus on one's own experience can be a by-product of the individual assuming that the other's thinking is identical to her or his own, thus speaking such that he or she believes the other individual understands the utterances just as intended. Or, an individual operating entirely from the perspective of her or his experience may presume that another's thinking has a rationality of its own, but the individual makes no concerted attempt to differentiate her or his own thinking from the other's thinking, thus interacting in an unreflective way.

As defined by Thompson (2000), an individual who interacts reflectively, makes a conscious effort to understand how other individuals might be thinking, constructs second-order models with respect to the interaction. Second-order models are, "[the models] observers may construct of the subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (Steffe et al. 1983, p. xvi). Because second-order models are "constructed through interaction with the subject … they are the observer's models of the observed…to explain the mathematical knowledge and learning of children as understood by the observer" (Steffe and Thompson 2000, p. 205), we contend that the act of decentering—attempting to adopt a perspective that is not one's own—is essential to the construction of sophisticated second-order models.

Speaking in more detail on differences between first- and second-order models, Steffe and Thompson (2000) and Thompson (2000) explained that an individual who creates a first-order model "does not intentionally analyze the mental structures of a child relative to his or her own mental structures" (Steffe and Thompson 2000, p. 202). In contrast, an individual who interacts with others by attempting to put aside her or his own ways of operating in order to construct an image of the other participant's ways of operating creates a second-order model. This individual might also attempt to construct an image of, "what they [the observer] understand about what the other person *could* understand" (Thompson 2000, p. 423, emphasis added) including the "ramifications of positing alternative ways of thinking that might prove more profitable were students to think in those ways" (Thompson 2000, p. 424). As Silverman and Thompson (2008) described, it is through an individual constructing second-order models of her or his students' thinking, including the ramifications of those and alternative ways of thinking, that personal meanings can be transformed into sophisticated mathematical knowledge for teaching. Whereas first-order models allow the individual to characterize students' current state of understanding, second-order models require that the individual also construct an image of alternative student ways of thinking and how students might develop these ways of thinking given their current state of understanding.

Returning to student–teacher interactions, teachers who are oriented to decenter understand that their students have idiosyncratic understandings; therefore, the teacher attempts to gain insights into the thinking of her or his students by interacting reflectively with students. These insights may be gained by listening to the student when responding directly to the teacher or another student, and the teacher might subsequently use her or his evolving second-order models to determine what questions to pose or statements to make as he or she works to further understand and advance student learning. We note that a decentering teacher always assumes that a student has some viable system of meanings that contribute to her or his actions. For this reason, notions of incorrect and correct are mostly irrelevant beyond informing the teacher's future actions. This can be summarized by the following statement: There are no such things as misconceptions, "misconceptions are just conceptions that, for a variety of possible reasons, you do not want the other individual to hold" (Patrick W. Thompson, personal communication). In other words, misconceptions are judgments an observer makes (and possibly acts on) relative to her or his constructed second-order models, but the observer views these misconceptions as viable conceptions to the student.

The following examples contrast two teacher–student interactions to demonstrate teachers attending to student thinking and building what we classify as first- or second-order models of students' thinking. Consider the problem in which a class is tasked with determining the measure of an angle that subtends an arc length of 7 inches on a circle with a radius of 3.5 inches.

Interaction 1 (interacting unreflectively—building a first-order model):

Teacher: Ok, who wants to give an answer?
John: 7 divided by 2 pi times 3.5 equals x over 360 and I got x to be 114.6.
Teacher: Great, and how do you know that works?
John: It's a proportion.
Teacher: Good, a proportion. Katie, you solved it a different way. What did you do?
Katie: I divided 7 by 3.5 and got 2 radians.
Teacher: And that's the same as 114.6 degrees, right?
Katie: Yes.
Teacher: Good.

Interaction 2 (interacting reflectively—building a second-order model):

Teacher: Ok, who wants to give an answer?
John: 7 divided by 2 pi times 3.5 equals x over 360 and I got x to be 114.6.
Teacher: Great, and how do you know that works?
John: It's a proportion.
Teacher: A proportion, OK. Can you say a little more about that?

John: Well, like, if we divided 7 by 2 pi times 3.5, we get about 0.32. So that times 360 gives me the answer.

Teacher: Yes, that does give an answer. Back to the 0.32, what does that number mean and why does multiplying that by 360 give us the answer?

John: The arc length is 32 % of the circle's circumference, so the angle measure should be 32 % of 360 degrees because this corresponds to a circle's circumference.

Teacher: Good. Katie, I see that you solved it a different way. What did you do?

Katie: I divided 7 by 3.5 and got 2 radians.

Teacher: And why does that work?

Katie: Well, radians tell me how many radii are cut off by the angle, and 7 divided by 3.5 tells me that the arc length is 2 times as large as it.

Teacher: And what do you mean by 'it'?

Katie: The arc length is 2 times as large as the radius.

Teacher: OK. Now, John and Katie came up with different numbers for their answers using different calculations. How can we connect both of these answers to John's idea of proportion?

Dave: 114.6 degrees is the same as 2 radians.

Teacher: What do you mean by the same? Does 114.6 equal 2?

Dave: They both equal the same percentage of the total, 2 divided by 2 pi is the same as 114.6 divided by 360.

