On the use of dynamic animations to support students in reasoning quantitatively

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This study addresses the well-documented issue that students struggle to write meaningful expressions and formulas to represent and relate the values of quantities in applied problem contexts. In developing an online intervention, we drew from research that revealed the importance of and processes involved in conceptualizing quantitative relationships to support students in conceptualizing and representing quantitative relationships in applied problem contexts. The results suggest that the use of dynamic animations with prompts that focus students' attention on conceptualizing and relating quantities can be effective in supporting students in constructing meaningful expressions to represent the value of one quantity in terms of another, and formulas to define how two co-varying quantities change together.

Key words: Quantitative Reasoning; Problem Solving; Online Learning; Dynamic Imagery

Introduction

This study investigated student learning in the context of online lessons that were designed to support students in conceptualizing and relating quantities in applied contexts. It is well documented that students have difficulty knowing how to approach word (or applied) problems (Schoenfeld, 1992; DeFranco, 1996; Geiger and Galbraith, 1998; Carlson & Bloom, 2005; Moore & Carlson, 2012). A major difficulty for students in defining meaningful formulas or functions to model how quantities are related in an applied context results from their lack of effort to conceptualize the quantities in the problem and consider how they are related and change together (Moore & Carlson, 2012). Writing formulas to represent how two varying quantities change together further requires that students conceptualize variables as a means of representing the varying value that a quantity assumes, as opposed to only seeing a variable as an unknown value to solve for (Trigueros & Jacobs, 2008; Jacobs, 2002). This study leveraged these research findings to design an instructional intervention to support student learning in an online instructional environment. Furthermore, past research findings informed the development of research-informed and adapted instructional sequencing to support students in employing reasoning abilities needed to construct formulas and graphs that are meaningful to students as ways of conveying how two varying quantities change together in applied problem contexts. In this article we describe the online instruction. We then report the results of a study that examined a student's thinking as he interacted with the online dynamic animations, responded to instructional prompts, and viewed videos designed to support students in conceptualizing and relating quantities and expressing these relationships symbolically.

Theoretical perspective

The orientation phase in problem solving has been generally described by Polya (1957) and Carlson & Bloom (2005) as making sense of the problem, organizing relevant information, and developing a plan for producing a solution. In more recent studies it has been revealed that the mental process of orienting to a problem context involves initially conceptualizing the quantities in the situation and imagining how the relevant quantities are related and change together (Carlson & Moore, 2015; Moore and Carlson, 2012). The conceptualization of quantities in a situation and how they are related has been described by

Thompson (2002) as quantitative reasoning; the act of analyzing a situation into a network of quantities and relationships between quantities. Thompson further describes a *quantity* as a conceived attribute of an object that one envisions as being measureable. Integral to reasoning quantitatively is the act of quantification, "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship … with its unit" (Thompson, 2013). Upon deciding on a unit of measure, the *value* of a quantity is the numerical value assigned to the measurement of that quantity. According to Thompson (2013) it is important that students are able to perform quantitative operations - mental operations on quantities for the purpose of characterizing new quantities - as opposed to only performing numerical operations on real numbers. An example of a quantitative operation is comparing the lengths of Car A and Car B additively to determine how much longer Car A is than Car B, using subtraction to determine (numerically) this difference, and knowing that *the length of Car A – the length of Car B* represents the amount by which the length of Car A exceeds the length of Car B. An example of a numerical operation is performing the calculation 5–12 for the purpose of finding a number.

Writing meaningful formulas further requires that one consider how the two varying quantities to be related with a formula change together. Carlson et al. (2002), Saldanha & Thompson (1998), and Thompson (1992) have described *covariational reasoning* to be the cognitive activities involved in coordinating two varying quantities while simultaneously attending to the ways in which they change in relation to each other. In the context of this study, we examine covariational reasoning in the context of two continuously varying quantities. In general, we use the term *covariational* as an adjective to describe an entity that involves two quantities varying simultaneously.

