An Investigation into the Relationships Among Teachers' Mathematical Meanings for Teaching, Commitment to Quantitative Reasoning, and Actions

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An Investigation into the Relationships Among Teachers' Mathematical Meanings for Teaching, Commitment to Quantitative Reasoning, and Actions

by

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ABSTRACT

Over the past thirty years, research on teachers' mathematical knowledge for teaching (MKT) has developed and grown in popularity as an area of focus for improving mathematics teaching and students' learning. Many scholars have investigated types of knowledge teachers use when teaching and the relationship between teacher knowledge and student performance. However, few researchers have studied the sources of teachers' pedagogical decisions and actions and some studies have reported that advances in teachers' mathematical meanings does not necessarily lead to a teacher conveying strong meanings to students. It has also been reported that a teacher's ways of thinking about teaching an idea and actions to decenter can influence the teacher's interactions with students.

This document presents three papers detailing a multiple-case study that constitutes my dissertation. The first paper reviews the constructs researchers have used to investigate teachers' knowledge base. This paper also provides a characterization of the first case's mathematical meaning for teaching angle measure and the impact of her meaning on her interactions with students while teaching her angle measure lessons. The second paper examines another instructor's meaning for an angle and its measure and illustrates the symbiotic relationship between the teacher's mathematical meanings for teaching and decentering actions. This paper also characterizes how an instructor's commitment to quantitative reasoning influences the teacher's instructional orientation and instructional actions. Finally, the third paper includes a cross-case analysis of the two instructors' mathematical meanings for teaching sine function and their enacted teaching

practices, including their choice of tasks, interactions with students, and explanations while teaching their sine function lessons.

DEDICATION

To all young women in STEM, if not you, who?

ACKNOWLEDGMENTS

I would like to first thank my family for all their love and support during my childhood and throughout my continued education. Everything that I am today is because of each of you! Second, my fellow Hamm Slab, Caren Burgermeister, I can't thank you enough for the support and laughs you provided, especially during my first year of graduate school when I needed it most. Thank you for being my family in Arizona! Third, to my dog Boomer, thank you for your endless cuddles, licks of support, and the constant reminder to take a break and go outside! Without you this journey would have been a lot less fun.

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CHAPTER 1
STATEMENT OF THE PROBLEM

The need to improve mathematics education in the United States has been broadly documented, with research identifying procedurally focused curricula and poor-quality teaching as two primary sources of the problem (Stigler & Hiebert, 1999; Thompson & Thompson, 1994; Thompson & Thompson, 1996; Thompson, 2013). Researchers have reported that students in the United States are not learning as much or as deeply as their European and Asian counterparts and are scoring significantly lower on international assessments (National Council of Teacher of Mathematics, 2014; Stigler & Hiebert, 1999). Lack of resources and professional development for teachers (Stigler & Hiebert, 1999), along with a cultural inattention to mathematical meaning and coherence, has contributed to U.S. students' low scores (Thompson, 2013).

Over the past thirty years, research on teachers' *mathematical knowledge for teaching* (*MKT*) has developed to improve teaching and students' learning. Ball and many of her colleagues have investigated the various types of knowledge teachers use when teaching (Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2005) and the relationship between teacher knowledge and student performance (Ball, Thames, & Phelps, 2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004, Hill et al., 2008). This area of research has primarily focused on investigating what teachers do in their teaching and the knowledge required to engage in these teaching actions (Ball & Bass, 2003). Although many mathematics educators have investigated the relationships among the mathematics that teachers know, their instruction, and students' learning, few researchers have studied the sources of teachers' pedagogical decisions and actions, teachers' *mathematical meanings for teaching* (*MMT*) (Thompson, 2016).

The affordances of attending to teachers' *meanings* as opposed to teachers' *knowledge* is illustrated in the following example. If two teachers, Teacher A and Teacher B were asked to determine the value of $sin(\theta)$ in Figure 1 below, both Teacher A and Teacher B may correctly answer $sin(\theta) = \frac{1.78}{3.96} = 0.45$ indicating that they *know* how to answer this question. However, this correct answer provides mathematics educators with limited insight into the teacher's approach to teaching students how to think about and answer this question.

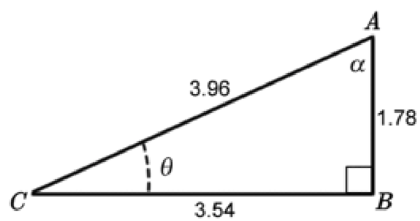

Figure 1. The Value of the Sine Function

However, a shift in focus to what teachers mean by $\sin(\theta) = 0.45$ can uncover more about what a teacher might say or do in a classroom, and thus, the meanings students have the opportunity to construct. For instance, imagine that Teacher A uses the commonly applied trigonometric ratios (SOHCAHTOA) to answer this question. With this way of thinking, Teacher A might only convey that the word sine is a command to divide the length of the side opposite an angle in a right triangle by the triangle's hypotenuse. In contrast, imagine that Teacher B conceives of $\sin(\theta)$ as a function that represents how a length varies with an angle's measure. Namely, that sine is a function that relates the measure of an angle (whose vertex lies at the center of a circle) swept out from the 3 o'clock position of a circle with the vertical distance of a terminal point above the horizontal diameter of the circle in radius lengths. A teacher with this meaning may conceive of $sin(\theta) = \frac{1.78}{3.96} = 0.45$ as a relative size measure that represents a vertical distance of a point on the terminal ray of the angle that is approximately 0.45 radii or 0.45 times as large as the circle's radius when the terminal ray of an angle has been rotated θ radians counterclockwise from the 3 o'clock position on the circle; and will likely have the goal to convey this meaning to her students (Silverman & Thompson, 2008). As such, uncovering teachers' meanings for an idea is essential for understanding a teacher's instructional goals and actions. Although attention to teachers' mathematical meanings for teaching has been repeatedly called for (Byerley & Thompson, 2017; Musgrave & Carlson, 2017; Tallman & Frank, 2018; Thompson, 2013; 2016), few researchers have investigated teachers' MMT for teaching specific ideas.

In response to calls for more investigations of teachers' MMT and the need to improve U.S. mathematics teachers' instruction, I investigated the relationships between teachers' MMT, their decentering actions, and subsequent instructional practices. Thompson (2013) proposes that a teacher's mathematical meanings for an idea constitute their image of the mathematics they teach, their pedagogical decisions, and the language they use to cultivate similar images in students' thinking. More recently, Tallman (2015, 2021) has provided empirical support for Thompson's claims that a teacher's teaching actions are strongly related to their mathematical meanings. Thus, it seems reasonable that a teacher's MMT may be related to their ability to attend to and effectively leverage students' thinking while teaching. The primary questions motivating this dissertation study are:

1. What mathematical meanings for teaching angle measure and sine function do teachers construct when using a research-based curricula?

2. What is the relationship between a teacher’s mathematical meanings for teaching angle measure and sine function, and the teacher’s instructional practices, including their instructional decisions, explanations, and interactions with students when teaching?
3. What is the relationship between a teacher’s mathematical meanings for teaching an idea and their decentering actions?

CHAPTER 2
INTRODUCTION TO THREE PAPERS

In the following three chapters, I present three papers detailing a multiple-case study that constitutes my dissertation. As part of this study, I investigated two pre-calculus instructors' mathematical meanings for and teaching of angle measure and the sine function. The first paper begins by distinguishing two constructs *Mathematical Knowledge for Teaching (MKT)* and *Mathematical Meanings for Teaching (MMT)*, that researchers have used to characterize teachers' knowledge base as it relates to their teaching. After introducing these constructs and their uses, I discuss the affordances of attending to a teacher's mathematical meanings for teaching an idea. Following this, I provide a characterization of a teacher's mathematical meanings for teaching angle measure and an example of the impact of her MMT for angle measure on her interactions with students while teaching her angle measure lessons.

In the second paper, I introduce a construct my colleagues and I have used to characterize teachers' ways of thinking as they relate to teaching. I then illustrate the usefulness of this construct by presenting an example of a second instructor's ways of thinking about teaching angle measure. In this paper, I also present data that illustrates the symbiotic relationship between the teacher's mathematical meanings for teaching and decentering actions. The data included in this paper also provides empirical evidence of the affordances of a teacher's engagement in quantitative reasoning and the ways his commitment to quantitative reasoning supported his instructional actions.

The third paper presents a cross-case analysis of the two instructors' mathematical meanings for teaching the sine function and their enacted teaching practices. In particular, this paper illustrates the relationship between each instructor's mathematical meanings for teaching sine function and the nature of their (1) goals for students' learning, (2) explanations, and (3) interactions with students. This paper illuminates the ways in which a teacher's mathematical meanings for teaching an idea influence their images of ways to support students' learning of an idea, how these images influence their instructional choices and actions, and ultimately how their instructional choices and actions influence the meanings that students have the opportunity to construct.

Collectively these three papers make a case for the critical role a teacher's mathematical meanings for teaching an idea have on the mathematics instruction that a student experiences. Furthermore, a comparison of the two teachers' MMT for sine function and professional development experiences suggests that the process of supporting a teacher in constructing strong MMT needs further investigation.

CHAPTER 3
PAPER 1: INVESTIGATING A TEACHER'S MATHEMATICAL MEANINGS FOR TEACHING ANGLE MEASURE AND HER ACTIONS WHILE TEACHING: THE CASE OF SHIRA

LITERATURE REVIEW

Research on Teachers' Mathematical Knowledge for Teaching

In the last three decades, researchers have turned their attention to teachers' MKT to improve the quality of mathematics teaching and learning in the United States. The literature on teachers' MKT is vast, with diversity in ontological stances and research foci among studies.

A Practice-Based Theory of MKT

Ball and colleagues (e.g., Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2005) sought to answer the question, "what do teachers do in teaching mathematics, and in what ways does what they do demand mathematical reasoning, insight, understanding, and skill?" (Ball, Hill, & Bass, 2005, p. 17). Ball's work on teacher knowledge began in 1990 when she studied 252 pre-service elementary teachers' understanding of division with fractions. In this study, most teacher candidates could correctly complete the procedures of dividing fractions. However, few were able to describe a situation that represented the division of fractions accurately to students (Ball, 1990).

Following Ball's (1990) findings of teachers' limited mathematical understandings, Ball and Bass (2003) proposed focusing on the mathematical work of teachers, including how and where teachers use their mathematical knowledge in practice. Ball and Bass' analyses revealed that the mathematical knowledge needed for teaching differs from that of mathematicians and other practitioners of mathematics. Namely, mathematics teaching requires teachers to have knowledge of (1) the work involved in problem-solving, (2) how to “unpack” their mathematical understandings, and (3) how to help students in making connections across mathematical domains (ibid). These empirical analyses lead to Hill, Schilling, and Ball’s (2004) proposal that teachers’ MKT is multi-dimensional, and their refining of Shulman’s (1986) original category of subject matter knowledge to include teachers’ *common content knowledge (CCK)* and *specialized content knowledge (SCK*).

In the past twenty years, these researchers have explored the multi-dimensional nature of teachers’ knowledge when teaching and classified teachers’ MKT into five types of specialized knowledge (Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2005). These researchers have also identified positive links between teacher knowledge and student performance (Ball, Thames, & Phelps, 2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004; Hill et al., 2008).

A Piagetian Theory of MKT

While Ball and colleagues (e.g., Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2003; Hill, Rowan, & Ball, 2005) conducted empirical studies that categorized qualitatively-distinct domains of teachers’ knowledge, Silverman and Thompson (2008) developed theories on the nature of teachers’ MKT and how it develops. Silverman and Thompson (2008) made two propositions, (1) teachers’ MKT is grounded in personally powerful understandings of mathematical concepts, and (2) teachers’ MKT is created through the transformation of those concepts from an understanding having pedagogical potential to an understanding that has pedagogical power.

Mathematical understandings are personally powerful if they carry throughout an instructional sequence, are foundational for learning other ideas, and play into a network of ideas that do significant work in students' reasoning (Thompson, 2008). According to Silverman and Thompson (2008), *Key Developmental Understandings (KDUs)* (Simon, 2006) are an example of mathematical understandings that are personally powerful. A KDU is an individual's understanding of an idea that is critical to their learning of other related ideas (Simon, 2006). We say an individual has developed a KDU for a mathematical idea if they (1) make a conceptual advance or have "a change in [their] ability to think about and/or perceive a particular mathematical relationship" (Simon, 2006, p. 362) and (2) develop an understanding for the idea through their own activity and reflection on their activity.

Silverman and Thompson (2008) propose that "developing MKT involves transforming [personal] KDUs of a particular mathematical concept to an understanding of: (1) how this KDU could empower their students' learning of related ideas; (2) actions a teacher might take to support students' development of it and reasons why those actions might work" (p. 502). More concretely, the development of MKT involves transforming teachers' KDUs into *Key Pedagogical Understandings (KPUs).* A KPU is a mini theory a teacher has regarding how to help students develop the meanings she intends (Byerley & Thompson, 2017). As such, these researchers developed a framework to describe the process by which teachers' MKT-their KDUs and awareness of their KDUs- develop.

Silverman and Thompson's (2008) developmental framework enables teacher education to shift its focus from developing teachers' MKT to helping teachers develop practices that support their ability to continually develop their MKT throughout their career. "These practices include development of KDUs, becoming reflectively aware of them, and placing them within a model of student learning in the context of instruction (Silverman & Thompson, 2008, p. 509)."

In his 2015 dissertation study, Tallman presented results that support Silverman and Thompson's claim that the mathematical knowledge needed to teach involves more than powerful understandings of mathematics. Tallman's analysis revealed that a teacher's particular ways of understanding, as demonstrated in a clinical interview, differed from and sometimes contradicted the meanings the instructor conveyed while teaching. Tallman (2015) described the inconsistencies in the teacher's mathematical knowledge as a consequence of the teacher's MKT consisting of "disorganized and disconnected cognitive schemes" (p. 592). As such, Tallman (2015) suggested that mathematical knowledge for teaching involves an awareness of the mental actions and operations that constitute productive ways of understanding mathematical ideas. Tallman further elaborated,

> A teacher's awareness of the mental actions and operations that constitute powerful ways of understanding mathematical ideas supports him or her in: (1) defining instructional goals and objectives in cognitive rather than behavioristic terms, (2) designing and/or selecting curriculum materials that seek to engender intended mental activity, (3) employing pedagogical actions that support students in engaging in—and thereby constructing internalized representations of—the mental actions that constitute productive mathematical meanings, (4) developing a disposition to attend to students' thinking, and (5) constructing models of students' epistemic ways of understanding (p. 598).

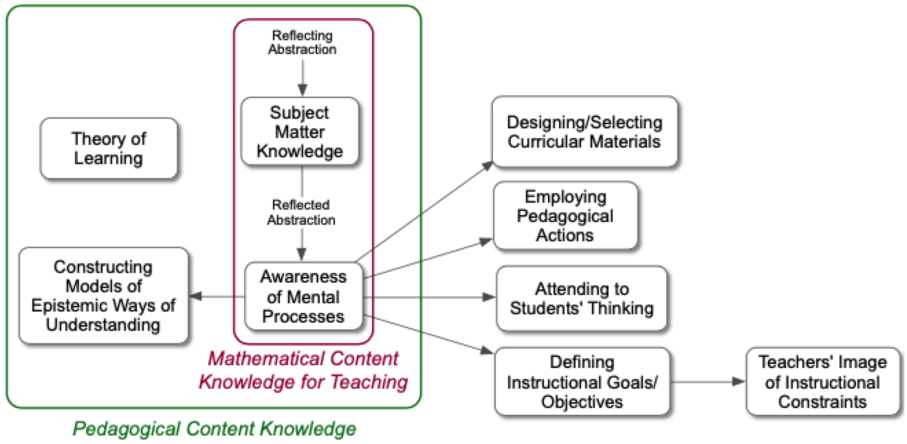

Figure 2. Tallman's Theoretical Framework for Mathematical Knowledge for Teaching

As such, Tallman (2015) proposed a framework that suggests teachers' awareness of the mental actions and operations that constitute productive ways of understanding mathematical ideas is a critical component of teachers' MKT as it supports them in reorganizing their mathematical knowledge to engage in effective teaching practices (see Figure 2).

Comparing Research on Teachers' MKT

There are three distinguishing characteristics in research programs focused on teachers' MKT. First, Ball and colleagues conducted empirical studies of teachers' knowledge. In contrast, Silverman and Thompson (2008) proposed a framework for developing teachers' MKT based on theory and their research experiences. The second involves the focus of each program. For example, Ball and colleagues' investigations primarily focused on what teachers do in their teaching and the knowledge required to engage in these teaching actions (Ball & Bass, 2003).