Teacher: OK. What do you think about that John? Does that fit with what you said?

John: That's like my solution. It's always 32 % of the circle's circumference. No matter if it's radians, degrees, or inches.

In the first interaction, the teacher questioned his students; however, the students' reasoning and meanings that led to their answers are left unclear. The teacher's responses suggest that he is focused on key phrases or procedures that led students to the correct answer—the teacher is acting unreflectively. For instance, John mentions that he used a "proportion," but *his* meaning of the word proportion, or how he used a proportion is not clear. As research in the area has revealed (Lamon 2005; Thompson and Saldanha 2003), students can hold numerous meanings for "proportion," many of which are limited to a procedural execution of calculations. Also, students shared two different solutions, yet the teacher did not follow up on either student response in a way that drew their meanings to the surface or helped students connect the two ways of thinking about angle measure and how they are related to the arc length and the radius. As a result, in some students' minds, there could be two different answers to this question—114.6 degrees and 2 radians—that have only superficial connections (Carlson et al. 2013; Moore 2013). Thus, while the teacher did ask questions, we do not interpret the teacher to be interacting reflectively in an attempt to decenter, thus constraining the teacher to what we consider to be a first-order model of student thinking that did not assist him in revealing student understanding.

In the second interaction, we interpret the teacher to be interacting reflectively, thus creating a second-order model of her students' thinking. She then uses that model to help her students connect two different numerical answers for an equivalent measure of an angle's openness. The teacher's prompt to have John explain what he meant by "proportion" initiated a question–response sequence that offered insights into the students' meanings for angle measure, particularly in terms of relationships between a circle's radius, circumference, and subtended arc length. This understanding of her students' thinking enabled the teacher to build on and extend students' thinking in a way that prompted students to draw connections among their different ways of thinking for angle measure. An observer can only infer the thinking behind the teacher's questioning, but one explanation for the teacher's actions is that her initial question was intended to build a second-order model of how John was thinking and understanding "proportion." Supporting this interpretation, when John provided a calculational response to the teacher's question, she followed up with a second question ("Back to the 0.32, what does that number mean and why does multiplying that by 360 give us the answer?") directed at revealing John's thinking. As a result, John raised the idea of proportion as a multiplicative comparison between quantities. The teacher was subsequently able to use her second-order model of John's thinking to raise the critical idea that an angle measure quantifies the fractional amount of a circle's circumference subtended by the angle. Collectively, by reflecting on the students' responses and building second-order models of their thinking, the teacher was able to pose questions to help students see that Katie's and John's solutions represented two different correct ways of representing an angle's measure.

Although the decentering construct has rarely been used explicitly in research to analyze student–teacher interactions within a classroom, there are several frameworks and constructs related to decentering. One relevant example is that of *professional noticing* (Jacobs et al. 2010), which refers to teachers reflecting on student strategies that emerged during videotaped lessons, interpreting student understanding by analyzing student written work, and thinking about how to respond to student understanding in the hypothetical situation of working with the students in a classroom. Jacobs et al. described, "…attending, interpreting, and deciding how to respond—happen in the background, almost simultaneously, as if constituting a single, integrated teaching move" (p. 173). We agree that the collective constructs of professional noticing are best approached as an integrated teaching move, and we contend that a teacher's decentering actions play an important role in such a move and that this move is not necessarily in the background to the teacher, particularly in cases when students' understandings are not a transparent feature of their activity. Whereas Jacobs and colleagues were primarily interested in what teachers noticed when watching a videotaped lesson and how teachers interpreted student written work, the construct of decentering offers a novel perspective that focuses on what is being built in the teacher's mind, particularly in the moment of teaching. When decentering, the teacher is attempting to take on the perspective of the student through interacting reflectively in the moment of teaching for the purpose of building second-order models of her or his students' thinking relative to a specific mathematical idea. This activity enables the teacher to pose questions intended to help students make essential connections related to learning a concept.

Another relevant framework related to decentering is *teacher follow-up* (Franke et al. 2011). Franke et al. specifically analyzed student–teacher interactions in elementary classrooms to determine how teacher follow up on student explanations differed across teachers and student achievement. The authors found, "it was how they used these moves in relation to the students and the mathematics that seems to support students to provide more explanation, take an incorrect explanation to correct, or an explanation from incomplete to complete" (p. 6). The authors characterized teacher actions that promoted a complete and correct explanation as, "connecting with students' mathematical ideas and pressing students to detail their ideas for themselves and the class" (p. 12). As with much of the discourse literature, this framework is focused on the way in which teachers respond to students (e.g., what teachers *do*), but not on the reflective, nuanced model building that a teacher must mentally engage into connect with students' ideas and productively use these ideas in interactions. Teacher follow-up on students' explanations is certainly important, and hence, our goal is to better understand and then characterize what the teacher constructs in her or his mind and to illustrate the subtle ways in which the teacher's model of students' thinking influences the quality of the teacher's questions and various instructional decisions made during teaching.

Background

Project Pathways is a National Science Foundation Math and Science Partnership (MSP) project (No. EHR-0412537) designed to support secondary mathematics teachers improving their instruction and students' learning. Early in the project, we identified state assessments, teacher knowledge, and school leaders as obstacles for teachers in shifting their teaching to be more inquiry based and conceptually focused (Moore and Carlson 2012). After confronting these obstacles in select schools, teachers continued to make minor shifts in their teaching, with our continuing to measure only minimal gains in student learning. This led to our piloting a research-based conceptually focused precalculus curriculum in select classrooms. We refer to these materials as the *Pathways* curriculum (Moore 2013).