Conceptual analysis and design

Stemming from the work of Von Glasersfeld (1995), Thompson (2008) presents two meanings for the term *conceptual analysis* – to build a model of knowing that might help the researcher understand how a person might know an idea, and to devise a way of understanding an idea such that if a student had such a way of understanding, it would likely support that student in dealing mathematically with his or her environment. We refer to these as conceptual analyses of the first and second type, respectively. In this section, we use the term in the latter manner – to propose a way of thinking that may be powerful for supporting a student in building particular mathematical meanings.

Lesson 1 was designed to support students in identifying and relating quantities in a given applied problem-context. The over-arching student-learning goal for Lesson 1 of the intervention is for students to be able to write an expression that describes the value of one quantity in terms of the value of another quantity. Past research has shown that the act of conceiving of a situation in terms of quantities and determining how the conceived quantities are related and vary together is imperative for writing meaningful expressions and formulas (Moore & Carlson, 2012; Carlson & Moore, 2015; Moore, 2013). As such, in this lesson the student watches an animation of a dynamic situation in which at least two quantities are varying together (See Figure 1). They are prompted to determine values of one quantity when specific values of another quantity are known, and are then asked to consider changes in one of the varying quantities when given that the other varying quantity changes from one specific value to another. We have evidence from past studies that the mental imagery required to respond to these types of questions requires students to conceptualize how the value of some known fixed quantity can be combined with specific values of one varying quantity to determine values of the other varying quantity (e.g. Moore & Carlson, 2012; Carlson & Moore, 2015). Therefore, we provide the student with multiple occasions to engage in covariational reasoning and imagine combining a fixed quantity and a varying quantity to obtain another varying quantity in various non-complex problem contexts.

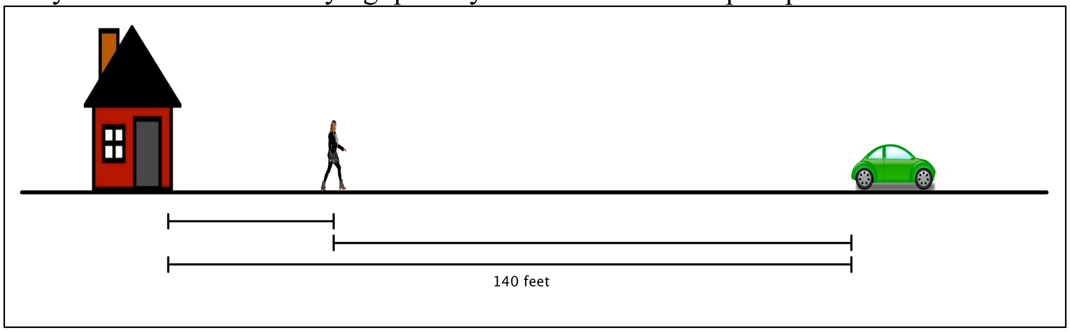

Figure 1: Jo Walking to Car

Lesson 2 introduced ideas of variable, expression, and formulas. Since past studies have documented that students have difficulty interpreting what is being asked when a problem requests students to define "one variable in terms of another," specific prompts were included to support students in developing this meaning. Past studies have also revealed that students have a strong tendency to view variables as always representing a single unknown value to solve for (Trigueros & Jacobs, 2008). A student who only thinks about a variable as an "unknown to solve for" will likely have difficulty using variables meaningfully to write formulas that define how two co-varying quantities change together. We introduce a variable as a letter or symbol that represents the varying value that a quantity can assume, and provide an emphasis on the varying nature of variables. The use of variables is motivated by the notion that trying to precisely relate the values of varying quantities via words can quickly become very cumbersome. Subsequently, the student is engaged in describing the meaning of variables in dynamic problem contexts. The student is also provided with tasks that involve determining how the value of one variable changes when the value of another variable changes from some initial value to some final value. Once the student has been provided with an intellectual need to use variables, we introduce the notion of *formula* as a mathematical statement that both expresses the value of one quantity using the value of another quantity and describes how the values of two quantities co-vary. In this intervention, we present formulas in a function-like manner. We reserve the phrase "in terms of" to describe the directionality of the formula – that is, to describe which quantity's value is being explicitly represented using the value of the other quantity. For example, we take a formula to express x in terms of y if the formula determines *the* value of x given any valid value of y; as a convention, such a formula is in the form x = <some expression in y>. The student is provided with instructional videos discussing the idea of formula, as well as the surrounding conventions. Amongst the probing questions in this lesson, the student is given practice computing the value of one quantity given the value of another quantity, and writing formulas to express one quantity "in terms of" another.