In comparison, Silverman and Thompson (2008) and Tallman (2015) focus on the cognitive mechanisms that enable teachers' behaviors. These researchers seek to uncover the nature of teachers' knowledge and how it develops. According to Silverman and Thompson (2008), the third distinguishing feature of MKT research involves their ontological stance that an individual's knowledge is constructed and is idiosyncratic. Ball and colleagues (e.g., Ball 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2005) describe teachers' knowledge as a truth about a reality external to the knower. In comparison, Silverman and Thompson (2008) and Tallman (2015) describe teacher knowledge in the sense of Piaget and von Glasersfeld (1995)- as schemes and ways of coordinating them that explain how a person might act when teaching. Moreover, Silverman and Thompson describe a teacher's knowledge base in terms of her individual cognitive structures and thought patterns. These researchers use the phrase *Mathematical Knowledge for Teaching (MKT)* to describe teachers' schemes or meanings for the ideas they teach and hold at a reflected level.

Research on Teachers' Mathematical Meanings for Teaching

In 2016, Thompson proposed using the construct *Mathematical Meanings for Teaching (MMT)* rather than MKT to make explicit that he used the word knowledge to describe an individual's schemes or meanings for an idea. An individual's meaning is the space of implications resulting from assimilation to a scheme (Thompson, 2016). Thus, to say an individual has a meaning for a word, symbol, expression, or statement means that the individual has assimilated that word, symbol, expression, or statement to a scheme. A scheme is a mental structure that "organize[s] actions, operations, images, or other schemes" (Thompson et al., 2014, p. 11). When defining *Mathematical Meanings for Teaching,* Thompson (2016) extended the construct of mathematical meaning to account for characterizations of teachers' actions related to teaching. As such, a teacher's mathematical meanings for teaching an idea include (1) the meanings (schemes) the teacher uses while teaching or thinking about teaching and (2) the teacher's image of the meanings (schemes) they want students to develop for the idea.

The construct of *Mathematical Meanings for Teaching* differs from the construct of *Mathematical Knowledge for Teaching* in two distinct ways. First, Thompson's use of the word meaning connotates something personal (i.e., meanings existing in the knower's mind) to readers rather than knowledge, which seems less personal and disjoint from the knower (Thompson, 2016). Second, arguably most important, the construct of *Mathematical Meanings for Teaching* describes the status of an individual's understanding relative to teaching. Although Ball and Thompson defined teachers' MKT differently, both researchers used the construct MKT to describe teacher knowledge as a target for teachers to achieve, not a state of teacher's knowledge that may be advanced. For instance, Ball, Hill, and Bass (2005) use the example of multiplying two integers to demonstrate that a teacher's sole ability to compute 35 x 25 correctly is insufficient knowledge. These researchers argue instead that teaching involves knowledge of an effective way to represent the meaning of the algorithm to multiply. In this example, Ball, Hill, and Bass (2005) state what teachers must be able to do and know, but they do not make explicit the ways of thinking that enable the teacher to do or know. As a second example, Silverman and Thompson (2008) outline a framework for developing teachers' knowledge that supports conceptual teaching of a particular mathematical topic. This framework frames teachers' MKT as something to be attained instead of a status of teachers' knowledge that may be advanced.

The Affordances of Attending to Teachers' MMT

Attending to teachers' mathematical meanings for teaching an idea has many affordances. Investigations into teachers' meanings enable researchers to focus their attention on the mathematical conceptions teachers have and the implications of these understandings on students' learning. In particular, focusing on teachers' MMT enables researchers to shift their focus from the behaviors that a teacher exhibits while teaching to the cognitive mechanisms, meanings, and ways of thinking that enable these behaviors. This shift in focus to teachers' meanings allows mathematics educators to explain why teachers act as they do and how we can improve their teaching (Thompson, 2013; 2016).

> "A focus on MMT would also foster the field's conceptualization of bridges among what teachers know (as a system of meanings), how they teach (their orientation to high-quality conversations), what they teach (meanings that an observer can reasonably imagine that students might construct, over time, from teachers' actions), and what students learn (the meanings they construct)" (Thompson, 2013, p. 82).

By investigating instructors' meanings for ideas, we, as mathematics educators, are positioned to enhance our understanding of teachers' practice and develop ways to improve it. Researchers who study teachers' MKT describe teacher knowledge as a target for teachers to achieve, not a state of teacher's knowledge. These researchers focus their lens on teacher performance instead of studying and characterizing the meanings teachers exhibit in relation to their instructional practices (Thompson, 2013).

Research on Mathematics Teaching Practices

Research on secondary and post-secondary mathematics teachers has predominantly focused on examining the relationship between various teaching practices and student learning. A teacher's practice includes everything a teacher does that contributes to their teaching (planning, assessing, interacting with students) and everything teachers think about, know, and believe about what they do (Simon & Tzur, 1999).

In 2010, Speer, Smith III, and Horvath proposed a framework that describes the dimensions of post-secondary instructors' practice and differentiates teaching practices from instructional activities. These researchers propose that a teachers' practice includes "what teachers do and think daily, in class and out, as they perform their teaching work" (Speer et al., 2010, p.99). In contrast, Speer et al. describe instructional activities as "the organized and regularly practiced routines for bringing together students and instructional materials (textbooks, whiteboards, overhead projectors, computer-generated graphic displays, etc.) to support students' learning of mathematics" (Speer et al., 2010, p.101). Speer et al. contend that investigations into collegiate mathematical instructors' practice could benefit from researchers attending to the following practices included in their framework: (1) allocating time within lessons, (2) selecting and sequencing content within lessons, (3) motivating specific content, (4) posing questions, using wait time, and reacting to student responses, (5) representing mathematical concepts and relationships, (6) evaluating and preparing for the next lesson, and (7) designing assessment problems and evaluating student work.

Since 2010, research on various teaching practices has become more prominent in mathematics education literature. Jacobs, Lamb, and Philipp (2010) introduced the construct of professional noticing to describe teachers' attention to, interpretation of, and response to students' mathematical thinking as it is expressed in writing or classroom video recordings. Jacobs and Empson (2016) extended Jacobs et al.'s (2010) work by introducing teacher responsiveness, "a type of teaching in which teachers' instructional decisions about what to pursue and how to pursue it are continually adjusted during instruction in response to children's content specific thinking, instead of being determined in advance to noticing" (Jacobs & Empson, 2016). They have also captured aspects of teaching moves characteristic of teacher responsiveness in a framework that can be used in the moment with students to support and extend students' mathematical thinking.

Several researchers have also extended Jacobs et al.'s (2010) work by investigating teachers' abilities to attend to and leverage others' thinking when interacting with another (Bas Ader & Carlson, 2021; Carlson, Bowling, Moore, & Ortiz, 2007; Teuscher, Moore, Carlson, 2016). In particular, these researchers have argued that decentering or setting aside one's thinking in an attempt to understand what students understand (Steffe & Thompson, 2000) is a useful construct for explaining and characterizing teachers' responses and interactions with students while teaching. This perspective differs from the research on teacher noticing as it "moves beyond discourse analysis by including a focus on the teacher's interpretations of students' verbal and written explanations to make decisions in the moment of teaching" (Teuscher, Moore, Carlson, 2016, p. 437).

THEORETICAL PERSPECTIVE

Quantitative reasoning- the analysis of a situation into a quantitative structure, a network of quantities, and quantitative relationships (Thompson, 1990, 1993, 2011), has been identified as a critical way of thinking that supports students' development of coherent meanings for angle measure and trigonometric functions (Hertel & Cullen, 2011; Moore, 2010, 2014; Tallman, 2015; Thompson, 2008).

Within the theory of quantitative reasoning, a quantity is a quality of something that one has conceived as admitting some measurement process (Thompson, 1990). Quantities exist in the mind of the individual conceiving them. To comprehend a quantity, an individual's conception of "something" must be elaborated to the point that they "see" characteristics of the object that are admissible to the process of quantification (ibid). Quantification is a direct or indirect measurement process that results in a value. A quantity's value is the numerical result of a quantification process. Numerical operations are used to calculate a quantity's value, however, numerical operations differ from quantitative operations. A quantitative operation is the conception of two quantities taken to produce a new quantity. Put another way, a quantitative operation is a description of how quantities come to exist (Thompson, 1990). One cannot have a completed quantitative operation without having created a quantitative relationship. A quantitative relationship involves conceiving three quantities, two of which determine the third by a quantitative operation. However, one cannot construct a quantitative relationship without conceiving a quantitative operation.

A Quantitative Meaning for Angle Measure

An angle is a geometric object formed by two rays that meet at a common vertex. Many students and teachers recognize that measuring an angle involves quantifying the openness between two rays that meet at a common vertex. However, the openness of an angle is a quantity only to those who conceive of an attribute to measure, an appropriate unit of measure, and a process by which to measure it (Thompson, 2008; Tallman & Frank, 2018). For example, one may conceptualize measuring the openness of an angle by measuring the length of the arc of a circle subtended by the angle whose vertex lies at the center of the circle. However, for the measure of the angle to be independent of the size of the circle, the arc length needs to be measured in units that covary with the length of the subtended arc so that the ratio of the *subtended arc length* to a *unit length* is always constant for an angle with a fixed amount of openness (Tallman, 2015). Thus, the unit of measure must be proportional to the subtended arc length and, by extension, the circle's circumference (Thompson, Carlson, & Silverman, 2007).

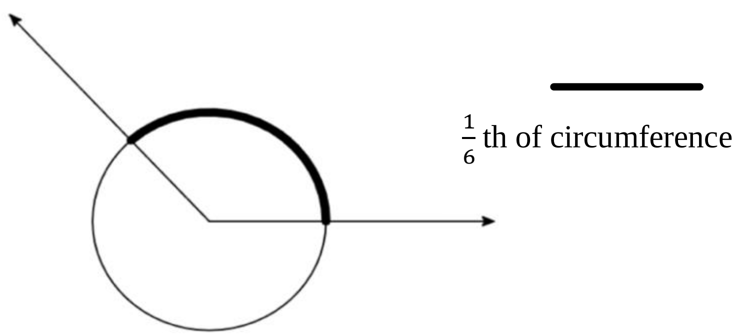

Figure 3. Measuring an Angle in Units Proportional to the Circumference

For example, we could measure the angle in Figure 3 by multiplicatively comparing the length of the subtended arc to the length of the circle's circumference. As such, the angle would have a measure of 1/3 as it subtends an arc that is 1/3 times as large as the circumference of the circle centered at the vertex of the angle. One could also imagine measuring the angle's openness in Figure 3 by multiplicatively comparing the length of the subtended arc to the length of 1/6th of the circumference of the circle centered at the angle's vertex. Again, the angle would have a measure of 2 since the angle subtends an arc that is always two times as large as 1/6th of the circumference of the circle centered at the angle's vertex. Thus, a productive conceptualization of angle measure involves conceiving angle measure as a measurement process that defines a multiplicative relationship between a subtended arc and some unit proportional to the circumference of the circle centered at the angle's vertex (Moore, 2014).

However, to understand the need for the unit of measure to be proportional to the circumference of the circle that contains the subtended arc requires one to realize that a fixed angle always subtends the same fraction of the circumference of *any* circle centered at the vertex of the angle (Moore, 2010). For example, an angle measures one radian if it subtends a class of arcs that each measure one radius length or are $1/2\pi$ times as long as the circumference of the circle (for which the arc is a part) centered at the vertex of the angle. Similarly, an angle measures one degree if it subtends a class of arcs that are 1/360 times as long as the circumference of the circle (for which the arc is a part) centered at the vertex of the angle. As such, one could measure the angle in Figure 4 by multiplicatively comparing the length of either of the three subtended arcs (red, green, or blue) to the length of the respective circle's radius. Thus, the angle in Figure 4 measures 1.2 radians as it subtends a class of arcs that are each 1.2 times as large as the length of the radius of the respective circle that contains the subtended arc.

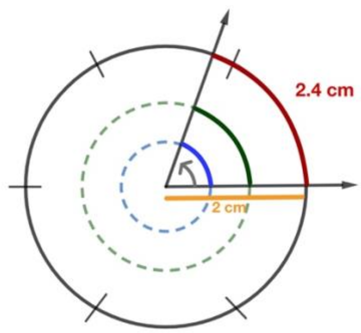

Figure 4. Class of Arcs

METHODS

This study aimed to investigate the relationship between a teacher's mathematical meanings for teaching angle measure and her instructional practices. To this end, I conducted an exploratory multiple-case study (Stake, 1995; Yin, 2009). The primary goal of an exploratory case study is to expand and generalize theories (ibid). As such, the reader must keep in mind that case studies are less about accurate descriptions of "the case" and more about generalizing theoretical propositions. I chose to do an exploratory multiple-case study because the data I had access to was limited to contemporary events (e.g., classroom observations and clinical interviews). This methodology also served as the most appropriate method for closely investigating the interaction between a teacher's MMT and instructional actions by allowing me to focus in-depth on two cases (Yin, 2009). Lastly, I selected to conduct a multiple case study to provide contrasting cases of two instructors' MMT angle measure and sine function and the interaction between their meanings and their practices.

This paper presents an in-depth analysis of one instructor, Shira, who was a graduate student instructor (GSI) in her first year (second semester) of teaching precalculus using the Pathways to Calculus research-based curriculum (Carlson, Oehrtman, Moore, & O'Bryan, 2020) at a large research university in the southwest United States. Shira was a Ph.D. student studying applied mathematics at the time of this study. I selected Shira as the case for this study because:

1. she was in her first year of teaching precalculus using the Pathways curriculum.
2. she had limited experience (one and a half semesters) attending professional development seminars led by the leaders of the Pathways Precalculus research team.
3. she was willing to let me video-record her classroom instruction.

Concurrent with this study, Shira was enrolled for the second semester in a required professional development seminar that met weekly for ninety minutes. The purpose of the professional development seminar was to (1) support the graduate student instructors in developing coherent meanings and ways of thinking about the ideas to be taught during the upcoming week and (2) support the GSIs in clearly explaining their meanings for these ideas to others. The professional development seminar primarily focused on the mathematical content and what was entailed in understanding and learning the key ideas of each lesson. During the seminar, GSIs and the seminar leader regularly discussed and engaged in the Pathways Conventions for supporting quantitative reasoning that included reinforcing patterns for speaking about quantities and relationships among quantities (speaking with meaning), conceptualizing a graph as a record of how two quantities' values vary together (quantity tracking tool), representing the quantitative structure of a problem context (quantitative drawing), and consistent expectations and methods for defining variables, constructing algebraic expressions and defining formulas (emergent symbolization) (Carlson, O'Bryan, & Rocha, 2023). During the weekly seminar, the GSIs were also asked to demonstrate (to the seminar leader and their teaching peers) how they intended to support their students in learning the key ideas in the upcoming lessons. Materials for such discussions and presentations stemmed from the instructor materials for the Pathways Precalculus curriculum and were supplemented by the PD leader's research background and online lessons that were developed for entirely online courses during the COVID-19 pandemic.

The Pathways Precalculus curriculum is a research-based curriculum that was designed as a result of many years of research on students' learning and instructors' teaching of precalculus ideas (Carlson, 1995; 1997; 1999; Carlson, Jacobs, Larsen, Coe, & Hsu, 2002; Carlson, Madison, & West, 2015; Carlson, Oehertman, & Engelke, 2010; Engelke, Carlson, & Oehertman, 2005; Kuper, 2018; Marfi, 2017; Moore, 2010; O'Bryan, 2019; Tallman, 2015). The Pathways curriculum was designed to (1) increase the retention and success of students in STEM and (2) shift instructors' practice to have a greater focus on developing students' understandings of precalculus ideas (Carlson, 2019). In particular, the Pathways curriculum materials (Carlson & Oehrtman, 2010) were designed to support instructors in fostering productive reasoning patterns in their students that research has revealed to be essential for students' construction of meaningful function formulas (e.g., Moore & Carlson, 2012; Thompson, 1988; 1990, 1992) and graphs (e.g., Carlson et al., 2002; Moore & Thompson, 2015).

Experimental Methods

The first phase of this study consisted of a task-based clinical interview (TBCI) (Clement, 2000; Goldin, 1997; Hunting, 1997) designed to elicit Shira's (1) meanings and ways of thinking about angles and their measures, (2) commitment to quantitative reasoning, and (3) image of how to support students in developing coherent meanings for angles and their measures. The second phase of this study consisted of a series of lesson observations and semi-structured clinical interviews that I engaged Shira in before and after her instruction on angles and their measures.