The Pathways curriculum is informed by theory on learning the concept of function (Carlson 1995, 1998), the processes of covariational reasoning (Carlson et al. 2002), and literature about mathematical discourse (Carlson et al. 2007; Clark et al. 2008) and problem-solving (Carlson and Bloom 2005). The curriculum contains modules based on research of student understanding and learning of key concepts of precalculus, including proportion, rate of change, function, function composition, and has a focus on using these ideas to represent how two quantities change together to represent exponential, polynomial, and periodic growth patterns. The curriculum also supports a problem-solving approach to mathematics, where students are expected to apply quantitative reasoning (Carlson et al. 2012; Moore and Carlson 2012) to construct images of applied problems, and develop meaningful formulas and graphs to characterize how quantities in a situation are related and change together.

Selecting and analyzing student–teacher interactions

As part of the project, we videotaped and observed secondary mathematics teachers who used the Pathways curriculum during the school year. Research team members who observed lessons took field notes and answered questions for teachers after their lessons. After observing classrooms with different teachers and working with these teachers over the course of 1 year, we noticed a substantial shift in how one teacher, Claudia, was effectively using student thinking in her classroom. Thus, we decided to analyze her student–teacher interactions (i.e., a sustained conversation between the teacher and a student or group of students) over the course of the year to better understand how she was thinking and what led to her posing questions that were highly effective in advancing her students' thinking and understanding of the key course ideas.

Using conceptual analysis techniques (Steffe and Thompson 2000; Thompson 2008), we analyzed the student–teacher interactions by first characterizing the students' mathematical thinking during an interaction. This involved attempting to discern students' ways of thinking by examining their utterances and written work (e.g., we developed, to the best of our ability, second-order models of the students' thinking). We then examined Claudia's utterances and written work to characterize her mathematical thinking including whether she built a first- or second-order model of her students' thinking. With specific interactions analyzed, we then compared and contrasted these interactions to further characterize student thinking, teacher thinking, and the degree to which we inferred that Claudia interacted reflectively for the purpose of building second-order models of student thinking. Analyzing across interactions enabled us to identify shifts in student thinking as well as shifts in Claudia's decentering. For instance, at times it was difficult to discern whether or not Claudia developed a second-order model during an interaction where it was apparent (from our perspective) that she held different ways of thinking than her students. Comparing the broader set of interactions enabled us to determine whether Claudia later engaged with that group (either during group work or during whole class discussion) in a way that implied she had developed a second-order model of students' thinking or at least acknowledged that their thinking was not identical to her own. For the purpose of this article, we select two representative student–teacher interactions from Claudia's Pathways high school precalculus course to illustrate the nature of interactions that emerged when she attempts (or does not attempt) to build a second-order model of student thinking (i.e., decenter).[5]

Illustrations of classroom interactions

The mathematical lens by which we observed and analyzed the interactions is related to the Pathways curriculum learning goals and instructional practices. Therefore, we begin each illustration by discussing the instructional activity and learning goals as described in the Pathways curriculum. We then discuss the interactions in terms of the teacher's decentering actions (or lack thereof) for the purpose of building and leveraging a second-order model of student thinking.

Illustration 1

Our first illustration is an interaction from Claudia's classroom at the beginning of the school year. The initial Pathways unit introduces the idea of constant rate of change by prompting students to explore how two quantities change together in situations where the two quantities are changing together at a constant rate of change. The lessons support students in understanding the meaning of constant rate of change—that is, two quantities are changing together at a constant rate of change when the changes in the two quantities are proportional. The idea of average rate of change of one quantity with respect to another (e.g., distance with respect to time) follows by prompting students to explore the meaning of average rate of change in a context where a car moves from point A to point B at a varying rate of change. Students are supported in understanding that the average rate of change over that time interval is the constant speed the car would need to travel to achieve the same change in distance over the same interval of time that the car actually traveled as it moved from point A to point B. As an example, an average speed of 34 mph of an object as it travels from point A to point B is the constant speed required to travel from point A to point B in the same amount of time as it actually took the object to move from point A to point B. At this point in the class, students were given the diver task (Fig. 1) to address a common way of thinking about average rate of change that can be problematic—that is, the

5 We are not attempting to characterize a teacher as being a decentered or non-decentered teacher, but rather we are attempting to illustrate how decentering allows the teacher to build a second-order model of student thinking, which can inform the teacher's decisions.

1. In a diving competition, a diver received the following scores from 4 judges after making a dive.

Judge 1	Judge 2	Judge 3	Judge 4
8.7	9.3	8.0	8.2

a. Determine the diver's average score for the four dives.
b. What is the meaning of *average* in the context of computing a diver's average score for a dive?
c. How does the meaning of the word *average* when computing a diver's average score compare with the meaning of the word *average* when computing a diver's average speed?

Fig. 1 Average score for a diver task (Carlson and Oehrtman 2010)

idea that the average rate of change of one quantity with respect to another is the arithmetic mean of instantaneous rates of change.