Methods

An online intervention was developed to support students in first conceptualizing quantities in an applied context and then consider how two varying quantities in the context change together. The design of the intervention was guided by past research on writing meaningful formulas to relate two co-varying quantities. To justify the content and scaffolding of the online intervention we performed a conceptual analysis (of the second type) of the mental processes involved in: conceptualizing and relating relevant quantities in a problem context, defining variables, and writing meaningful formulas to relate the values of two co-varying quantities. The online intervention in this study consisted of two consecutive "lessons." Each lesson was of a similar structure, consisting of carefully scaffolded instructional videos, mathematical animations, interactive applets, and probing questions. Once a learning trajectory was developed, I designed animations and applets using GeoGebra (Hohenwarter, 2001), a dynamic geometry software. We then recorded the instructional videos using a screen-recording software. We embedded these videos, animations, applets and probing questions into a math-assessment and course platform, IMathAS (Lippman, 2006).

The lessons were piloted with two Pre-Calculus level students who completed a series of three clinical interviews in which they worked through the online intervention while the interviewer watched the student and periodically asked questions to elicit the student's thinking. After the interview data was collected, we analyzed the data and performed a conceptual analysis (of the first type) for the purpose of building a model of the student's thinking. We then made inferences about the ways in which the online intervention shifted the student's meanings. We conclude by offering suggestions for using these research findings to guide future revisions of the online instructional materials. The subject discussed in this manuscript is Alex[1], an undergraduate student studying pre-law at a large university in the Southwestern United States.

Results

The first page of Lesson 1 is grounded in the following context: "*Jo walks the 140 foot distance from the front door of her house to her car*." The student was given the animation portrayed in Figure 1. He was then asked a series of questions designed to support him in conceptualizing and relating pairs of related quantities in the problem context. The first two tasks prompted the student to select from a list of quantities that vary and those that are constant within the given problem context. Alex had no difficulty identifying the constant and varying quantities in this situation. The next task provided the student (consecutively) with three values for the distance from Jo to her front door (in feet), and asked the student to compute the corresponding distance from Jo to the car (in feet). The resulting interactions are provided in Excerpt 1.

Excerpt 1:

1	Alex	How far is Jo when she is 40 feet from her front door. She's 40 feet from the door [on the animation Alex uses mouse to point from front door to somewhere between the door and the car] and this is 140 feet [points to line segment with 140 marked on it] means she is 100 feet from her car [with mouse, points from somewhere between the door and the car to the car].
2	Alex	When she is one-hundred and fifteen feet. [1-second pause] One hundred and forty minus one hundred and fifteen is twenty-five.
3	Alex	[Alex computes 140 – 55.3 in head] Normally I would be writing this down. But the thought process is one-forty minus that number.

In Line 1 of Excerpt 1 Alex used the mouse to point at the quantities in the animated diagram that he used to determine Jo's distance from her car. While doing so he noted the 40 feet Jo had walked from the front door and the 140 feet between the front door and car, and

[1] Alex is a pseudonym.

realized that the distance Jo needed to walk to her car could be determined by considering what he must add to Jo's distance from her front door (40 feet) to obtain the total distance of 140 feet. We consider this mental process to be a quantitative operation, wherein Alex envisioned comparing two quantities to obtain a new quantity; subtraction was the appropriate numerical operation to evaluate the value of the quantity resulting from the quantitative operation. Lines 2 and 3 of Excerpt 1 then show Alex performing the same numerical operation without reference to the diagram. We interpret this as Alex generalizing the quantitative operation and resulting numerical operation he enacted in Line 1 to other situations wherein the distance from Jo to the front door is known and the distance from Jo to the car is to be determined. Alex then went on to correctly answer a prompt to select a worded expression that represents the quantity "the distance from Jo to her car." After Alex completed Page 1 of Lesson 1, the interviwer (Grant) asked him for his reactions to the page. This interaction is displayed in Excerpt 2.