Before Shira's teaching, I conducted short semi-structured clinical interviews in which she posed questions that probed Shira's meanings for angle measure, her approach to planning lessons, and her image of the understandings she wanted students to construct. During these pre-teaching clinical interviews, I also inquired about the activities and conversations about the activities Shira anticipated having with students during her lessons on angle measure. Following this pre-teaching interview, I observed and video-recorded Shira's teaching. Immediately following Shira's teaching, I conducted short semi-structured clinical interviews in which I probed her rationale for specific instructional actions that I observed during her teaching.

Two days after Shira's respective lesson on angle measure, I conducted "video-analysis" clinical interviews with Shira. These video-analysis clinical interviews were conducted immediately before the subsequent pre-teaching interview with Shira. During the video analysis clinical interview, I showed Shira short video segments from her previous lesson. For these interviews, I selected clips in which the instructor:

1. interacted with a student or multiple students.
2. leveraged students' thinking in their teaching.
3. used an applet, table, or diagram to support their discussion of a mathematical idea.
4. deviated from their original lesson plan.

While watching the video segments with the instructor, I prompted her to explain their motive for specific comments, actions, and questions. I also posed questions to understand and characterize the mental actions the teacher engaged in when interacting with students. Table 1. outlines the timeline of the research activity conducted in each phase of the study.

Date	Research Activity	Duration
	Phase 1	
3/27	Angle Measure TBCI	90 Minutes
	Phase 2	
3/28	Pre-Teaching Clinical Interview 1	30 Minutes
3/28	Classroom Observation 1	50 Minutes
3/28	Post-Teaching Clinical Interview 1	20 Minutes
3/30	Video Analysis 1& Pre-Teaching Clinical Interview 2	60 Minutes
3/30	Classroom Observation 2	50 Minutes
3/30	Post-Teaching Clinical Interview 2	20 Minutes
4/1	Video Analysis 2 & Pre-Teaching Clinical Interview 3	30 Minutes

Table 1. Schedule of Phase 1 and Phase 2 of Data Collection

Analytical Methods

Preliminary Analysis

The angle measure TBCI constituted the only data for my preliminary analysis. During the TBCI, I took notes on the meanings Shira expressed. At this stage of my analysis, I was alert to moments when Shira (1) engaged in quantitative reasoning or (2) expressed a desire to support students' engagement in quantitative reasoning. After the interview, I took notes to record my initial thoughts and impressions of Shira's meanings for angles and their measures. Within 24 hours of the TBCI, I watched the video recording of the interview and wrote detailed memos that described Shira's responses and the understandings she expressed when responding to each task included in the TBCI. These memos constituted my initial model of Shira's MMT for angle measure.

Ongoing Analysis

My initial analysis occurred during the data collection phase of the study. Throughout the pre-teaching, post-teaching, and video analysis clinical interviews, I wrote detailed notes about Shira's responses and the understandings I interpreted her to be expressing. I also noted instances when she attempted to support her students' engagement in quantitative reasoning and interacted with students while teaching. In addition, I made note of instances when Shira discussed the meanings she wanted students to have for an angle and its measure. During the post-teaching and video analysis interviews, I noted any instances when Shira expressed interest in a student's thinking or discussed her image of how she thought a student(s) was thinking. Following each clinical interview, I wrote detailed memos on (1) how I hypothesized Shira to be reasoning about angles and their measures, (2) expressions that provided insights about the degree to which she engaged in quantitative reasoning, and (3) her attention to and expressions of how a student was thinking.

My ongoing analysis of Shira's instruction involved taking detailed notes as I observed her teaching. As I observed Shira's teaching, I noted moments when she (1) expressed a meaning for angles and their measures, (2) interacted with students in their groups, and (3) attended to or leveraged student thinking while teaching. Within twenty-four hours of a lesson, I watched the video recording of Shira's instruction. During this time, I selected segments of her instruction according to the three criteria above. These selected segments constituted the video clips Shira watched with me during the clinical interview video analysis.

Retrospective Analysis

The goal of my retrospective analysis was to i) construct a model of the instructor's meaning for angle measure, ii) characterize her interactions with students while teaching, and ii) document instances when the teacher's mathematical meaning for teaching angle measure appeared to influence her actions while teaching, and when her interactions with students while teaching appeared to influence her mathematical meaning for teaching angle measure.

My lens for analyzing the data included: i) the conceptual analysis of the idea of angle measure, described in the prior section, and ii) Bas Ader and Carlson's (2021) decentering framework (see Table 3). My conclusions were primarily based on my retrospective analysis of the data. My implementation of Simon's (2019) three-phased method for performing a qualitative analysis began with a line-by-line analysis of the TBCI to generate hypotheses about the subject's thinking. In this phase, I wrote detailed memos about my conjecture for (1) what the subject was doing and why they might have been doing it, (2) what the subject might have meant by what they said at this point, (3) what the subject might have been thinking, and (4) what this might show about the subject's understanding (Simon, 2019).

I then used the memos constructed in the first phase of analysis as data for the second phase. Specifically, I coded each memo using the codes shown in Table 2. My decision to use the codes in Table 2 was informed by prior research that has identified quantitative and covariational reasoning as essential ways of thinking for supporting teachers' conveyance of coherent meanings while teaching (Carlson, O'Bryan, & Rocha, 2023; Tallman, 2015; 2021; Tallman & Frank, 2018; Thompson, 2008; 2013) and students' construction of meaningful function formulas (e.g., Moore & Carlson, 2012; Thompson, 1988; 1990, 1992) and graphs (e.g., Carlson et al., 2002; Moore & Thompson, 2015).

Code	Description
Quantitative Reasoning (QR)	Instructor engaged in quantitative reasoning.
Quantitative Reasoning Students (QRS)	Instructor encouraged students to engage in quantitative reasoning (QR), took actions to support students' engagement in QR, or talked about wanting to support students' engagement in QR; OR students engaged in QR.
Covariational Reasoning (CR)	Instructor engaged in covariational reasoning
Covariational Reasoning Students (CRS)	Instructor encouraged students to engage in covariational reasoning (CR), took actions to support students' engagement in CR, or talked about wanting to support students' engagement in CR; OR students engaged in CR
Angle Measure (AM)	Instructor expressed a meaning for angle measure
Angle Measure Students (AMS)	Instructor described meanings they want students to have for angle measure OR student expressed meaning for angle measure.
Sine Function (SF)	Instructor expressed a meaning for the sine function.
Sine Function Students (SFS)	Instructor described meanings they want students to have for the sine function OR student expressed meaning for the sine function.
Instructor-Student Interaction (ISI)	Instructor interacted with student while teaching, discussed an interaction with a student from previous teaching, discussed an anticipated interaction with students while teaching.

Image of Student Thinking (ImgStu)	Instructor described their image of how students were thinking.

Table 2. Codes Used During Analysis

The goal of the second phase of analysis was to hypothesize (1) the understandings the subject was exhibiting and (2) how the subject was thinking as they progressed from one task to the next (Simon, 2019). I refined the angle measure codes to further categorize the subjects' thinking about angles and their measures. As one example, if a subject discussed radians or degrees, these instances were coded using the "Measurement unit" code. Similarly, if a subject discussed measuring the length of an arc subtended by an angle using some unit, these instances were coded using the "Measurement Process" code.

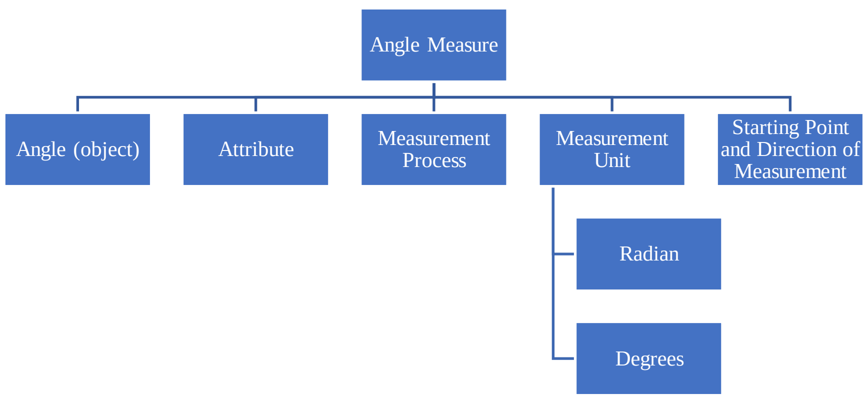

Figure 5. Angle Measure Subcodes.

Following my initial coding, I conducted a second pass that involved validating and refining my initial coding of the data. In the third and final analysis phase, I identified themes in the conceptual analysis that best characterized and modeled the instructor's MMT for angle measure. In this analysis phase, I reviewed the lines of the transcript that exemplified each theme and wrote a detailed summary that elaborated on each theme. I recorded each of these summaries beneath the transcript of the individual interview. These summaries constituted my emerging model of the instructor's MMT for angle measure.

When analyzing the instructor's classroom instruction, I used instructor-student interactions, including instructor-led discussions, as my unit of analysis. In the first round of analysis, I identified segments of the instructors' teaching in which (1) they expressed a meaning for angles and their measures or (2) interacted with students. During this round of analysis, I wrote detailed memos of each selected segment that detailed my hypotheses of how the instructor was thinking and/or why they interacted with students in the way they did. In the second round of analysis, I coded the previously selected segments using the codes shown in Tables 2 and 3 and Figures 4 and 5. When coding instructor-student interactions, I broke the instructor-student interaction code down into the subcodes shown in Table 3 and Figure 6. I then analyzed each code from the second phase of analysis and looked across the codes for themes to model how the instructor was thinking and/or the nature of the instructor's interaction with students.

Code	**Description**
Decenter 0 **(D0)**	Instructor shows no interest in the student's thinking but shows interest in the students' answer. Makes no attempt to make sense of the student's thinking but takes actions to get the student to say the correct answer.
Decenter 1 **(D1)**	Instructor shows interest in student's thinking but makes no attempt to make sense of the students' thinking. Attempts to move the student to their way of thinking without trying to understand or build on the expressed thinking and perspective of the student.
Decenter 2 **(D2)**	Instructor makes an effort to make sense of the student's thinking and perspective but does not use this knowledge in communication.

Decenter 3 **(D3)**	Instructor makes sense of the student's thinking and/or perspective and makes general moves to use the student's thinking when interacting with the student.
Decenter 4 **(D4)**	Instructor constructs an image of the student's thinking and/or perspective and then adjusts their actions to take into account both the student's thinking and how the student might be interpreting them.

Table 3. Decentering Framework (Bas Ader & Carlson, 2021)

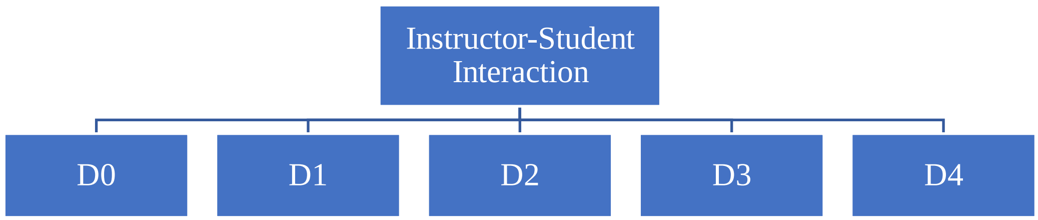

Figure 6. Codes Used to Characterize Instructor's Image of Student Thinking

RESULTS

Shira's Meanings for Angle Measure

Shira participated in one task-based clinical interview (TBCI) prior to her lessons on angles and their measures. During the TBCI, Shira completed a series of tasks designed to elicit her meanings and ways of thinking about angles and their measures. As described in the conceptual analysis section, a quantitative meaning for angle measure necessarily entails:

1. an understanding that the openness of an angle can be quantified by measuring the length of the arc subtended by an angle whose vertex lies at the center of the circle the angle subtends.
2. an understanding that an angle subtends the same fraction of the circumference of all circles centered at the vertex of the angle.
3. an understanding that the length of the subtended arc must be measured in a unit proportional to the circumference of all circles centered at the angle's vertex.

4. and an understanding that the measure of an angle can be determined by multiplicatively comparing the length of the subtended arc and some unit proportional to the circumference of the circle containing that arc.

Throughout the TBCI, Shira consistently distinguished an angle from its measure. She also consistently expressed that an angle's measure quantifies the openness between the angle's rays (see Excerpt 1).

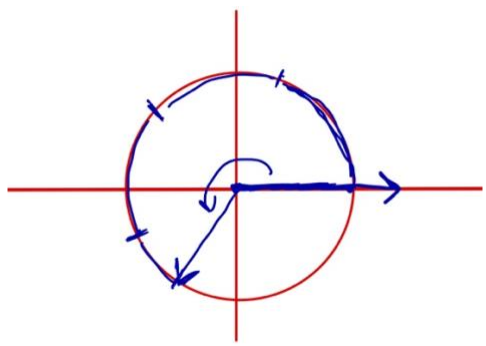

Figure 7. Shira's Drawing of an Angle That Measures 3.5 Radians

Excerpt 1

Shira: It says, what does it mean for an angle to measure 3.5 radians. Umm I am trying…. [long pause]. So, If I had an angle, we would need like a concept of how a radius relates to… I don't want to say relates to an angle, because a radius doesn't necessarily relate to an angle. So, if we have a radius length that measures *r* then for an angle to measure 3.5 radians means that it is 3.5 radius lengths. So, I could take this distance [points to r] and kind of imagine overarching it over an arc 3.5 times. So that would be about 3.5.

Int: Could you draw an angle that measures 3.5 radians?

Shira: I can try. So, it would be approximately it wouldn't be exact (see Figure 7).

Int: Can you tell me a little bit about how you drew that? How do you know where to put those tick marks?

Shira: I thought about the length of this distance [points to the radius of the circle] and kind of thought about placing the length of this distance [the radius] around the circle and doing that 3 and a half times.

Int: Why did you do that?

Shira: Because that is what the definition of what a radian is.

Int: What meaning do you want students to have for what the definition of a radian is?

Shira: A radian is a unit of measurement for measuring the openness of an angle.

But one radian corresponds to one radius length along the subtended arc.

When asked to draw an angle that measures 3.5 radians, Shira did so by drawing a circle and placing the vertex of an angle at the center of the circle. She then determined where to draw the terminal ray of the angle by envisioning laying the length of the circle's radius along the circle's edge three and a half times. These actions indicate that Shira had conceptualized the radius of a circle as a unit of measure for quantifying the openness of an angle and a process (laying the radius along the circle's arc) for measuring an angle. However, Shira's statement of "one radian corresponds to one radius length along the subtended arc" (Excerpt 1, line 21) suggests that she may have been thinking about a radian as a unit length (1 radian = 1 radius length).

She expressed similar thinking when she described what a degree was. As one example, Shira described a degree as the length of an arc that results from "[cutting] up a circle into 360 pieces evenly" (Excerpt 2, lines 8-9). Shira's descriptions of radian and degree in Excerpts 1 and 2 indicate that she was thinking about these units of measure as specific lengths to iterate along the arc of a circle. Although Shira recognized that radians and degrees correspond to particular lengths (a radian is one radius length, and a degree is the length of an arc that results from cutting a circle into three hundred and sixty pieces), as is supported in a later section, her statements and actions did not demonstrate an understanding that these measures (degrees and radians) always subtend the same fraction ($1/360^{th}$ or $1/2\pi$th respectively) of the circumference of *any* circle centered at the vertex of the angle.

Excerpt 2

Int: What would it mean for an angle to measure 30 degrees?
Shira: I could draw you a thirty-degree angle [draws a 90-degree angle and

partitions it into three smaller angles]. I would say that this angle [highlights one of the three smaller angles] is 30 degrees.
Int: Okay, and how do you know that?
Shira: Because I know this is a 90-degree angle [points to right angle] and if I were to divide it by three I would get this thirty-degree angle. But now that I am thinking about it, if I am not mistaken, one degree is… if you imagine a circle being cut up into 360 pieces evenly. Then I guess you could say one of those, arc lengths would correspond to one degree.
Int: What are you imagining cutting up?
Shira: I would say kind of like [pauses]…
Int: Can you draw where you are imagining cutting the circle?
Shira: Yeah it would kinda deal with cutting the circle in half, or just making sure the line goes through the center.
Int: Okay, so then you said this angle right here is 90 degrees [draws red rectangle near corner of the angle] so, how'd you know that?
Shira: It's just something that I've always been told. It is not something that I necessarily know.
Int: Yeah how do we determine that openness? Like how do we know it's 90 and not 110?
Shira: Yeah, I don't think I could tell you.