When approaching the diver task, students initially determine the average score for the diver based on four judges' scores using the calculation for determining the arithmetic mean of summing the scores and dividing by the total number of scores. The students are asked to explain the meaning of this number. For teachers to build a second-order model of student thinking that they can use to work toward the instructional goals for this task, they must be knowledgeable of the different meanings associated with the word average and how these meanings are connected to the calculation for determining an average. It is common for students to think of the average as the middle of the data set. Students may also describe average as some estimate of the entire data set, which is most consistent with the arithmetic mean. In such a case, the teacher must be prepared to assist students in investigating how to quantify and interpret this meaning for average (e.g., it is the score that the diver would receive had all four judges awarded the diver the same score).

Analysis of the student–teacher interaction during the diver task revealed that students determined the average score (or arithmetic mean) of the diver based on the four scores given by the judges. While circulating around the classroom, Claudia stopped to discuss part b of the task (see Fig. 1) with a group of students (Excerpt 1).

The students attempted to verbalize two ideas. In line 5, Student 1 briefly related the meaning of 8.55 to some idea of 'constant.' In lines 6, 7, 9, 11, and 14, the students referred to the average as the 'approximate score' that the judges awarded the diver, with one student mentioning the idea of 'mean.' We also observed that Claudia's first move was to ask her students to describe their response to the task. After prompting the students to continue their discussion (lines 4 and 8) and telling a student to not use 'mean,' she questioned the students on particular aspects of the situation (e.g., the number of judges and dives). Although Claudia asked her students questions related to the task, we interpret her actions to suggest that she was focused on the procedure for finding average and she interpreted the students' actions (i.e., what they did) with respect to this focus, as opposed to participating reflectively in an attempt to gain insights into how the students thought about their activity. That is, Claudia was oriented to her first-order model of the mathematical ideas and instructional goals.

Our analysis of the entire class session revealed that Claudia did not return to ideas of approximate score, constant rate of change, average, or mean. Considering that the purpose of the lesson was to explore connections between various meanings of the average, Excerpt 1 (and the subsequent interactions) does not provide sufficient evidence that Claudia was acting reflectively for the purpose of understanding her students' thinking (i.e.,

Excerpt 1 Average, Mean and Approximate Scores

Line number	Speaker	Transcription
1	S1	(*walking up to the group*) Do you know the meaning of average in this problem?
2	T	*Tell me what you have so far.*
3	S2	If the average is 8.55…[a]
4	T	*Okay*, *so*…
5	S1	I said that 8.55 would mean average would be like the constant or the, I don't want to say constant.
6	S2	You said approximate.
7	S1	The approximate score.
8	T	*Keep going.*
9	S3	Can we use mean to describe average?
10	T	*No.* (*laughs*)
11	S1	The approximate score to get your score.
12	S4	No, cause…
13	T	*Your*…
14	S4	…the approximate score given by the judges for that dive…
15	T	*Keep going*, *the approximate score by the judges*, *what*
16	S4	…for the dive.
17	T	*Okay, for what dive*?
18	S2	For the diver's dive.
19	T	*For just one dive*?
20	S	Yeah (*students in unison*).
21	S2	They only did one dive.
22	S1	If each, if one judge per dive or isn't it…
23	T	*Four judges for the one dive*, *okay*, *alright. Okay so tell me again what you said.*
24	S4	The approximate score given by the judges for the one dive.
25	T	*Okay, by which judges*?
26	S1	By the four judges.
27	T	*By what*?
28	S2	By the four judges.
29	T	(*shaking her head confirming the student's answer and leaving the group*)

[a] The ellipsis represent when the speaker is interpreted by another speaker

decentering). That is, we consider Claudia's actions to not be indicative of her reflectively trying to build a second-order model of the students' meanings for 'mean,' 'constant,' or 'approximate score' independent of her own understanding of these ideas and her instructional goals.

There are several explanations for Claudia's actions in Excerpt 1. First, she may have assumed that the students held particular ways of thinking when they used the words 'approximate' or 'constant' score and moved forward with these presumed ways of thinking. Second, she may not have understood the underlying mathematical ideas of the lesson; therefore, she was unsure which direction to take her questioning or how the students' utterances might have related to the lesson goals. Third, she may not have understood her students' explanations, and instead of continuing to question the students to discern their thinking, she redirected the conversation in ways focused on her own understanding. As a result of one or more of these reasons, Claudia's questions limited the potential for the students to explore relationships and differences between their ideas.

Illustration 2

Our second illustration draws on an interaction from Claudia's classroom near the beginning of a unit focused on quantitative relationships and representing these relationships with formulas, graphs, and tables. Collectively, the lessons support students in understanding how different representations (e.g., formulas, graphs, and tables) are connected and how to obtain information about the problem situation using each representation. As part of the unit, students were given different situations and prompted to represent the given information using formulas, graphs, and tables in order to make predictions about the situations. We draw on the hotel renovation task in which students were to determine the cost of adding a number of rooms for a hotel renovation given a fixed cost to build six additional rooms (Table 1). As part of this task, students were to sketch a graph to represent the relationship between the number of rooms that can be rented in terms of the amount of money spent on renovating.