Excerpt 2

1	Grant	Alright, good. So what. What are your initial thoughts on this page? Is there anything that seemed confusing to you other than using words instead of x and y?
2	Alex	No. I thought the diagrams really, really helped. And that's kind of what I was visualizing before. A constantly moving person, but the 140 stays fixed. Knowing those two things grounds pretty much everything else that I thought about in this problem.
3	Grant	Okay, good. So what about… what do you think of these line segments that change as she walks [moves mouse to the diagram in the instructional video]?
4	Alex	I really like that. Because it shows that you have two different variables changing as she's walking, or moves. And I like that in relation to that 140, which doesn't move. In my mind, I was taking that a step further and picture numbers on the two distances, one increasing and one decreasing. To show the relationship between the two, not just in lines but also in distances.

In Lines 2 and 4 of Excerpt 2 Alex expressed that the quantitative images provided by the animation supported him in reasoning about the problem context. In Line 2, Alex's comments suggest that knowing and visualizing how the three relevant distances varied in relation to one-another grounded his thinking for "pretty much everything else that I thought about in this problem." In Line 4, Alex's comments suggest that he visualized the moving distance segments as representing "two different variables changing as she's walking," one whose value is increasing while the other's is decreasing. We infer that the animated diagram supported Alex in both conceptualizing the relevant quantities as well as imagining how they co-vary.

The first page of Lesson 2 is grounded in the following context: "A 14-inch long candle is lit and steadily burns until it is burned out." This page starts with an instructional video designed to motivate the usefulness of variables as a means of representing the varying values that a varying quantity assumes. In this context the variable b was defined to represent the varying number of inches that have burned away from the candle; the variable b is portrayed on the animated diagram next to a steadily increasing line segment representing the burned length of the candle. This video was followed by prompts to explain the meaning of the variable b and the expression $14 – b$. The last of these tasks prompts the student: "According to the video, what does the letter b represent in the context of the candle-burning problem?" Alex's verbal reaction to this task is presented in Excerpt 3.

Excerpt 3

1	Alex	…the length that has been burned. And before I'm looking at the answers,

		I'm thinking that's what I have in my head. And it is an unknown value, so [option] A looks right. But I'm going to scan the other answers. It's not the remaining length. Oh, but it's not! I did that wrong. I was focusing on the 'unknown' versus 'known,' but that's not the contrast. The contrast is a single unknown versus a varying value, and I am looking for a varying value. That would be the burned length.

In Excerpt 3, Alex's response revealed that he conceived of the variable *b* as representing varying values, rather than a single unknown value. The second instructional video describes how one could use a value of *b* to determine the remaining length of the candle (in inches). The variable *r* is also defined to represent the values of the remaining length of the candle (in inches). Following this video is a series of tasks designed to engage the user in determining how the value of *r* changes if *b* changes from some initial value to some final value. Alex responded to the prompt "As *b* varies from $b = 7.5$ to $b = 12$ inches, how does *r* vary?" by determining that the remaining length of the candle would decreases from 7.5 to 2 inches. The next task on this page began with the prompt: "What does your answer to the above question mean?" The student is then given a sentence with four missing words that can be selected via dropdown select-menus (displayed in Figure 2 below). Alex's response to this task is presented in Excerpt 4 below.