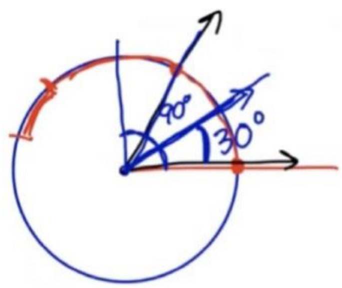

Figure 8. Shira's Drawing of an Angle That Measures Thirty Degrees

In Excerpt 2, I asked Shira what it means for an angle to measure thirty degrees. Shira responded by drawing a right angle and partitioning it into three smaller angles. Shira then said that one of the three smaller angles measures thirty degrees. When I asked Shira how she knew that the openness of a right-angle measures ninety degrees, she was unable to answer. Shira's inability to explain why a right-angle measures ninety degrees as opposed to one hundred and ten degrees further supports that she was solely thinking about degrees as a unit length (measuring 1/360th of the circumference of a circle centered at the angle's vertex) as opposed to also thinking about a ninety-degree angle as subtending a class of arcs that measure ninety-three hundred sixtieths or one-fourth of the circumference of any circle centered at the vertex of the angle. I also noticed Shira's inability to recognize that a fixed angle always subtends the same fraction of the circumference of any circle centered at the angle's vertex when she asked Shira to draw angles with various measures. For example, when asked to draw an angle measuring 2.7 radians, Shira drew an angle with the initial ray at a 3 o'clock position and the terminal ray in the third quadrant (see Figure 9).

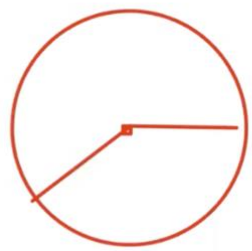

Figure 9. Shira's Drawing of an Angle That Measures 2.7 Radians

At the end of the first TBCI, I asked Shira to explain why degrees and radians are appropriate units for measuring angles (see Excerpt 3). Shira responded by stating radians and degrees are "independent of distance" (Excerpt 3, lines 3-5). However, when probed to explain why radians and degrees were independent of distance, Shira was unable to do so. I hypothesize that Shira was unable to answer this question because she did not conceive of these units as always subtending an arc that measures the same fraction (1/360th or 1/2πth, respectively) of the circumference of any circle centered at the vertex of the angle.

Figure 10. Shira's Drawing to Illustrate That Degrees and Radians Are Independent of Distance

Excerpt 3

Int: So, you have talked a lot today about radians and degrees. So why are we allowed to use those? Why are they appropriate units of measure?

Shira: Umm… [long pause] I would say it goes back to the fact because those are…independent of the… [long pause] hmmm… I guess you could say they are independent of distance. When I say distance, I mean like in essence I wouldn't be able to measure it with this [draws short blue vertical line] because this is a fixed distance and that is not necessarily [draws longer blue vertical line between angle's rays] but yet this is still as open here [draws inner blue arc] as it is over here [draw outer blue arc]. And likewise, with again thinking about the openness it's independent of the arc length because this length [highlights inner blue arc] is obviously smaller than this length [highlights outer blue arc].

Int: Would you be able to articulate why it is independent? Like with degrees if we measured that angle that you just drew, why is it that using degrees would be okay?

Shira: [very long pause] I really don't know.

Int: Okay, that's okay.

I posed the task in Figure 11 for the purpose of determining if Shira recognized that she could measure any angle by multiplicatively comparing the length of the arc subtended by the angle's rays and any unit proportional to the circle's circumference (including the circumference itself). Excerpts 4, 5, and 6 include Shira's response to this task.

Suppose you give students the angle below. Lamonte, a student in your class claims that the measure of the angle is 3/8^{ths}. Arlyse, another student in your class, claims that the measure of the angle shown below is 3. How is Lamonte thinking about measuring the angle? How is Arlyse thinking about measuring the angle? Are they both correct?

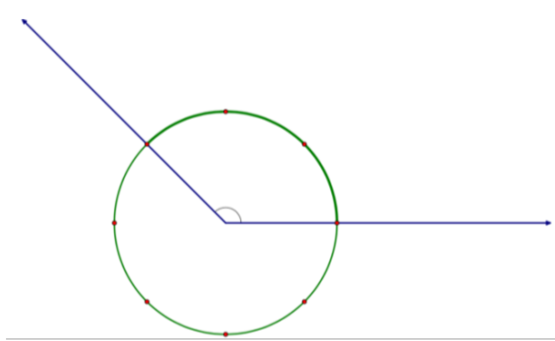

Figure 11. Angle Measure TBCI Unit of Measure Task (Adapted from Tallman, 2015).

Excerpt 4

Shira: 3/8ths so I am assuming there is 8 dots? [counts to check] Okay [long pause].
Int: What are you thinking?
Shira: I am trying to think if… Like earlier I made the claim that cutting up the circle into 360 pieces I guess then the openness of one of those is one degree… Umm and so, I am trying to think if that is what Arlyse is doing?
Int: When you say “that”, what do you mean?
Shira: If Arlyse is cutting the circle into 8 pieces instead of three sixty.
Int: Is that a valid way of measuring the angle?
Shira: In part I want to say yes, because even if I had a bigger circle, umm the angle is still the same. The angle would still equal 3 whatever name units we want to give it. And then I am also trying to think about Lamonte.

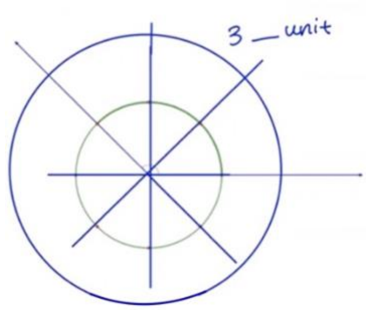

Figure 12. Shira’s Description of How Arlyse is Thinking

Shira responded to this task by relating Arlyse’s (hypothetical student) activity to her own activity of splitting a circle into three hundred and sixty pieces to determine the measure of an angle that is one degree. Shira then expressed that Arlyse’s splitting the circle into eight pieces is a valid way of measuring the angle because she could iterate one of those eight pieces on a larger circle and still get a measure of 3. Shira’s initial response to this task further indicates that she was thinking about measuring angles by iterating units; however, when asked what Arlyse’s unit of measure was, Shira expressed that Arlyse’s unit was the openness of one of the eight pieces (see Excerpt 5).

Excerpt 5

Int: Before we jump to Lamonte, what unit is Arlyse using?
Shira: Umm I would say, I don’t know. I feel like we can’t give it a name. I think we would have to give it a name.

Int: Like, we could just make anything up? Like doggos?
Shira: Yeah [laughs]
Int: What is one doggo then?
Shira: One doggo is this [highlights arc of 1/8th of circumference of small circle].
Well, it wouldn't necessarily be the arc length…
Int: So, what is one doggo?
Shira: This openness [draw arced arrow]. Yeah….

Although Shira correctly identified that Arlyse was measuring the angle using one of the eight pieces of the circle, Shira did not identify Arlyse's unit of measure as the length of an arc that subtends one-eighth of any circle's circumference centered at the angle's vertex. Furthermore, Shira's activity of iterating one of the eight pieces of the circle indicates that she may have been thinking about iterating a slice or sector of the circle that measures one-eighth of the circle's area as opposed to the length of an arc that measures one-eighth of the circle's circumference.

Excerpt 6

Int: Okay, so let's go to Lamonte. How is he thinking?
Shira: He is thinking about it in terms of a fraction. So, he says this [highlights
arc in blue] measures 3 of the whole. Ummm… [long pause].
Int: Yeah, what is Lamonte's unit of measure?
Shira: I would say the distance from here to here. Like the length of one of these
segments [highlights black arc].
Int: Okay. So, is Lamonte's unit of measure the same as Arlyse's?
Shira: I want to say no, but I am not sure that I have the full reasoning behind it
umm…
Int: Do you think Lamonte and Arlyse are both correct? Or do you think one
or both of them are incorrect?
Shira: I want to say that Arlyse is correct. I am not sure if Lamonte is correct.

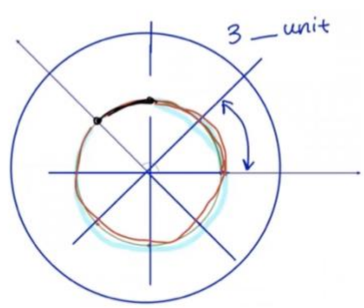

Figure 13. Shira's Description of Lamonte's Unit of Measure

In Excerpt 6, Shira expressed that Lamonte was measuring the length of the arc subtended by the angle's rays using the length of the black arc (see Figure 13) instead of the circle's circumference. When asked if Lamonte's unit of measure was the same as Arlyse's, Shira said no. This further supports that Shira may have thought Arlyse's unit of measure was a sector of the circle instead of an arc length. When asked if Lamonte's answer was correct, Shira said she was unsure. Shira's response to this task indicates that she had not conceptualized the circumference of a circle as an appropriate unit for measuring an angle's openness. Her response to this task also provides further evidence that she was solely thinking about units of angle measure as unit lengths (or potentially unit sectors) to be iterated and not simultaneously as measures of an angle's openness that subtend a class of arcs that always measure the same fraction of the circumference of *any* circle centered at the vertex of the angle.

Shira's Meanings for Degree as Expressed While Teaching

Following the angle measure TBCI, I attended Shira's classes in which she was leading lessons on angles and their measures. Shira had about thirty-five students in her class. The students were seated at six round tables around the classroom. Each table had approximately six students seated together. During Shira's class, students worked together at their tables most of the time.[1] They worked with their group to complete the workbook investigations that included conceptually focused problems and scaffolded questions that prompted them to explain their thinking. After completing specific questions in their groups, Shira typically led a discussion by asking particular groups or

[1] The students sitting at one table worked together as one group.

individuals to share their answers and thinking. Shira repeated this pattern of group work followed by a short class discussion for the duration of the 50-minute class period.

I collected the data in the next section during Shira's second lesson on angles and their measures. At the start of this class, Shira introduced degree angle measure. Shira's description of a degree is in Excerpt 7.

Excerpt 7

Shira: What you can imagine, right, is that degrees come from cutting up this circle into 359 equal pieces [draws multiple diameters through center of circle]. The reason why it would be 359, right, is because we want it to be cut into 360 pieces. Does that make sense? Okay, so then one of those, so then one degree is gonna be a really small sliver [motions with her hands to indicate the whole sector]. And that would measure one degree.

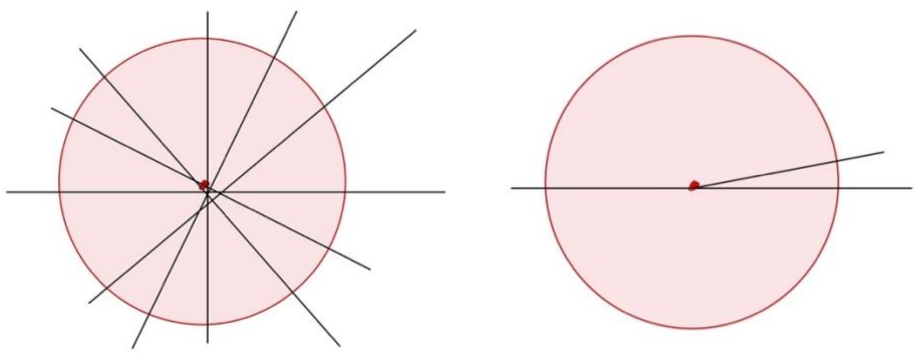

Figure 14. The Image Shira Drew on the Board While Introducing Degree Angle Measure

Shira followed her introduction of one degree by prompting students to work on the task shown in Figure 15 with their groups. As students worked on the task, Shira walked around the room and observed the students' work. Excerpt 8 shows Shira's interaction with a group of students as they worked through the task shown in Figure 15.

Describe what it means for an angle to have a measure of 10 degrees.

Figure 15. Degrees Task That Students Completed in Groups

Excerpt 8

Shira: So, what are we thinking? What does it mean?

Student 1: It has a measure of 10 degrees.
Shira: What has a measure of 10 degrees?
Student 2: The circle.
Student 1: The angle.
Shira: Okay… Umm…
Student 1: The space between the two rays.
Shira: What do you mean by space?
Student 1: Like right here [can't see what the student is pointing to].
Shira: Okay, so the openness?
Student 1: Yeah, the openness.
Shira: Okay, but we just defined what a degree was, right, what was a degree?
Student 3: A unit of…
Student 2: Of angle measure!
Shira: Okay, so it is a unit of angle measure, but like specifically, where does it come from?
Student 1: from splitting it 359 ways.
Shira: Okay, good so then 10 degrees would equal…
Student 3: Ten slices
Shira: Ten slivers right, ten slices. Alright. Okay keep it going [walks toward group 2].

In this interaction, Shira asked students to describe what it means for an angle to measure ten degrees. A student in the group responded by explaining that ten degrees come from splitting a circle three hundred and fifty-nine ways. Another student in the group further clarified that ten degrees would equal ten slices that result from splitting the circle three hundred and fifty-nine ways. Shira responded to students by confirming that ten degrees are "ten slivers [or] ten slices" (Excerpt 8, line 21). Shira's response to these students further suggests that she may have been thinking about degrees as sectors of a circle. Shira concluded this interaction with students by telling them to continue working on the remainder of the task (which contained two other parts). She then moved to a second group of students. Excerpt 9 shows Shira's interaction with the second group of students.

Excerpt 9

Shira: Alright what did we say over here?

Student 4: Ten over three sixty

Shira: Ten over three sixty, what do you mean ten over three sixty, what does that mean?

Student 4: Umm, you know how each degree is one over three sixty?

Shira: Well, one over three sixty of the circle’s circumference, yeah.

Student 4: So, ten degrees would be like ten over three hundred and sixty. So, it would be like ten of the little dashes [motions with hands in an arc motion].

Shira: Okay, so I would make the argument that it would be ten of those arc lengths, right, but I wouldn’t say it’s like, like ten degrees does not mean ten over three sixty [moves to group 3].

Following her discussion with group 1, Shira immediately moved to group 2. When Shira prompted group 2 to explain what it means for an angle to measure ten degrees, Student 4 said, “ten over three sixty,” to which Shira disagreed (Excerpt 9). Shira’s response of “it would be ten of those arc lengths right, but I wouldn’t say… ten degrees does not mean ten over three sixty” (Excerpt 9, lines 10-12) supports that Shira was thinking about degrees solely as units to iterate.

While working with group 2, a student from a nearby table overheard Shira tell Student 4 that ten degrees does not equal ten over three sixty. After hearing Shira say this, Student 5 immediately raised their hand, and Shira moved to the third group to assist Student 5 (see Excerpt 10).

Excerpt 10

Shira: Student 5 how are we doing?

Student 5: We’re confused now that you said it wouldn’t just be ten over three sixty because that is what it shows. Because one degree is one over three sixty so, wouldn’t ten degrees be ten over three sixty?

Shira: So, we’re saying we have ten of those one over three sixtieths but we’re not saying… It’s like if I had a pizza and I cut it up into like eight things. And I had a slice. Right, I take a slice out. I could say I only have one slice. Now granted I could say I have one of the eight slices, right, but you would not say I have one eighths of a slice. So, what I am saying, the angle measure would be ten degrees. That’s without dividing by three sixty. What you're finding there is a portion of the whole when you say ten divided by three sixty, but the angle measure is just ten degrees. It wouldn't be ten divided by three sixty.

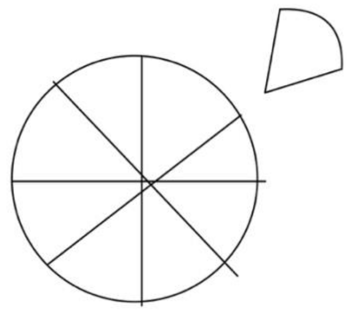

Figure 16. The Image Shira Drew on the Board When Discussing Pizza Slices

Shira's response to group 3 further shows that she was thinking about degrees as units to iterate and not as a unit of measure that subtends an equivalence class of arcs that each measure one three hundred and sixtieth of the circumference of any circle centered at the angle's vertex. Shira's response to group 3 also shows the impact an instructor's mathematical meaning for teaching angle measure can have on their interaction with students and the meanings students can reasonably construct. In this example, Shira's conception of degrees solely as units to iterate led to her confusing students about the meaning of ten degrees. It is also noteworthy that Shira was inconsistent in describing degrees as measures of "slivers or slices" and as "little arc lengths."

Immediately following Shira's lesson on degree and radian angle measure, I conducted a semi-structured clinical interview. During this interview, I asked Shira about her discussion of the meaning of ten degrees during class and probed her rationale for discussing slices of pizza. Excerpt 11 shows my discussion with Shira about her teaching of the task shown in Figure 16.