The hotel renovation task was designed with the intention that students use their knowledge from the first unit that if the changes in two quantities are proportional (e.g., $\Delta y = m \cdot \Delta x$), then those quantities model a linear relationship. That is, the students were to use the information that for every $5550 the hotel spent on renovation costs, they could rent an additional six rooms to conclude that the situation can be modeled using an increasing, continuous linear relationship with a rate of change of 5550/6 and an initial point of (0, 28). From there, the students were expected to determine other coordinate pairs by reasoning about changes in the quantities (e.g., a change in the renovation cost of $5550 yields a change in the number of rented rooms of six; hence, the coordinate pair (5550, 34) satisfies the linear relationship).

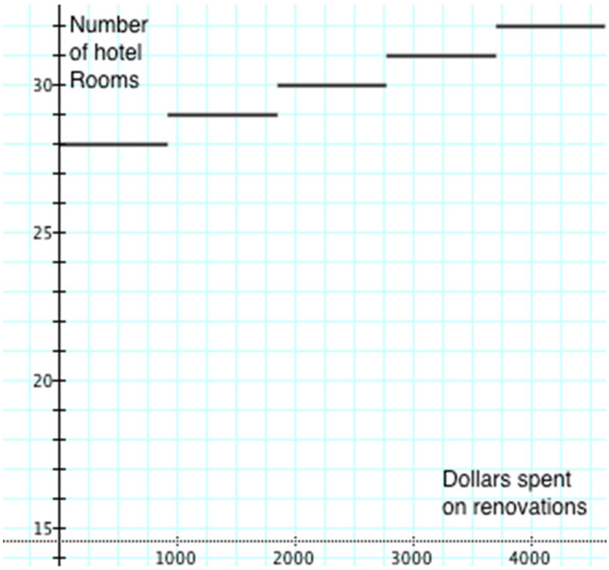
While we expected Claudia to draw on her knowledge of rate of change and linear relationships to support the students in reaching the above conclusions, analysis of the student–teacher interactions during the hotel renovation task revealed that she had not anticipated the reasoning expressed by some students. This is revealed when Claudia circulates around the classroom monitoring student progress and stopped at one group to question the students' construction of a step-function graph (Fig. 2). The interaction (Excerpts 2a, 2b) reveals the varying degree of which Claudia acted reflectively in an attempt to build a second-order model of the students' thinking.

The students created a step function to model how the number of rooms rented changed with the amount of money spent on renovations (lines 2, 12, 19, and 24). Student 1 provided a valid explanation for why she chose to use a step function as opposed to a continuous linear function: "if I spend half of the money on the renovation you cannot have ½ a room so the rooms must increase by one" (line 2). That is, the students conceived that the number of rooms rented could only change in discrete amounts. Thus, any spent

Table 1 Hotel renovation task (Carlson and Oehrtman 2010)

A local hotel currently rents an average of 28 rooms per night. The hotel management estimates that for every $5550 spent on hotel renovations, they will be able to rent an additional 6 rooms each night. Sketch a graph that represents the relationship between the number of rooms that can be rented in terms of the amount of money spent on renovations

Fig. 2 Students' graph of the hotel problem



amount less than that required for one room does not result in an additional room (or partial room). However, this was not the graph that Claudia was expecting, nor was it how she interpreted the situation and the students' equation.

An important facet of this interaction is that the students maintained a focus on their graph and how a step function was the appropriate model for the situation. In fact, Student 1 acknowledged her difficulty with writing a formula for a step function (line 4) and understood that the group's formula, which was written in the form of a continuous linear function, did not match their graph or conception of the situation. Despite the students' focus on their graph and their admitting that the formula did not model the situation as they had conceived, Claudia repeatedly returned the students to their formula (lines 3, 5, 7, and 13). Claudia's questions suggest that she expected them to conclude that the situation was modeled by a continuous linear function, which was compatible with *her* interpretation of the task situation and the students' formula. As the students maintained their conception of the situation, as opposed to adopting Claudia's perspective, Claudia questioned them about changes in the two quantities apparently for the purpose of concluding that because the changes in the quantities were proportional, the relationship was linear (lines 15, 17, 20, 25). Claudia's questioning about changes suggests that she did not have a well-developed second-order model of how her students were thinking; she did not realize there was not a conflict between how the students and she envisioned the changes for a change of one room (lines 15–16). Hence, her questioning did not address the conflict between how she and the students envisioned the situation with respect to a partial renovation payment and the number of rooms rented.

We note that Claudia *was* attentive to the students' work, at least to the extent that she realized the students' solution was inconsistent with her first-order model. For instance, in line 9, she attempts to interpret Student 1's claims (line 2). Additionally, Claudia asked a question that may have helped her build a second-order model of how the students were thinking when she asked, "…according to her equation, you can pay for one room at a time, right" (line 13)? Yet, we interpret Excerpt 2a to suggest that Claudia was not acting

Excerpt 2a Linear or Proportional?