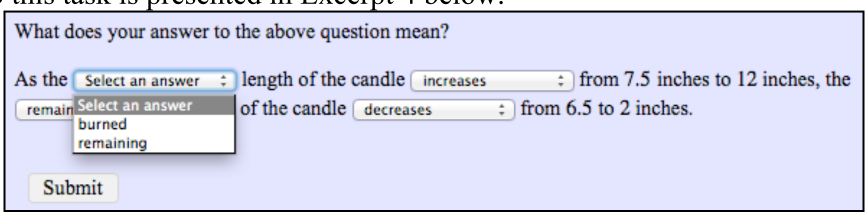

Figure 2: Candle Burning Task

Excerpt 4:

1	Alex	So what does the above answer mean? So I look at the above answer and try to predict first, as opposed to looking at the answers. So that means as the burned length of the candle increases, the length, *r*, decreases. I'm seeing a similar structure… And that is how we initially conceptualized it. And that makes sense, because you're essentially just saying as one variable goes up the other variable goes down.
2	Grant	And does that make sense to you… like if you imagine the candle, does that make sense?
3	Alex	Yes, especially from the video, when they showed how one variable was increasing while the other one was decreasing. That wording is how I would have worded that from the video.

In Line 1 of Excerpt 4, Alex determined that as the value of *b* increases, the value of *r* decreases. In Line 3, Alex explained that this statement aligned with his dynamic image of the variables changing together. In Line 3 Alex mentioned the dynamic diagram and how it "showed how one variable was increasing while the other was decreasing," suggesting that the dynamic diagram supported Alex in conceptualizing how the relevant quantities vary together. From Line 3 of Excerpt 4 we infer that the dynamic diagram supported Alex in conceiving of the variables *b* and *r* as varying while imagining how they change together as the candle burns. Although it may be the case that, without the dynamic diagram, Alex could have conceived of *b* and *r* as varying wherein *r* decreases as the value of *b* increases, Excerpt 4 suggests that the dynamic diagram strongly supported Alex in bolstering these conceptions. A few minutes after the interaction in Excerpt 5, Alex responded to the task prompt: "What does the expression 14 – *b* represent in the context of the candle burning question?" by saying,

Excerpt 5:

1	Alex	So 14 is the initial length of the candle. *b* we know is the burned length. So that equation is going to be varying because we know *b* is a varying variable. And if you have that in an equation and you don't have *b*, you don't have a varying value. So, we know that A is correct because 14 minus *b* gives you the remaining length as a varying value.

Line 1 of Excerpt 5 suggests that Alex conceived of the expression 14 – *b* as representing the varying values of the remaining length of the candle. Alex did not only conceive of 14 – *b* as representing the remaining length of the candle, he also conceived of it as varying as the value of *b* varies. Alex continued working through this Page of Lesson 2, and was able to determine a formula that expresses *r* in terms of *b* and a formula that expresses *b* in terms of *r*.

Conclusions and implications

The findings suggest that the dynamic diagrams presented in the intervention supported Alex's construction of a coherent and useful image of the quantities in the problem context, and how relevant quantities change together. These images emerged as Alex conceptualized the quantities in the situation and determined how they were related. This suggests that the dynamic animations with accompanying prompts that support students in conceptualizing the quantities in a problem context, and how they change together, can be effective in helping students construct expressions and formulas that are meaningful to them.

From these results we draw the following implications relative to computer-based instructional design.

i. Dynamic diagrams can be useful tools for supporting students in conceptualizing relevant quantities in problem contexts. To further support students in conceptualizing relevant quantities in a problem context, these animations should be supported by questions that probe the student to consider various quantities in the context and describe how they vary together.
ii. The dynamic nature of the diagrams can be leveraged to support students in thinking in dynamic ways. For example, the candle burning animation was effective in supporting the student in conceptualizing variables as varying, not as single unknown values. This can be extended to other ideas, such as constant rate of change and proportionality, that students often think of statically but are perhaps more productively thought of as dynamic.
iii. Computer-based tools can be used to simultaneously represent the co-variation of two quantities via different representation systems. This capability can be used to support students in "seeing" the meaning of the symbols that they write on their paper, in the sense that in the student's mind, the symbols and expressions they write represent the values of quantities within the context.

Although the implications we mention above are framed in the context of computer-based instructional activities, we believe that such implications are relevant to in-class practice. Dynamic diagrams and applets can be used to center classroom conversation around particular mathematical ideas while simultaneously providing a dynamic view of how multiple quantities might vary together and how one might represent the varying values of these quantities and how they are related.

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