Excerpt 11

Shira: They wanted to say ten degrees meant ten divided by three sixty. So that was kind of…. I'm just not one hundred percent sure they understood.

Int: Were you surprised by that? When they said ten divided by three sixty?

Shira: Uh, in part I was, like I wasn't expecting them to say ten divided by three sixty, but in part I understood where it was coming from. Like the definition. But no, I wasn't expecting it.

Int: Yeah, why do you think they said that?

Shira: Umm… [long pause] I think they said that because, I think they were trying to say like we had ten of the one hundred and three sixtieths of the circle.

Int: So, you responded to students by talking about slices of pizza. What were you hoping to draw students' attention to by talking about slices of pizza?

Shira: Like if I say I have one slice of pizza, that does not mean that we have one eighth of a slice of pizza. And true if you look at the whole pizza, and you have one slice, then yes you do have one eighth of the entire pizza, but you don't have one eighth of a slice. Which I feel like is what they were doing, or what it meant for them to take ten degrees divided by three sixty degrees.

Shira conveyed that she wasn't sure if 10 degrees represented 10/360; and she was surprised when students said, "ten degrees meant ten divided by three sixty" (Excerpt 11, lines 1-6). Shira's response further supports that she was not thinking about ten degrees as the measure of an angle that subtends a class of arcs that are 10/360 times as large as the corresponding circumference of any circle centered at the angle's vertex. Excerpt 11 also shows that Shira interpreted students' statement of ten divided by three sixty to mean ten three hundred sixtieths of a degree (a fraction of a degree) instead of conceiving of ten divided by three sixty as representing a fraction of a circle's circumference. It is also noteworthy that when I asked Shira why she thinks the student might have said "ten divided by three sixty," she responded by stating what she believed the student meant to say (which also happens to be how Shira thinks about degrees) as opposed to considering that the student may be thinking differently than she was. As such, I characterized Shira's interaction with these students as Decentering level 1 behavior. Shira showed interest in the students' thinking but tacitly assumed the student was trying to express a meaning for ten degrees that aligned with her meaning for degrees.

Two days after her lesson on radians and degrees, Shira was shown a video clip of this segment of her teaching in a clinical interview setting. While watching this segment of her teaching, I asked Shira for the second time how she believed Student 4 (from Excerpt 9) to be thinking when they said, “ten divided by three sixty.” Shira said she believed the student was thinking about ten over three sixty as “ten of those little arc lengths.” Shira also expressed that she wanted the student to think about ten degrees as ten arc lengths that each measure 1/360th of the circle’s circumference and “not as a portion of the whole.” This further illustrates that Shira conceived of degree angle measure as unit lengths (measuring 1/360th of the circumference of a circle centered at the angle’s vertex) to iterate and not simultaneously a unit that always subtends a constant portion (1/360th) of the circumference of any circle centered at the vertex of the angle. Moreover, Shira’s second interpretation of Student 4’s statement of “ten over three sixty” as “ten of those little arcs” further supports that Shira made no attempt to understand student 4’s thinking and instead assumed they were thinking in the same way she was.

CONCLUSIONS AND DISUSSION

This paper illuminates the affordances of attending to an instructor’s mathematical meanings for teaching an idea by providing a model of an instructor’s meanings for angles and their measures and an example of how an instructor’s MMT for angle measure contributed to student confusion and ultimately limited the meanings students could reasonably construct. Shira’s conception of degrees solely as units to iterate and not simultaneously as a unit of measure that always subtends an equivalence class of arcs that measure one three-hundred and sixtieth of the circumference of any circle centered at the angle's rays led to her conveying meanings that were incoherent, inconsistent, and confusing for students.

During a lesson on degree and radian angle measure, Shira was inconsistent in describing a process for measuring angles. In one instance, Shira described ten degrees as subtending ten little arcs that each measured one three-hundred sixtieth of the circumference of a circle centered at the angle's vertex. However, in another instance, Shira described ten degrees as "ten slivers or slices" to iterate. Moreover, Shira's conception of degrees and radians as unit lengths (or sectors) to iterate led to her telling students that ten degrees are "ten little arc lengths or slivers." Thus, it is likely that students who took Shira's statement at face value conceptualized angle measure as the result of an iteration process (laying down ten slivers or arc lengths) and not as a measure of openness that subtends an equivalence class of arcs that each measure the same fraction (10/360th or 1/36th) of any circle centered at the vertex of the angle.

The results of this study also provide empirical support for Silverman and Thompson's (2008), and Carlson et al.'s (in press) claims that the nature of an instructor's meanings for a specific mathematical idea influences the nature of the model the instructor builds of a student's thinking and her subsequent instructional actions. In particular, this paper provides an empirical example of a teacher operating from a first-order model of students' thinking. This paper further illustrates the ways an instructor is constrained to using their own thinking and meanings to make sense of students' activities when operating from a first-order model of students' thinking.

The results of this study also echo Tallman and Frank's (2018) claims that teaching angle measure coherently requires teachers to have made subtle mathematical connections. Shira's instruction often lacked coherence because she was inconsistent in identifying quantities, attending to units of measure, and constructing quantitative relationships. As such, the results of this study suggest that a teacher's inattention to conceptualizing and relating quantities contributed to her conveying incoherent meanings for angle measure. These findings motivate mathematics teacher educators to engage teachers in experiences that support their development and conveyance of mathematical meanings grounded in quantitative reasoning.

Moreover, these findings challenge assumptions that graduate students in mathematics have strong meanings for foundational mathematical ideas (Ellis, 2014). These findings also challenge claims that requiring teachers to take more high-level mathematics classes will significantly advance their mathematical understandings and enacted teaching practices. Although Shira had taken numerous PhD-level mathematics courses and attended a weekly professional development seminar for almost two-full semesters before this study, she still struggled to develop and convey coherent mathematical meanings to students. As such, this paper echoes Carlson, O'Bryan, and Rocha's (2023) claim that transitioning teachers' mathematical meanings for teaching an idea is a slow process that requires sustained professional development that provides teachers with repeated opportunities to construct coherent meanings for the ideas they are teaching.

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CHAPTER 4
PAPER 2: AN INVESTIGATION INTO THE INTERACTION BETWEEN A TEACHER'S MATHEMATICAL MEANINGS FOR TEACHING ANGLE MEASURE AND HIS DECENTERING ACTIONS: THE CASE OF ENZO

INTRODUCTION AND LITERATURE REVIEW

Researchers have developed constructs in the last few decades to investigate teachers' knowledge base and how it relates to their teaching. These constructs include Mathematical Knowledge for Teaching (MKT) and Mathematical Meanings for Teaching (MMT). While research on teachers' knowledge base is vast, scholars' use of these constructs varies greatly. The constructs MKT and MMT differ in the nature of the phenomena they describe. Researchers have used the construct MKT to describe a knowledge base as a target for teachers to achieve. In contrast, the construct MMT has been used to describe the current state of teachers' mathematical understandings relative to teaching. While many researchers have advocated for more attention to teachers' MMT (Byerley & Thompson, 2017; Musgrave & Carlson, 2017; Tallman & Frank, 2018; Thompson, 2013, 2016), others have proposed that a sole focus on advancing teachers' MMT is insufficient for explaining a teacher's instructional planning and subsequent actions while teaching (Carlson, O'Bryan, & Rocha, 2023; Rocha, 2021; Tallman & Frank, 2018; Tallman, 2021).

Instead, scholars have proposed that teachers' *ways of thinking* are critical to their ability to develop, advance, and convey coherent meanings to students (Carlson, O'Bryan, & Rocha, 2023; Carlson et al., in press; Musgrave & Carlson, 2017; O'Bryan & Carlson, 2016; Rocha & Carlson, 2020; Rocha, 2022; Tallman, 2015; Tallman & Frank, 2018). As one example, Musgrave and Carlson (2017) investigated graduate teaching instructors' understanding of average rate of change (AROC) before and after an intervention designed to support instructors in engaging in conceptually oriented teaching (Thompson & Thompson, 1996).

Musgrave and Carlson (2017) found that only one of seven mathematics graduate-student instructors initially expressed a meaning for average rate of change as a tool for characterizing a function's change over some interval of the domain prior to engaging them in professional development focused on advancing their MMT for key ideas of precalculus mathematics. Following a summer workshop and weekly seminar that encouraged instructors to reason about quantities and provide conceptually oriented explanations of precalculus ideas and their problem solutions, most instructors expressed meanings for AROC that were conceptually focused and aligned with what Musgrave and Carlson (2017) deemed to be a productive meaning. Musgrave and Carlson argued that the impoverished nature of graduate student instructors' initial meanings for average rate of change may be widespread. As such, they call for researchers to attend to graduate instructors' meanings for foundational mathematical ideas and interventions to support their development of productive meanings for these ideas.

Moore & Carlson (2012) and Tallman & Frank (2018) have documented the importance of quantitative reasoning in learning and teaching precalculus ideas, and O'Bryan and Carlson (2016) have provided evidence that a teacher's engagement in quantitative reasoning and commitment to supporting their students' engagement in quantitative reasoning can lead to advancement in the teacher's mathematical meanings and enacted teaching practices. These researchers engaged a teacher in multiple professional development sessions focused on supporting the teacher in reasoning quantitatively, stating her goals for student learning in terms of student thinking, reasoning about mathematical expressions as representing relationships between two quantities' values, and thinking about how mathematically equivalent statements can represent individuals' different ways of conceptualizing a relationship among quantities' values. The results of their study revealed that a teacher's fluency in using quantitative reasoning and commitment to support her students' engagement in quantitative reasoning was associated with her being mentored in:

1. Expressing her meanings for algebraic expressions and processes.
2. Designing activities to create similar opportunities for students to reason quantitatively.
3. Interpreting and responding to student thinking during classroom interactions.

Recently, Carlson, O'Bryan, and Rocha (2023) proposed a set of conventions for supporting teachers in reasoning quantitatively and representing quantitative relationships. These conventions reinforce patterns for speaking about quantities and relationships among quantities (speaking with meaning), conceptualizing a graph as a record of how two quantities' values vary together (quantity tracking tool), representing the quantitative structure of a problem context (quantitative drawing), and consistent expectations and methods for defining variables, constructing algebraic expressions, and defining formulas (emergent symbolization) (Carlson, O'Bryan, & Rocha, 2023). These researchers propose that the design of conventions for representing quantitative relationships can support instructors' construction of robust meanings for the key ideas taught in precalculus. Carlson, O'Bryan, and Rocha (2023) also propose that teachers' consistent implementation of the Pathways Conventions may advance their meanings for the key ideas of precalculus to include quantitative reasoning as a critical way of thinking. These researchers have also found that many teachers who have consistently implemented the Pathways conventions have also come to recognize the affordances of their consistent engagement in quantitative reasoning on students' thinking and learning. (Carlson, O'Bryan, & Rocha, 2023).

Thompson, Carlson, and Silverman (2007) and Silverman and Thompson (2008) argued that quantitative reasoning would not become a meaningful part of instructors' teaching practices until they have an image of the conceptual affordances of this way of reasoning for students' learning. Carlson and colleagues reported that an instructor's enactment of the Pathways Conventions can be effective in supporting their students' engagement in quantitative reasoning. In particular, the Pathways Conventions specify expectations and actions for supporting students in i) conceptualizing and speaking about quantities and how their values vary together, ii) representing how two quantities change together using a graph, and iii) representing quantitative relationships with expressions and formulas (Carlson, O'Bryan, & Rocha, 2023).

Researchers have also identified the nature of a teacher's decentering actions as a mechanism for advancing teachers' MMT and their ability to convey coherent meanings to students while teaching (Bas Ader & Carlson, 2021; Carlson et al., in press; Rocha & Carlson, 2020; Teuscher, Moore, & Carlson, 2016). Researchers have argued that decentering or setting aside one's thinking to understand what students understand (Steffe & Thompson, 2000) is useful for explaining and characterizing teachers' responses and interactions with students while teaching (Bas Ader & Carlson, 2021; Carlson et al., in press; Rocha & Carlson, 2020; Teuscher et al., 2016). As one example, Teuscher et al. (2016) propose that a teacher's ability to decenter is related to the nature of their interactions with students and the development of their mathematical meanings for teaching. Teuscher et al.'s., (2016) empirical study revealed that teachers' explanations advanced to become more conceptually oriented as the instructors made their speaking and listening to others a specific object of focus and reflection. In 2020, Rocha and Carlson corroborated Teuscher et al.'s (2016) proposal by providing empirical evidence of the reflexive relationship between teachers' meanings and their decentering actions. Rocha and Carlson's (2020) analysis of an instructor's teaching revealed that the teacher's actions to decenter advanced her MMT by providing the teacher with more refined images of a student's ways of understanding the sine function.

Based on these claims of the relationship between a teacher's MMT and their decentering actions (Carlson, Bowling, Moore, & Ortiz, 2007; Carlson et al., in press; Rocha & Carlson, 2020), there is a need for research to investigate teachers' decentering actions and the impact of these actions on advancing teachers' MMT and ways of thinking about learning and teaching an idea. This paper provides an example of the interaction between a teacher's MMT for angle measure and their ability to construct an image of students' thinking and spontaneously leverage it when interacting with students during classroom instruction.

THEORETICAL PERSPECTIVE

Meanings and Mathematical Meanings for Teaching an Idea

To Piaget, meanings and understandings were synonymous and grounded in the knower's schemes (Montanegro & Maurice-Neville, 1997). Individuals construct their meanings through assimilation to a scheme (Thompson, 2013). Thus, to say an individual has a meaning for a word, symbol, expression, or statement means that the individual has assimilated that word, symbol, expression, or statement to a scheme. A scheme is a mental structure that "organize[s] actions, operations, images, or other schemes" (Thompson et al., 2014, p. 11). As such, an individual's meaning for a word, symbol, expression, or statement is the space of implications resulting from assimilation to a scheme (Thompson, 2016).

In 2016, Thompson extended the construct of *mathematical meani*ng when defining *mathematical meanings for teaching* (MMT) to account for characterizations of teachers' actions related to teaching. More specifically, a teacher's mathematical meanings for teaching an idea include (1) the meanings (schemes) the teacher uses while teaching or thinking about teaching and (2) the teacher's image of the meanings (schemes) they want students to develop for the idea. The construct of Mathematical Meanings for Teaching differs from the construct of Mathematical Knowledge for Teaching in two distinct ways. First, Thompson's use of the word meaning connotates something personal (i.e., meanings existing in the knower's mind) to readers rather than knowledge, which seems less personal and disjoint from the knower (Thompson, 2016). Second, arguably most important, the construct of Mathematical Meanings for Teaching describes the status of an individual's understandings relative to teaching.

Conveyance of Meaning

Mathematics educators who adopt a radical constructivist perspective contend that an individual's meanings are idiosyncratic and reside in the individual's mind. With these theoretical assumptions, it is natural to ask, "how does a teacher convey meaning to students?" Thompson (2000; 2013) proposed a theory for the negotiation of meaning based on Piaget's notion of intersubjective operations and Pask's conversation theory.

According to Thompson (2013), person A in Figure 17 holds something in mind that she intends person B to come to understand. Figure 17 shows person A considering how to express what they intend to convey and how person B might interpret them. Assuming both individuals act as second-order observers (Steffe & Thompson, 2000), person A constructs their model of how they think person B might interpret them, and person B does the same thing. Namely, person B constructs their understanding of what person A said by thinking of what they (person B) might have meant had they (person B) said it. Thus, person B's understanding of what person A said comes from what they know about person A's meanings; thereby, person B's understanding of person A's utterance need not be the same and likely is not the same as what person A meant.

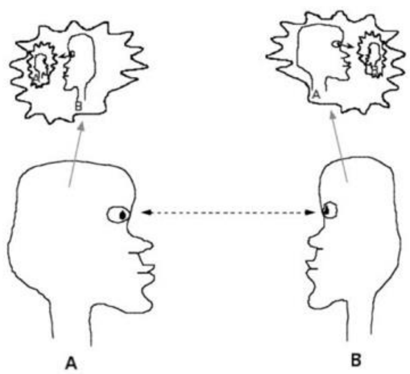


Figure 17. An Example of a Reflective Interaction (Thompson, 2013, p. 64)

The above example is a description of a reflective interaction. A teacher acts reflectively if they decenter - attempt to understand their students and use their understanding of students' thinking (second-order model) to guide their instructional actions (Bas Ader & Carlson, 2021). However, teachers do not always act reflectively. A teacher interacts with a student unreflectively if they are constrained to using their meanings of an idea (first-order model) when interacting with students (Bas Ader & Carlson, 202; Teuscher et al., 2016). In either interaction (reflective or unreflective), the teacher's meanings create a space for their students' meanings (Thompson, 2013). In the case of a reflective interaction, if a teacher's image of what students are to learn entails weak meanings or no meanings, then intersubjectivity can be attained with students collectively possessing a wide variety of meanings that fit the discourse, many of which we would identify as problematic. When a teacher's image of what students should learn entails a strong system of meanings, the space for possible student meanings is much smaller, assuming that the teacher and students mutually adapt their understanding of the other (Thompson, 2013, p. 69). As such, a teacher who has coherent meanings for an idea and a strong image of how they intend to support students in constructing compatible meanings is more likely than a teacher with weak meanings composed of disconnected facts and procedures to place themselves in positions to be interpreted (by students) in the ways they intend (Thompson, 2000).