Line number	Speaker	Transcription
1	T	(*Walks up to group and looks at students' work for a few seconds*) *Okay*, *so tell me what your graph* (Fig. 2) *means*?
2	S1	Because if I rent ½ a room, even if you spend ½ the extra 5500 dollars (*cost per six rooms*), you can't have ½ room so it would increase by 1.
3	T	(*Pause*) *Okay*, *but according to your equation*, *well actually where is your equation*?
4	S1	It is right here, I know I have problems writing stepwise equations.
5	T	*Okay*, *so is this* (*points to student's formula*) *a step function*?
6	S1	(*begins to erase her graph*)
7	T	*Don't erase your graph*, *no this is not a step function right here* (*pointing to a formula*), *this is what kind*?
8	S1	It is just two equations (*referring to same formula*).
9	T	*Okay*, *so let's go back here*, *so if they spend, uh, so you are saying if they spend anywhere between 5550 up to 11*,*100, but not including 11*,*100 but anything up to that*, *then they are only going to get…*
10	S1	I am sorry this is wrong, this should be up here.
11	T	*Okay*, *so let's well it should be at…*
12	S2	But if she is graphing the situation then it would be stepwise.
13	T	*Okay*, *but her*, *according to her equation*, *you can pay for one room at a time*, *right*?
14	S1	Yes.
15	T	*So if you can pay for one room at a time that would mean*, *as soon as you pay the next 925 dollars* (*cost per one room*) *you would get an additional room. Right*?
16	S1	Correct.
17	T	*Okay*, *what kind of relationship is that*?
18	S1	(*thinking*)
19	S2	Step function.
20	T	*No.* (*Laughing*)
21	S2	It is not proportional.
22	S1	It is not proportional and it is not linear because…
23	T	*Okay*, *why is it not proportional*?
24	S1	Because you start at 28 (*number of rooms*), and you can't spend negative money on the next room.
25	T	*Are the changes proportional though*?
26	S1	No.

reflectively with respect to how the students were thinking, *independent* of her interpretation of the situation, her instructional goals, and her interpretation of the students' products. Hence, Claudia was operating from a first-order model of her students' thinking. She primarily focused on how she could use the students' formula to guide them to the thinking and conclusion that she was expecting, which led to her making inferences and decisions that were not compatible with or attentive to (from our perspective) the students' thinking.

By the completion of Excerpt 2a, Claudia and her students' conceptions of the situation remained in conflict (from our perspective and apparently the students' perspective). While Claudia continued attempting to impose a particular way of thinking based on her

Excerpt 2b Piecewise and adding rooms

Line number	Speaker	Transcription
1	T	*So for each additional room built are they always going to pay nine hundred*?
2	S1	I don't know, we are trying to think about how to describe this.
3	S2	Well you presume, because they're not only going to build half a room.
4	S1	If for every apple you need one orange and you have half an apple then you need half an orange, but this is not the case here because you spend half the money here you are still going to get one, or you can't spend half the money here.
5	T	*Okay, are you assuming that you have to spend the 5550 dollars* (*amount per 6 rooms*) *in your graph, is that what you are assuming*?
6	S1	Yes, oh I am sorry, this is all wrong anyways, this should be, I am sorry this should all be in 925 s (*price per one room*), I am making errors all over this paper (*explaining the increments on the stepwise graph should be 925*)…
7	T	*Okay, but this graph is saying that if I spend less than 925 dollars I am still getting a room, is that true*?
8	S1	No this is starting at 28 so you are not adding your room yet until you spend 925 dollars at which point you get the 29th room.
9	T	*Okay*, *but this is saying that according to this graph if I am spending this much money*, *which let's say that it is four something whatever it is*, *that I am getting this room. Is that true*?
10	S1	But you already have that room because this is the 28th room that you already had in the problem.
11	T	*Okay, so go to this one—oh, oh oh I see what you are saying. So this…*
12	T	*So if you are paying this much money* (*pointing to a point on the student's graph*) *you still don't have the next room*?
13	S1	Yep.
14	T	*So once you pay 925 dollars you have the next room*?
15	S1	You get the 29th room.
16	T	*Okay, so it doesn't matter how much money you pay after that until you get to 925 dollars you're paying this amount, okay I see what you are doing.*

understanding of the situation and the instructional goals of the lesson provided in the curriculum, as opposed to trying to understand and adopt the perspective of the students, the students persisted in maintaining their conception of the situation (Excerpt 2b).

In an attempt to clarify their reasoning to Claudia, the students used a fruit analogy, claiming that it was possible to have ½ of a piece of fruit, but in the situation of hotel rooms it was not possible to have half a room (line 4). It was at this point in the interaction that Claudia's actions shifted in focus. In line 5 of Excerpt 2b, Claudia changed her focus by asking a question related to the *students*' graph and understanding of the situation. Claudia continued asking questions that we interpreted to be for the purpose of understanding how the students conceived the situation and graph. That is, Claudia appears to be reflectively considering the students' activity and thinking, as opposed to questioning the students for the purpose of guiding them to a specific understanding based on her own understanding and goals. Hence, we infer that Claudia's questions (lines 7, 9, 12, and 14) and the student responses led her to build a second-order model of how the students thought about the situation (lines 11 and 12). As Claudia questioned and listened to her students, she discovered that although her students were not thinking about the situation in a way that was compatible with her initial conception of the situation, they were thinking about the problem in a viable way (line 16). Although not included in this excerpt, and adding further evidence that Claudia constructed a second-order model of her students' thinking, Claudia later had these students present their solution, the step function, to the whole class and she facilitated a class discussion about their thinking and the feasibility of their representation in relation to the continuous, linear representation.

We conjecture that Claudia's interest in how her students thought about the situation assisted her in decentering, which allowed her to listen and build a second-order model of the students' thinking about the situation in a viable and different way then she had anticipated. This illustration exemplifies the importance of approaching student thinking as viable and seeking to understand students' ways of thinking. The task was intended to focus on linear functions in terms of how changes in quantities are proportional. However, as seen in Illustration 2, these students determined a more viable way of thinking about the situation: "because they're not only going to build half a room." One can imagine that, had Claudia not acted reflectively in order to understand the meanings her students had constructed, the students would have been left perplexed as to why a continuous function was used in a non-continuous situation. Instead, the students' success in conveying their meanings to Claudia led to her subsequently having them present their way of thinking to the class.[6]

Discussion

Returning to Piaget's (1955) explanation of decentering, and specifically his explanation of someone interacting in a way that does not account for another's perspective, he stated: "the explainer believes from the start that the reproducer will grasp everything, will almost know beforehand all that should be known, and will interpret every subtlety" (p. 116). Reframing this in terms of teaching, a teacher (consciously or sub-consciously) not oriented to engage in decentering acts from the start as if he or she will grasp everything his or her students do, will know beforehand all that should be known, and will interpret every subtlety with ease. Likewise, he or she acts in ways that assume his or her students interpret all explanations and actions just as intended. On the other hand, teachers oriented to decenter appreciate the instructional power gained by considering and attempting to make sense of student thinking. These teachers are more likely to build and leverage second-order models of their students' thinking when teaching, thus understanding that they do not know beforehand almost all that should be known, especially relative to subtleties in their students' thinking including their students' interpretations of the teacher's actions.

Based on our research, we claim that further exploration of the role of decentering in teaching is important. More specifically, we hypothesize that a teacher's ability and propensity to build a second-order model of her or his students' thinking in the moment of teaching is connected to the quality and nature of her or his classroom interactions, as well as the development and enactment of her or his mathematical knowledge for teaching. In this section, we outline connections in these areas.

Professional noticing, follow-up, and in-the-moment interactions

We find the decentering construct useful in extending discourse constructs including professional noticing (Jacobs et al. 2010) and teacher follow-up (Franke et al. 2011) for two

[6] As a result of this interaction, the curriculum developers removed the hotel renovation task from the linear functions module, as the task was more appropriately suited for exploring piecewise functions.

primary reasons. First, decentering enables situating teachers' actions within a psychological perspective, which has more explanatory power when attempting to characterize student–teacher interactions. Not only does the decentering construct support a focus on the nuances of student thinking as Hiebert and Grouws (2007) called for, investigating teachers' decentering actions also entails researchers developing hypotheses about teachers' thinking as teachers interact with students, including the depth at which teachers understand their students' thinking. For this reason, the decentering construct provides a way to distinguish between different ways that teachers follow up on student explanations; decentering allows more detailed cognitive accounts of how teachers "[connect] with students' mathematical ideas and [press] students to detail their ideas for themselves and the class" (Franke et al. 2011, p. 12). In some cases, teachers may not engage in decentering, thus not reflectively considering students' mathematical ideas. In other cases, teachers may decenter by reflectively considering and connecting with students' mathematical ideas. In the former cases, a teacher might press her or his students for explanations, but the teacher's actions are likely focused on students' procedures and actions (i.e., Excerpt 1) as related to the teacher's instructional goals and thinking (i.e., Excerpt 2a). In the latter cases, a teacher presses her or his students so that he or she understands the students' thinking behind the procedures, actions, and solutions (i.e., Excerpt 2b). Such pressing enables a teacher to develop second-order models of her or his students' thinking and then make decisions based on these models, instructional goals, and the teacher's thinking.

A second way that decentering contributes to existing discourse literature is by addressing professional noticing *in the moment* of a teacher interacting with students in the classroom. Jacobs et al. (2010) suggested that future studies should, "…connect teachers' professional noticing of children's mathematical thinking with the *execution* of their in-the-moment responses" (p. 197, emphasis added). We argue that the decentering construct extends professional noticing to interactions in the moment of teaching including how professional noticing relates to the extent that teachers make intentional efforts to take on the perspective of students. When teachers are decentering during interactions with students, they are attending to, interpreting, and responding to student thinking in the moment of teaching. More specifically, teachers are attending to and interpreting student thinking in a way that enables them to be sensitive to their students' perspectives. As a result, teachers' decisions of how to respond to student thinking can be based on students' ways of thinking rather the teachers' ways of thinking. Hence, we consider teachers' decentering actions—their building second-order models of how students are thinking during their interactions with students—as a productive way to envision the in-the-moment execution of professional noticing *in the classroom*. Returning to Illustration 2, if Claudia had not attempted to adopt the perspective of her students, she would not have attended to and interpreted her students' thinking in ways compatible with how they were thinking about the situation, nor would she have been in position to productively leverage her students' way of thinking.