Decentering, First-Order Models, and Second-Order Models

Piaget (1995) introduced the construct of decentering to describe the cognitive and social processes that occur during a child's transition from the pre-operational to the concrete operational stages of development. Piaget used the term *decentering* to describe a child's transition from egocentric thought to the capability of (1) adopting a perspective of another and (2) coordinating multiple aspects of an object or situation simultaneously (Piaget, 1995). More recently, researchers have adopted the term "decentering" to describe an individual's attempts to understand another's actions from the perspective of the other (Carlson & Bas Ader, 2021; Carlson, Bowling, Moore, & Ortiz, 2007; Carlson, Bas Ader, O'Bryan, & Rocha, in press; Carlson, O'Bryan, & Rocha, 2023; Rocha & Carlson, 2020; Teuscher, Moore, & Carlson, 2016). More specifically, an individual is acting in a decentered way when they attempt to understand another's perspective by considering how the individual may have been thinking, which led to them acting in the way they did. In contrast, individuals act in a non-decentered way when they project their "own reasoning, goals, beliefs, worldview, or understandings onto another person to explain their actions" (Carlson, Bas Ader, O'Bryan, & Rocha, in press, p. 5).

As such, decentering may be best understood as an individual's ability to construct a model of another's thinking. A teacher's first-order models are "models the observed subject constructs to order, comprehend, and control his experience (i.e., the subject's knowledge)" (Steffe et al., 1983, p.xvi). When forming a first-order model, a teacher is constrained to using her own thinking and meanings to make sense of students' activities. As a result, a first-order model is always constructed by an actor in an interaction rather than by an observer. In contrast, a teacher constructs a second-order model of a student's thinking when they "[put themself] into the position of the student and attempts to examine the operations that they (the teacher) would need and the constraints [they] would have to operate under in order to (logically) behave as the student did" (Thompson, 1982, p.159). The result of this construction is the teacher's second-order model of the student's thinking that explains the teacher's observations of the student's states and activities. Moreover, suppose the teacher constructs a second-order model of a student's mathematics by attempting to understand the student's actions and operations and uses their (the teacher's) understanding of the student's thinking to guide their (the teacher's) actions. In that case, they are said to have decentered (Bas Ader & Carlson, 2021).

Ways of Thinking About Teaching an Idea

Carlson, Bas Ader, O'Bryan, and Rocha (in press) proposed a theoretical framework to make explicit the role teacher's decentering actions play in advancing their MMT to include images of teaching that include ways of thinking about students' ways of learning an idea. In particular, these researchers leveraged Harel's (1998) construct of a way of thinking to describe a "habitual form of reasoning that governs the application of a variety of specific mathematical schemes" (Thompson et al., 2014, p. 12). These researchers proposed that a teacher has a way of thinking about teaching an idea if they have an image of how an idea might be productively understood and learned and image(s) of the thinking a student might engage in when learning an idea. Moreover, a teacher's Way of Thinking about Teaching an Idea (WTTI) includes the following:

1. How the teacher reasons about and understands the idea (the teacher's MMT, which includes the ways of thinking the teacher engages in when reasoning about the idea) (Thompson, 2016).
2. The teachers' images (second-order models) of students' thinking about and learning that idea (Steffe et al., 1983; Thompson, 2000).
3. The teachers' image of ways of thinking students may engage in to develop and refine their understanding of an idea (ways of learning an idea) (Carlson et al., in press; Silverman & Thompson, 2008).

Carlson, Bas Ader, O'Bryan, and Rocha (in press) also claim that a teacher's WTTI becomes more connected and refined as the teacher engages with students while attempting to model students' thinking. More specifically, Carlson and her colleagues propose that there is a symbiotic relationship between teachers' MMT for an idea and their decentering actions. These researchers claim that the teacher's mathematical meanings for an idea (first-order model) advance as the teacher's actions to decenter advance. Conversely, a teacher's decentering actions advance as the teacher's mathematical meanings for teaching become more stable and refined. That is, as a teacher's image of a student's mathematics (the teacher's second-order model) becomes more stable through reflecting on acts of decentering, the teacher's images of students' generalized ways of thinking (epistemic students, i.e., reflected abstractions) inform (are assimilated into) the teacher's first order model or mathematical meaning for teaching that idea (see Figure 18). Thus, Carlson, Bas Ader, O'Bryan, and Rocha propose that through perturbations and accommodations to the teacher's scheme for knowing, learning, and teaching an idea, the teacher constructs a more connected *way of thinking about teaching that idea*.

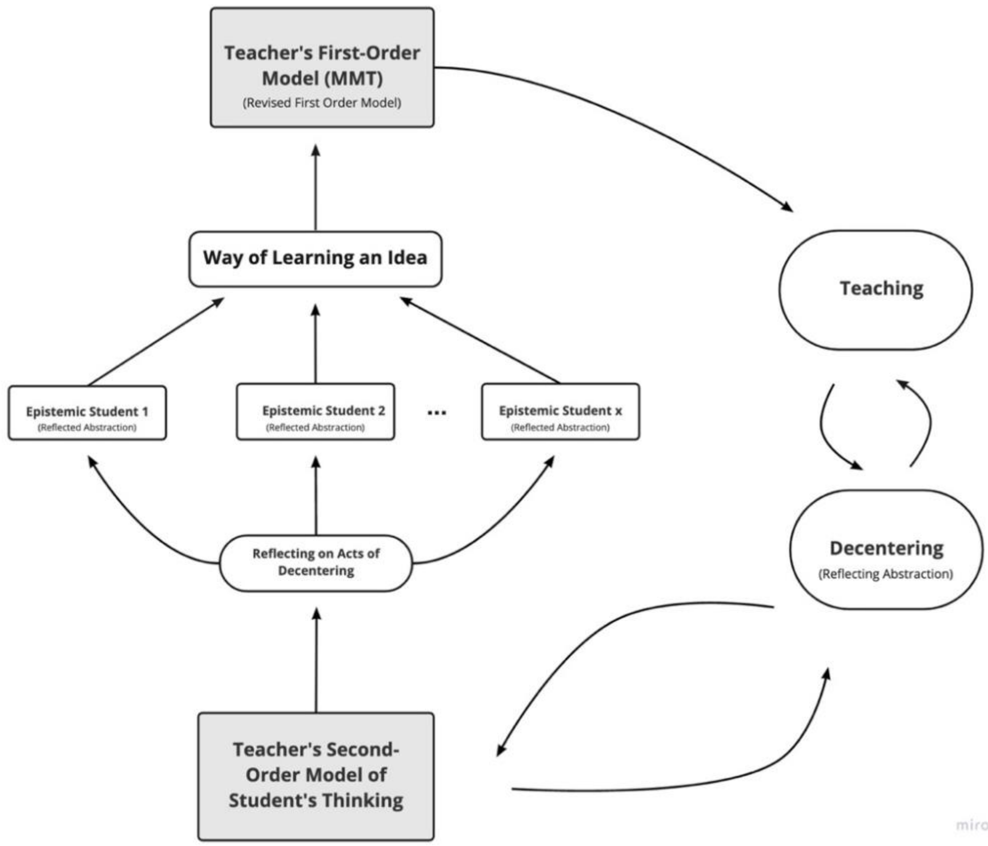


Figure 18. The Development of Ways of Thinking About Teaching an Idea- Included in Carlson, Bas Ader O'Bryan, and Rocha (in press)

CONCEPTUAL ANALYSIS

Researchers have proposed quantitative reasoning, the analysis of a situation into a network of quantities and quantitative relationships (Thompson, 1990, 1993, 2011), as a way of thinking for supporting students in conceptualizing angle measure (Hertel & Cullen, 2011; Moore, 2010, 2014; Tallman, 2015, 2021; Tallman & Frank, 2018; Thompson, 2008). Within the theory of quantitative reasoning, a quantity is a quality of an object or situation that one conceives of as admitting a measurement process. "A person comprehends a situation quantitatively by conceiving of it in terms of quantities and quantitative operations. Each quantitative operation creates a relationship: The quantities operated upon with the quantitative operation in relation to the result of operating" (Thompson, 1994, p. 14). A quantitative operation is the conception of two quantities taken to produce a new quantity. Put another way, a quantitative operation is a description of how quantities come to exist (Thompson, 1990). Although researchers have proposed quantitative reasoning as a way of thinking for supporting students and teachers in conceptualizing angle measure, students and teachers often develop meanings that do not involve this way of thinking (Tallman, 2015; 2021; Tallman & Frank, 2018; Thompson, 2008).

A Quantitative Meaning for Angle Measure

A key component of quantitative reasoning involves distinguishing an object from its measure. An angle is a geometric object formed by two rays that share a common vertex or endpoint. "A quantitative understanding of angle measure involves identifying an attribute of a geometric object to measure and conceptualizing a unit with which—and process by which—to measure it" (Tallman & Frank, 2018, p. 5). When determining the measure of an angle, one quantifies the openness of the angle's rays by measuring the length of an arc subtended by the rays of an angle whose vertex lies at the center of a circle. However, for the measure of the angle to be independent of the size of the circle, the arc subtended by the angle's rays must be measured in a unit that covaries with the length of the subtended arc so that the ratio of subtended arc length to unit length is always constant for an angle with a fixed amount of openness (Tallman, 2015). Thus, the unit of measure used to quantify an angle's openness must be proportional to the subtended arc length and, by extension, the circle's circumference. As such, a quantitative understanding of angle measure involves conceptualizing angle measure as a measurement process that defines a multiplicative relationship between the subtended arc length and some unit proportional to the circle's circumference (Moore, 2014).

For example, one could measure the angle in Figure 19 using any made-up unit, so long as the unit is proportional to the circumference of any circle centered at the angle's vertex. As such, suppose an angle measures "1 Oscar" if it subtends an arc that measures 1/9th of the circumference of any circle centered at the vertex of an angle. Then, the angle shown in Figure 19 would measure 3 "Oscars" as it subtends a class of arcs that are each three times as large as 1/9th of the circumference of the respective circle centered at the angle's vertex.

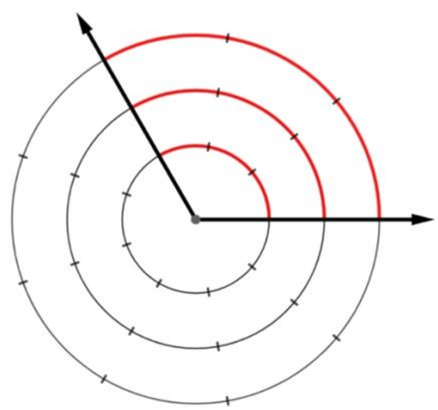


Figure 19. An Angle that Measures 3 "Oscars"

Similarly, we could measure the angle in Figure 19 in degrees. An angle measures one degree if it subtends a class of arcs that are 1/360 times as large as the circumference of any circle centered at the angle's vertex. The angle in Figure 19 measures 120 degrees since it subtends an arc that is 120 times as long as 1/360th (or $\frac{120}{360} = \frac{1}{3}$) of the circumference of any circle centered at the angle's vertex.

METHODS

A teacher's WTTI is a status of their current meanings and ways of thinking about teaching and learning. As such, a teacher's WTTI can be more or less productive for teaching. In this paper, I present an example of a precalculus teacher's ways of thinking about teaching angle measure and his teaching practice. More specifically, this paper illustrates how a teacher's mathematical meanings, ways of thinking, and decentering actions supported him in interacting productively with students while teaching.

Context

This paper presents one case from an exploratory multiple-case study (Stake, 1995; Yin, 2009). I chose to conduct an exploratory multiple-case study because this methodology provided an empirical inquiry into the interaction between a teacher's MMT and decentering actions by allowing me to perform an in-depth analysis of two cases (Yin, 2009). Furthermore, an exploratory case study methodology was most appropriate because it allowed me to (1) investigate phenomena over which I had very little or no control and (2) the data I had access to was limited to contemporary events (e.g., classroom observations and clinical interviews). The primary goal of the case study methodology is to corroborate, modify, reject, or advance theoretical propositions (Stake, 1995; Yin, 2009). As such, this method is productive to the degree it generates viable explanatory models of some phenomena. Thus, the reader must keep in mind that case studies are less about accurate descriptions of "the case" and more about generalizing theoretical propositions.

The subject for this study was a graduate student instructor, Enzo, who was in his second semester of teaching precalculus using the research-based Pathway

Precalculus curriculum at a large research university in the southwest United States. I selected each of the two cases in the multiple case study to provide contrasting examples of the teachers' MMT angle measure and sine function and the interaction between their meanings and practice. I selected Enzo as a subject for this study because:

1. He had experience teaching with the Pathways Precalculus and Algebra II Curriculum for approximately three years prior to the study.
2. He had been mentored in advancing his mathematical meanings for teaching precalculus ideas by participating in extensive professional development (over a 4-year period) led by the leaders of the Pathways Precalculus research team.
3. He was willing to let me video-record his classroom instruction.

Concurrent with the study, Enzo was enrolled in his first year of a Ph.D. program studying mathematics education. Enzo taught one section of precalculus that met for fifty minutes three times a week as part of his teaching assistantship with the university.

Before entering the Ph.D. program, Enzo earned a bachelor's degree in mathematics education from the same university. During his undergraduate studies, Enzo completed two courses for preservice secondary mathematics teachers that used the Pathways pre-calculus curriculum to support preservice teachers in advancing their mathematical meanings for key ideas of secondary mathematics and to explore lesson design and methods for teaching secondary mathematics. Enzo also taught high school mathematics for four years after completing his undergraduate degree. In the last three of these four years, Enzo taught both pre-calculus and algebra II using the Pathways Curricular Materials. In addition, as mentioned in item (2) above, while teaching with the Pathways curriculum materials in the high school, Enzo participated in approximately ten (three to four each school year) 2-hour professional development seminars led by a Pathways Project research team member.

The Pathways Project is an NSF-supported research project that includes curricular materials for Algebra I, Algebra II, Geometry, and Precalculus. The goals of the Pathways Project are to (1) increase student learning, success, and retention in STEM fields and (2) support mathematics instructors to shift their instruction to have a greater focus on student understanding (Carlson, 2019). The Pathways curricular materials were designed as a result of many years of research on students' learning and instructors' teaching of precalculus ideas (Carlson, 1995; 1997; 1999; Carlson, Jacobs, Larsen, Coe, & Hsu, 2002; Carlson, Madison, & West, 2015; Carlson, Oehertman, & Engelke, 2010; Engelke, Carlson, & Oehertman, 2005; Kuper, 2018; Marfi, 2017; Moore, 2010; O'Bryan, 2019; Tallman, 2015). The curricular materials are organized into modules that are further divided into investigations that students complete in small groups during class.

Concurrent with the study, Enzo was enrolled for the second semester in a required professional development seminar that met weekly for ninety minutes. The professional development seminar was led by a designer of the Pathways Curricular Materials and focused on (1) supporting GSIs in developing coherent meanings and ways of thinking about the ideas to be taught and (2) supporting the GSIs in clearly explaining their meanings for these ideas to others. During the seminar, GSIs and the seminar leader regularly discussed and engaged in the *Pathways Conventions* for supporting quantitative reasoning. These conventions included reinforcing patterns for speaking about quantities and relationships among quantities (speaking with meaning), conceptualizing a graph as a record of how two quantities' values vary together (quantity tracking tool), representing the quantitative structure of a problem context (quantitative drawing), and consistent expectations and methods for defining variables, constructing algebraic expressions and defining formulas (emergent symbolization) (Carlson, O'Bryan, & Rocha, 2023).

Experimental Methods

In addition to collecting video data of Enzo's instruction, I engaged him in a task-based clinical interview (TBCI) (Goldin, 1997) and a series of semi-structured clinical interviews (Clement, 2000). The purpose of the TBCI was to construct a model of Enzo's (1) meanings and ways of thinking about angles and their measures, (2) commitment to quantitative reasoning, and (3) image of how to support students in developing coherent meanings for angles and their measures. I also engaged Enzo in a series of semi-structured clinical interviews that occurred before, directly following, and two days after each of his lessons on angles and their measures.