We emphasize the implications of decentering with respect to investigating teachers' actions, including their professional noticing, in the moment of teaching. Focusing on student–teacher interactions in the moment of teaching is most germane to investigating a teacher's decentering actions and the extent that he or she builds second-order models of student thinking. The models of student thinking that result from decentering are based on a teacher's interpretations of students' ways of thinking, and teachers are best placed to take on the perspective of their students (e.g., decentering) when directly engaged with students. Hence, investigating teachers in the moment of teaching will ultimately provide researchers better opportunities to construct second-order models of teachers' decentering actions including how teachers envision relationships between their thinking and their students' thinking in the moment of teaching. Illustrating such an outcome, after Illustration 2, Claudia acknowledged that she needed to give more attention to how her students thought about a problem's context when having them explain their solutions, as opposed to primarily focusing on the instructional goals of the lesson, her own understanding, and the students' products.

Connections with mathematical knowledge for teaching

Researchers have conjectured that decentering is critical to the development and enactment of mathematical knowledge for teaching (Silverman and Thompson 2008). Teachers who develop second-order models of their students' thinking have the opportunity to learn from their students. Mason and Spence (1999) described this knowledge as *knowing-to*, a dynamic and evolving knowledge that is accessible at any time. The authors stated, "…the absence of knowing-to … blocks students and teachers from responding creatively in the moment" (p. 143). Other researchers (Mason 2002; Silverman and Thompson 2008; Walshaw and Anthony 2008) also argued that such opportunities could enable teachers to transform their own knowledge to knowledge that encompasses how students think and develop mathematical understandings, which can be used in future interactions, including in the design of instructional materials and in anticipating student thinking. As Johnson and Larsen (2012) discussed, without such knowledge, teachers are constrained in their ability to reflectively listen to their students in ways that inform their teaching. Hence, using the construct of decentering to characterize the observable actions of teachers in the moment of teaching may reveal information about not only teachers' mathematical knowledge, but also their development of the particular aspect of mathematical knowledge for teaching that includes knowledge of student thinking and development.

Limitations and future research

A limitation of our work is the level of inferences that we are able to make with respect to Claudia's decentering actions. Because our work is based on inferences of students and a teacher's mathematical thinking in the moment of teaching, and we were only able to draw these inferences as observers of student–teacher interactions, we do not claim a one-to-one correspondence between our characterizations and the students' or teacher's thinking. Instead, our characterizations of these interactions are based on our own interpretations of the interactions, which we partially created using the mathematical constructs and language of the Pathways curriculum and instruction goals. Because decentering emerged as an important construct during our retrospective analysis of teachers implementing the Pathways curriculum in their classroom, we did not have the opportunity to utilize other researcher tools to gain insights into teacher thinking and, particularly, the model building that a teacher might engage to understand her or his students' thinking. One way that future work in the area might benefit is through studies that utilize stimulated recall interviews with teachers following their lessons. Such work would provide additional insights into a teacher's thinking during instruction and would enable researchers to develop more viable inferences of the types of models teachers build of students thinking, ultimately supporting researchers in understanding how these models influence teachers' actions.

Although we did not work directly with Claudia to improve her propensity and capacity to decenter, we consider professional development and research studies that assist teachers in improving their ability to effectively develop second-order models of student thinking while teaching a potentially fruitful course of action. The use of the decentering construct in professional development provides a clearer purpose for teacher follow-up and questioning. The notion of interacting reflectively to *build a second-order model* of student thinking provides teachers with an image of what productive questioning accomplishes. Rather than teachers interpreting the purpose of questioning as achieving complete, clear, or correct explanations, they might find the model building focus to define the purpose of questioning as enabling those involved in the interaction to understand each other beyond just the actions and results one obtained. Research that reveals sources of teachers' decentering actions may provide insights into the mechanisms of decentering that lead to teachers' "building second-order models of student thinking" and then using those models to guide their instructional decisions.

Based on our observations of other Pathways teachers, we conjecture that a teacher's ability to decenter impacts the quality of classroom interactions and norms in the teacher's classroom. Approaching student thinking as viable independent of judgments of "correctness" has the potential to build classroom norms where students are willing to participate and share their thinking as well as consider others' thinking. For example, relative to Claudia's classroom in Illustration 2, her ability to develop second-order models of her students' thinking was associated with increases in student sharing and improvements in the quality of their explanations when sharing their thinking with other students. As a result, all individuals had an opportunity to learn from each other in the classroom; Claudia learned from her students and the students learned from each other. We suggest that researchers continue to investigate how a teacher's propensity and capacity for decentering influences classroom norms and individual student thinking.

Conclusion

In this article, we shared two representative illustrations of a secondary mathematics teacher attempting to model her student's thinking. We characterized what it means for a teacher to build a second-order model of student thinking during teaching (i.e., decenter), while revealing how teachers can leverage their students' thinking to improve the quality and nature of classroom interactions. In doing so, we demonstrated the usefulness of the decentering construct to focus on both teacher and student thinking while examining the nature and quality of student–teacher interactions. The illustrations provide evidence that a teacher who understands her students' thinking can make more deliberate in-the-moment decisions for the purpose of advancing student learning. Teachers' capacities to decenter impacts such things as teachers' decisions to pose (or not pose) a question, the nature of teachers' questions, the quality of their explanations, and their choices for student contributions. We have also provided evidence to support that teaching and learning opportunities go unnoticed (or can go unnoticed) when teachers pay more attention to the actions (e.g., procedures) that students *do*, rather than to the meanings that students construct or hold. We encourage other researchers to use the decentering construct as a lens to examine the quality and nature of mathematical conversations that occur during instruction.

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