In the pre-teaching clinical interviews, I posed questions to Enzo to probe his meanings for angle measure, his image of the understandings he wanted students to construct, and how he approached his planning of the upcoming lesson. During these interviews, I also asked Enzo to share his plan for class, including any activities and/or tasks he planned to use. I then inquired about the activities and Enzo's rationale for the selection and/or design of the activities and conversations about the activities he anticipated having with students.

Following Enzo's teaching, I engaged him in two different post-teaching clinical interviews. The first post-teaching clinical interview occurred immediately following his lessons on angles and their measures. During this interview, I probed Enzo's rationale for specific teaching actions I observed during his teaching. The second post-teaching clinical interview occurred two days after (on the day of his next lesson) his respective lesson on angle measure. I showed Enzo short video segments from his prior lesson during this second post-teaching clinical interview. While watching the video segments with Enzo, I prompted him to discuss his motive for specific comments, actions, and questions. I also posed questions to understand and characterize his mental actions when interacting with students. I selected the video clips for use in the second post-teaching clinical interview based on one or more of the following criteria: (1) Enzo interacted with a student or multiple students, (2) Enzo leveraged a student's thinking when teaching, (3) Enzo used an applet, table, or diagram to support his discussion of a mathematical idea, and (4) Enzo deviated from his original lesson plan.

Date	Research Activity	Duration
Phase 1		
3/25	Angle Measure TBCI	90 Minutes
Phase 2		
3/28	Pre-Teaching Clinical Interview 1	55 Minutes
3/28	Classroom Observation 1	50 Minutes
3/28	Post-Teaching Clinical Interview 1	20 Minutes
3/30	Video Analysis 1& Pre-Teaching Clinical Interview 2	60 Minutes
3/30	Classroom Observation 2	50 Minutes
3/30	Post-Teaching Clinical Interview 2	20 Minutes
4/1	Video Analysis 2 & Pre-Teaching Clinical Interview 3	30 Minutes
4/1	Classroom Observation 3	50 Minutes
4/1	Post-Teaching Clinical Interview 3	20 Minutes
4/4	Video Analysis 3 & Pre-Teaching Clinical Interview 4	60 Minutes
4/4	Classroom Observation 4	50 Minutes
4/4	Post-Teaching Clinical Interview 4	20 Minutes
4/6	Video Analysis 4	40 Minutes

Table 4. Data Collection Schedule

Analytical Methods

Preliminary Analysis

The angle measure TBCI constituted the only data for my preliminary analysis. During the TBCI, I took notes on the meanings Enzo expressed and moments when he either (1) engaged in quantitative reasoning or (2) expressed a desire to support students' engagement in quantitative reasoning. Directly following the interview, I took notes to record my initial thoughts and impressions of Enzo's meanings for angles and their measures. Soon after the TBCI, I rewatched the video recording of the interview and wrote detailed memos that described Enzo's responses to the tasks included in the TBCI. I then examined the memos to construct an initial model of Enzo's reasoning about angles and their measures.

Ongoing Analysis

My initial analysis of the data from Enzo's teaching sessions and interviews occurred soon after I collected the data. During the pre-teaching, post-teaching, and video analysis interviews, I took notes detailing moments when Enzo made an attempt to engage his students in quantitative reasoning. I also noted moments in which Enzo expressed interest in student thinking or discussed his image of how a student was thinking during his lessons on angle measure. Following each clinical interview, I wrote detailed memos on (1) how I hypothesized Enzo to be reasoning about angles and their measures, (2) Enzo's commitment to quantitative reasoning, and (3) Enzo's attention to and image of students' thinking.

My ongoing analysis of Enzo's instruction involved taking detailed notes as I observed Enzo's teaching. In particular, I noted moments in which Enzo (1) expressed a meaning for angles and their measures, (2) interacted with students in their groups, and (3) attended to or leveraged student thinking while teaching. Within twenty-four hours of Enzo's teaching, I watched the video recording of his instruction. During this time, I selected segments of his instruction according to the three criteria above. These selected video segments were later used during the clinical interviews in which Enzo watched clips of his teaching.

Retrospective Analysis

My retrospective analysis can be broken down into three phases. The first phase involved my retrospective analysis of the TBCI, pre-teaching, and post-teaching clinical interviews. The procedures I used to analyze the TBCIs and clinical interviews with Enzo are consistent with Simon's (2019) approach to analyzing qualitative data. I began my analysis of the clinical interviews by reviewing the memos I constructed during the ongoing phase of my analysis. I then used these memos to identify passages from the respective interviews that revealed Enzo's ways of thinking about teaching angle measure. This included his meanings for angles and their measures, his image of the understandings he wanted students to construct for these ideas, his commitment to quantitative reasoning, and his attention to and image of his students' thinking. Next, I transcribed these parts of the respective interviews and conducted a line-by-line analysis of the transcripts to generate hypotheses about Enzo's thinking. At this stage, I wrote detailed memos that elaborated (1) what the subject was doing and why they were doing it, (2) what the subject meant by what they said at this point, (3) what the subject was thinking and (4) what this might show about the subject's understanding (Simon, 2018). I then used the memos constructed in the first analysis phase as data for the second phase. Specifically, I coded each memo using the codes shown in Table 5.

Code	Description
Quantitative Reasoning (QR)	Instructor engaged in quantitative reasoning.
Quantitative Reasoning Students (QRS)	Instructor encouraged students to engage in quantitative reasoning (QR), took actions to support students' engagement in QR, or talked about wanting to support students' engagement in QR; OR students engaged in QR.
Covariational Reasoning (CR)	Instructor engaged in covariational reasoning.
Covariational Reasoning Students (CRS)	Instructor encouraged students to engage in covariational reasoning (CR), took actions to support students' engagement in CR, or talked about wanting to support students' engagement in CR; OR students engaged in CR.
Angle Measure (AM)	Instructor expressed a meaning for angle measure.
Angle Measure Students (AMS)	Instructor described meanings they want students to have for angle measure OR student expressed meaning for angle measure.
Sine Function (SF)	Instructor expressed a meaning for the sine function.
Sine Function Students (SFS)	Instructor described meanings they want students to have for the sine function OR student expressed meaning for the sine function.
Instructor-Student Interaction (ISI)	Instructor interacted with student while teaching, discussed an interaction with a student from previous teaching, discussed an anticipated interaction with students while teaching.
Image of Student Thinking (ImgStu)	Instructor described their image of how students were thinking.

Table 5. Codes Used During Analysis.

My decision to use the codes in Table 5 was informed by prior research that has identified quantitative and covariational reasoning as essential ways of thinking for supporting teachers' conveyance of coherent meanings while teaching (Carlson, O'Bryan, & Rocha, 2023; Tallman, 2015; 2021; Tallman & Frank, 2018; Thompson, 2008; 2013) and students' construction of meaningful function formulas (e.g., Moore & Carlson, 2012; Thompson, 1988; 1990, 1993) and graphs (e.g., Carlson et al., 2002; Moore & Thompson, 2015). Following my initial coding, I conducted a second pass that involved validating and refining my initial coding of the data. In the third and final analysis phase, I identified themes in the conceptual analysis that best characterized and modeled the instructor's MMT for angle measure. I followed this by extracting and reviewing the lines of the transcript that exemplified each theme. I then wrote detailed summaries of each theme, which constituted my emerging model of Enzo's mathematical meanings for teaching angle measure.

The second phase of my retrospective analysis involved analyzing Enzo's instruction. I began this analysis phase by reviewing my observation notes of each respective lesson. I then reviewed the short video clips of his teaching that I selected for the video-analysis interview. I then used my observation notes and the notes I constructed when I selected the short clips to identify particularly revealing moments of his instruction. At this analysis stage, I used student-teacher interactions, including instructor-led discussions, as my unit of analysis. In particular, I identified segments of Enzo's teaching in which (1) he expressed a meaning for angles and their measures or (2) he interacted with students. I then transcribed these revealing moments of Enzo's instruction. Following this, I wrote memos detailing my hypotheses of how Enzo was thinking and why he interacted with students in the way they did. In a second round of analysis, I coded the segments identified in the first stage using the codes in Figure 20 and Table 6. I then looked across each code for themes in how the instructor was thinking and the nature of the instructor's interaction with students. These themes constituted my emerging model of (1) Enzo's mathematical meanings as conveyed while teaching and (2) the nature of his interactions with students (decentering actions).

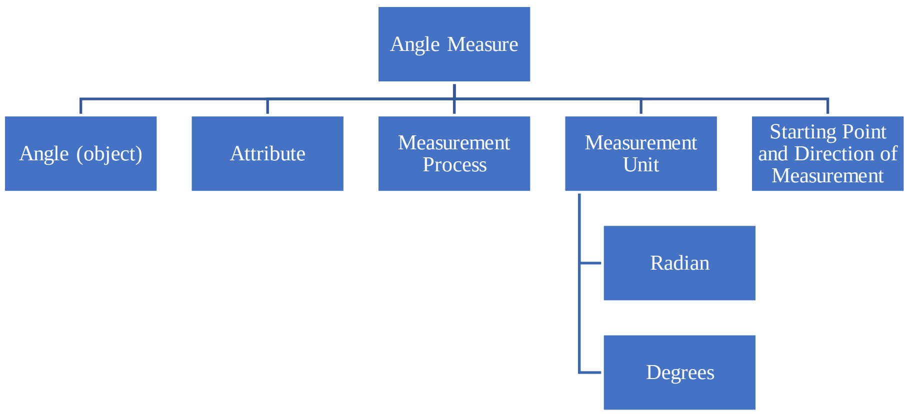


Figure 20. Angle Measure Subcodes

Code	Description
Decenter 0	Instructor shows no interest in the student's thinking but shows interest in the students' answer. Makes no attempt to make sense of the student's thinking but takes actions to get the student to say the correct answer.
Decenter 1	Instructor shows interest in student's thinking but makes no attempt to make sense of the students' thinking. Attempts to move the student to their way of thinking without trying to understand or build on the expressed thinking and perspective of the student.
Decenter 2	Instructor makes an effort to make sense of the student's thinking and perspective but does not use this knowledge in communication.
Decenter 3	Instructor makes sense of the student's thinking and/or perspective and makes general moves to use the student's thinking when interacting with the student.
Decenter 4	Instructor constructs an image of the student's thinking and/or perspective and then adjusts their actions to take into account both the student's thinking and how the student might be interpreting them.

Table 6. Decentering Framework (Bas Ader & Carlson, 2021).

My retrospective analysis's third and final phase involved reviewing the clinical interviews in which Enzo watched short video segments of his teaching. During this analysis phase, I reviewed the interview and wrote detailed notes on any instances in which Enzo (1) discussed his image of how students were thinking, (2) provided a rationale for his actions while teaching, or (3) discussed how he planned to sequence his next lesson based on the events in his previous lesson. I followed this by coding these moments using the framework in Table 6. After this iteration of coding, I looked across the codes for themes. These themes constituted my model of Enzo's decentering actions.

RESULTS

Enzo's Meanings for Angles and their Measures

Enzo consistently distinguished between an angle as an object and the measure of an angle's openness as a quantity (see Figure 21). Before teaching his first lesson on angle measure, Enzo said he wanted students to think about an object, how to quantify it, a unit of measurement, and a frame of reference (see Excerpt 12). Enzo also said he wanted students to begin thinking about the need to make a relative size comparison (Excerpt 12, lines 9-10). When prompted to explain what he meant by a relative size comparison, Enzo said, "you know, comparing how many times as large one quantity's value is to another, you know, like the output to a division problem. If we consider or conceive of a division as a form of measurement like we're trying to measure one quantity in units of another."

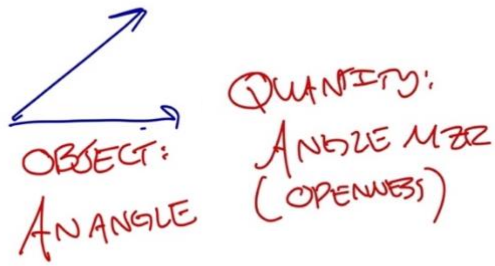


Figure 21. Enzo Distinguishes Between an Angle as an Object and Its Measure as a Quantity

Excerpt 12

Enzo: Well, umm... you know I want students to be able to interpret what is going on in some dynamic situation, like in an animation and to be able to do that they have to have some grounding in what is changing and what is not. Umm… and then beyond just that thing, ya know, how do we label it, how do we quantify it, you know are we thinking about an object, are we conceptualizing some unit of measure, are we thinking about a frame of reference? And so today, maybe those things might be a little bit loose, but I hope that by the end of today, by the end of the next lesson, that they might be able to be more precise about that. As far as the relative size comparison, like today I want to motivate the need for a relative size comparison rather than just tell them like “hey this is the really smart way to do it”.

Int: So, if asked you to just bullet point, what are the key meanings and understandings you want your students to leave your class with today, what would you say?

Enzo: Okay, so, I want them to understand that an angle is the union of two rays with a shared vertex. That, we measure one special thing about angles which is their openness. And that this is really hard to do without comparing it to something else… so the openness is the relative size of the arc that is cut off to the radius of the circle. That’s what I want them to get at. I want them to also realize that we can compare the relative size of the arc and the circumference. Umm…and being able to generalize that if an angle measures theta radians that this is the relationship between the arc length and the radius.

It is noteworthy that when discussing the meanings he wanted students to develop (see Excerpt 12, lines 16-22), Enzo identified an object (an angle), an attribute of the object to measure (the angle’s openness), a measurement process (comparing the relative size of the arc and the circumference/radius), and a unit of measure (both the circumference and the radius). Enzo's identification of an object, an attribute of the object, a measurement process, and a unit of measure indicates that his meaning for an angle measure was grounded in quantitative reasoning and that he was fluent in referencing the quantities when speaking. A quantitative understanding of angle measure also entails recognition that (1) an angle subtends the same fraction of the circumference of all circles centered at the vertex of the angle, and (2) for the measure of the angle to be independent of the size of the circle, the subtended arc length needs to be measured in units that are proportional to the circumference of the circle centered at the vertex of the angle.

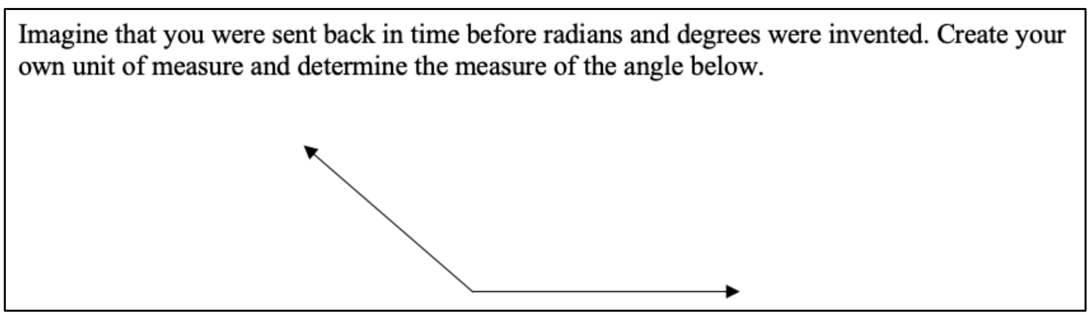


Figure 22. Task Included in the Angle Measure TBCI

Enzo expressed a quantitative understanding of the idea of angle measure when he invented two units of measure, an "Abby" and an "Abby Prime" (see Figure 23 and Excerpt 13), while responding to the task in Figure 22. Enzo defined an Abby as the "percentage or portion of one full revolution" the angle cuts off. When the interviewer asked Enzo to clarify what he meant by an "Abby," he invented a second unit of measure and called it an "Abby Prime," which measured one-quarter revolution (Excerpt 13, line 18). Enzo further expressed that an "Abby" or an "Abby Prime" works as a unit of angle measure because he could scale or dilate the picture (the size of the circumference of the circle centered at the angle's vertex) and still get the same measure since the percentage of a full revolution, or the ratio of the subtended arc to the radius and the ratio of the subtended arc to the circumference all remain the same (see Excerpt 14 lines 3-13).

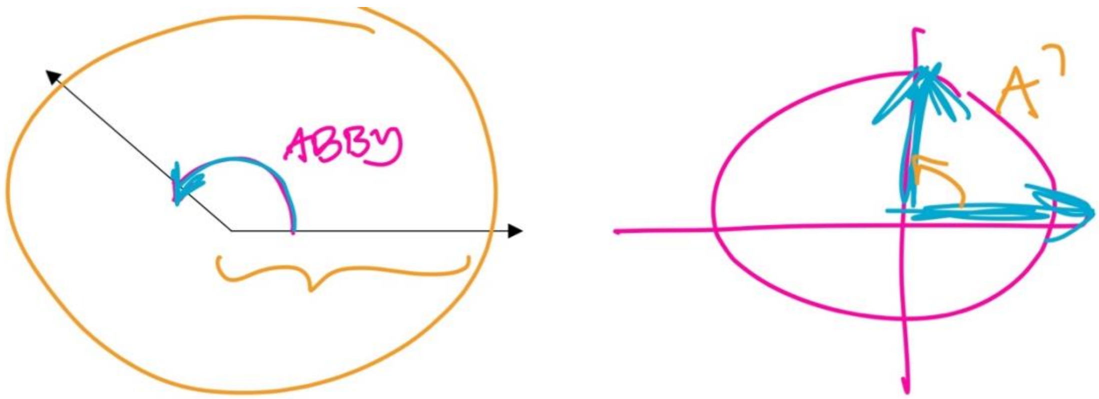


Figure 23. Enzo Chooses an “Abby” and an “Abby Prime” As Units of Measure

Excerpt 13

Enzo: The way to cheat is you know let’s call this [draws pink arc] an “Abby”, so its measure is one [laughs].
Int: Would that work?
Enzo: Why not?
Int: Like if we just pick any arc length and say the measure of the angle is one?
Enzo: Well, okay, what I am saying is that we’re comparing… there’s always a circle. So, if this is one Abby, there is two and some change in a full revolution. If we decide to care about full revolutions. It’s awkward, it’s funky, I don’t like it, but that is one way to do it. Another way to do it might be, umm….
Int: So, wait, hang on, before you move on. So, when you talk about an Abby, what is an Abby?
Enzo: Well, okay, so my unit is this rotation [highlights blue arc] is this percentage or portion of one full revolution. So yeah, I mean it’s a little bit clunky, but suppose you gave me a different one [draws a second angle]. Which was this and this one is an Abby prime. So, then I could say alright this angle is an Abby prime it measures one quarter revolution, there is four of them in one full revolution, whatever. So, it’s a little more awkward over here [points to original angle]because of the size the openness, but I would say the same thing.

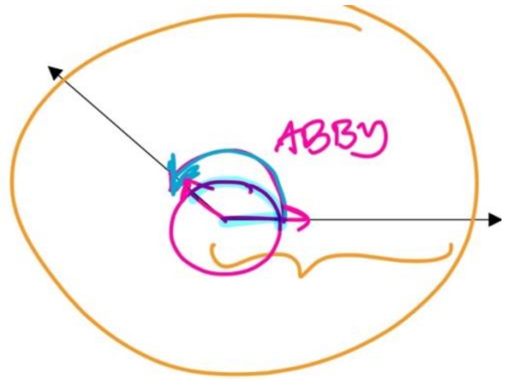


Figure 24. Enzo Shows That an Abby Is Invariant of the Size of the Circumference of Any Circle

Excerpt 14

Int: So why would an Abby, or an Abby prime work as a unit for measuring either of these angles?

Enzo: I would go back to like this whole invariant thing. Which is ya know we can scale, or ya know, dilate this picture and get the same measure. So, we could consider a smaller circle [draws smaller pink circle with the same center as the yellow circle] and then we have that guy [draws pink angle with vertex at the center of both circles]. So, then we could say the percentage of revolution is the same or that (highlights arc subtended by pink angle's rays in blue) to that (highlights radius of pink circle in blue) is the same, or the arc length to the whole circumference is the same. I mean what I am doing and the reason why I am arguing why it works is because it's the same as radians or degrees just taking a larger or smaller unit measure. So, I am saying implicitly that we're comparing something that is changing to something that scales proportionally with it.

Int: When you say, "we're comparing something that is changing to something that scales proportionally with it" what are those "somethings?"

Enzo: So, either the umm arc length to radius, or arc length to circumference, or though we don't like this one, shaded area to total area, or percentage of revolution to one full revolution [draws table] so if I change one of these (circles the left column of the table in Figure 25), these will change by the same factor (circles the right column of the table). And this only works because a circle is so magical, and it only has one measure [radius].

Enzo concluded his response to the task in Figure 22 by stating that the angle measure does not change because "we're comparing something that is changing to something that scales proportionally with it." When I prompted Enzo to explain what he meant by this, Enzo drew the table shown in Figure 25. Enzo then expressed that he could determine the measure of any angle by multiplicatively comparing the quantities on the left side of the table to the quantities on the right side of the table. As one example, Enzo expressed that he could determine the openness of an angle's rays by multiplicatively comparing the arc length subtended by the angle's rays (s) to the radius of the corresponding circle (r).

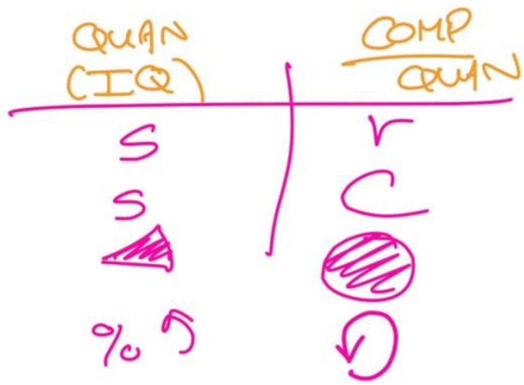
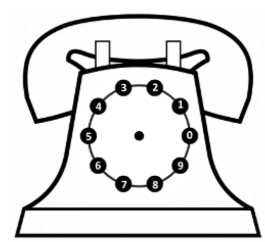


Figure 25. Enzo Draws a Table to Show Quantities He Could Compare to Determine an Angle's Openness

Enzo's description of an "Abby" and an "Abby Prime" as units that measure a "percentage or portion of one full revolution" indicates that he understood that an angle always subtends the same fractional portion of the circumference of any circle centered at the angle's vertex. Similarly, Enzo's statement of "we're comparing something that is changing to something that scales proportionally with it," together with the table he constructed (shown in Figure 25), support that Enzo understood that the unit of measure he chose needed to be proportional to the arc subtended by the angle's rays. As such, it appears Enzo conceived of the idea of angle measure as the result of a quantitative operation or the result of comparing two quantities' values (the length of the arc subtended by the angle's rays and any unit proportional to this arc).

It is noteworthy, however, that Enzo stated that he could determine the measure of an angle by multiplicatively comparing the percent of a full revolution an angle subtends to a full revolution (see Excerpt 14, lines 17-22 and Figure 25). Throughout the study, Enzo used the term "revolution" often. As one example, Enzo responded to part (a) of the task shown in Figure 26 by determining the fraction of a full revolution, that the angle with an initial ray at the number four and terminal ray at the number zero would subtend.

Imagine that you want to use a rotary phone to call a friend. Assume that the number 0 starts at the 3 o'clock position and that each number on the phone is equally spaced around the circle as shown below. To dial a number, you must rotate the number (in the counterclockwise direction) to the 3 o'clock position.



(a) Determine the measure of the angle (whose vertex lies at the center of the phone) needed to dial the number 4.
(b) If the center of the phone is 2.5 inches from the number 0, how many inches did you rotate the number 4 to dial it?

Figure 26. A second task included in the angle measure TBCI. A Second Task Included in the Angle Measure TBCI

Specifically, he said, to dial the number four, he would need to rotate the number a half revolution plus an additional fifth of a half revolution for a total of six-tenths of a revolution (see Figure 27). When asked what he meant by "revolution," Enzo expressed that he uses the term revolution to describe a "process." In particular, he said, "There is some dynamism in my mind. So, there's the process, and there is the result of the process.

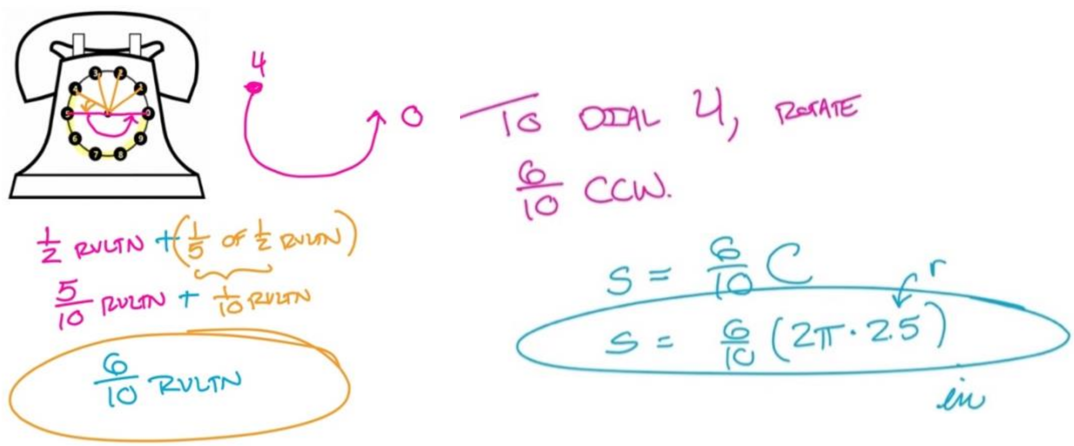


Figure 27. Enzo's Response to the Task Shown in Figure 23

And I can think of them separately and recognize that the end states are the same. So, the process is the revolution, and once I am done, I can think this arc length is six-tenths of the circumference." Enzo's description of the term "revolution" indicates that he is thinking about a static image of an angle as the result of varying the angle's terminal ray to some final position. Moreover, Enzo's statement of, "once I am done I can think this arc length is six-tenths of the circumference," indicates that he is coordinating the varying openness of the angle with the varying length of the arc subtended.

Excerpt 15

Enzo: So, I want the length of that arc [highlights arc in yellow]. So, this arc is going to be six tenths the size of the circumference. And the circumference is two pi times as large as the radius, so this many inches [writes 6/10(2pi*2.5)].

Int: So, you said that the subtended arc would be 6/10 the size of the circumference so how did you know that?

Enzo: Because I am trying to use that reciprocity that I've been talking about. So, since we have rotated six tenths of a full revolution, [pauses] if we revolve one full revolution then the arc length will be one times as large as the circumference. Right, the arc length will be the circumference. So, the fraction of the revolution is going to be the same thing as the fractional part of the circumference that is subtended. So, if I revolve six tenths of a full revolution, the arc length will be six tenths times as large as the circumference.

Enzo's coordinated conception of an angle's measure varying with the arc length subtended by the angle's two rays supported him in conceptualizing the measure of an angle as "the same thing as the fractional part of the circumference that is subtended" (Excerpt 15, lines 10-12). With this understanding, Enzo determined that "if [the angle's terminal ray] revolved six-tenths of a full revolution, the arc length [cut off by the angle's rays would] be six-tenths times as large as the [circle's] circumference." Enzo then used this thinking to determine that he would need to rotate the number four $\frac{6}{10}(2\pi * 2.5)$ inches to dial it.

Enzo's Teaching of Angle Measure

I also observed and recorded Enzo's teaching of lessons on angles and their measures. Enzo had approximately thirty-five students in his section of precalculus that met three times per week for fifty minutes. The classroom where Enzo taught contained six large round tables. Each table had approximately six students seated together. During class, the students sitting at a given table worked together. Enzo let each group pick their favorite fruit at the start of the semester. Each group's fruit of choice became their group name (e.g., Banana, Papaya, Dragon fruit, etc.). Students spent most of the class time working together in their groups on tasks that Enzo designed or chose from the workbook investigations. After students completed specific questions in their groups, Enzo typically led a discussion by calling on each group by fruit name to share their answers and thinking. Enzo repeated this pattern of group work followed by a short class discussion led by different groups for the duration of the 50-minute class period.

The data in this section is from Enzo's first lesson (out of four) on angles and their measures. Recall (from Excerpt 12) that Enzo's goals for this first lesson were to support students in (1) distinguishing angles from their measures and (2) recognizing the need to make a relative size comparison to determine the measure of an angle's openness. As such, Enzo started this lesson by asking students to define an angle. Once Enzo and the students agreed on their definition of an angle, he asked them to discuss with their groups how they might measure an angle. While students discussed how to determine the measure of an angle, Enzo circulated the room and prompted students to explain how they were thinking. After circulating the room for approximately five minutes, Enzo brought the class back together and began the group discussion by drawing the image in Figure 28 on the board. He then told students that he saw the image in Figure 28 on multiple groups' whiteboards as he circulated the room (see Excerpt 16).

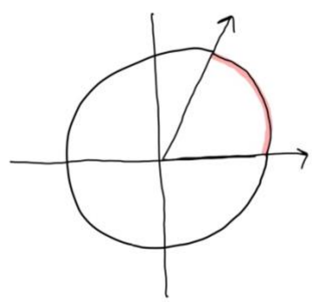


Figure 28. Enzo's Board Work- A Drawing of What He Saw on Students' White Boards

Excerpt 16: Enzo asks students "How do we measure angles?"

Enzo: So, I saw this picture [Figure 28] quite a lot and in the room there was sort of this confusion because some folks were saying what we're trying to do is measure this arc length [highlights arc subtended by the angle's rays in red]. Which is really nice. We're trying to measure that red arc length which is wonderful. But then, the size of circle that we started with was arbitrary. Okay, and arbitrary means we made a choice, but we could have made a different one. So here I am making a different one [draws circle with smaller radius]. But, what didn't change is the angle. And if the angle doesn't change, the measure can't change. So, what's happening here is we have this arc over here, let's call it "Big S" [labels large subtended arc] and we have this arc over here, let's call it "Little s" [labels smaller subtended arc]. Clearly, this arc length right here is much larger than this arc length right there [writes S > s]. So how do we reconcile this? I want you to take thirty seconds and think, how can we get one number out of this picture?

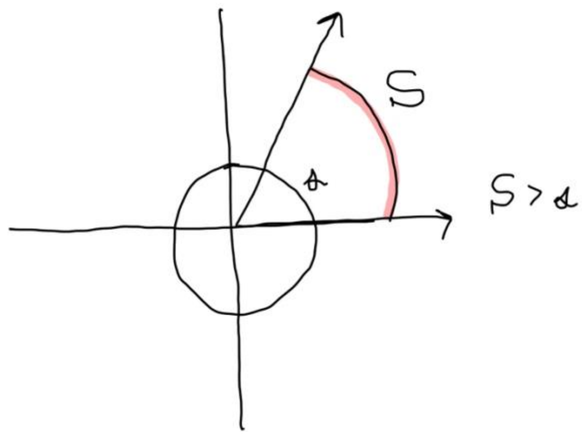


Figure 29. Enzo's Board Work as He Showed Students that S > s

Enzo followed this by telling students that it was "wonderful" that they were measuring the length of the red arc in Figure 28. He then told students that the size of the circle they drew was an arbitrary choice. Enzo then erased the original circle containing the red arc, drew a second circle, and labeled the subtended arc of this circle with a lowercase "s" (see Figure 29). Following this, he prompted students to discuss how they could get one measure for the angle, given that the two circles' corresponding subtended arcs ("big S" and "little s") were different lengths. Enzo's decision to draw a second circle and prompt students to determine "how to get one measure out of it" further suggests that he understood that an angle always subtends the same fractional portion of the circumference of any circle centered at the angle's vertex.

As students discussed how to get one measure for the angle's openness, Enzo circulated the room a second time. Excerpt 17 shows a conversation Enzo had with Group Banana. When Enzo joined Group Banana's discussion, he prompted them to explain how they could "get one number out of it." A student in the group immediately responded that they could divide the small, subtended arc by the small circle's circumference and the large, subtended arc by the large circle's circumference. This student then claimed that the result of these two divisions would be the same "because it's like a fraction of the whole, like a portion of it." Enzo responded to this student by asking him why this was true. Enzo's prompts for the students to explain their thinking suggests that he was interested in their thinking in addition to their answer.

Excerpt 17

Enzo:	Any ideas here of how to get one number out of it?
Banana Student 1:	Yeah, I was just thinking we could take the arc length and divide it by the circumference of whichever circle. So, like for the small one it would just be like little s over that circumference and for the big s it would just be the big s over the big circumference.
Enzo:	Why would that work?
Banana Student 1:	Because it's like a fraction of the whole, like a portion of it.
Enzo:	Uh huh uh huh.
Banana Student 1:	So, it should be the same.
Enzo:	I agree with you, but why? [walks away from group]

Following this, Enzo continued to circulate the room, and after a few minutes, Enzo brought the class back together to discuss how they could get one measure for the angle's openness. Excerpt 18 shows the class discussion that ensued. Once Enzo brought the class back together, he asked Group Banana to share their thoughts. After the student

Excerpt 18

Enzo:	Okay, Okay, so we had a cool idea. Actually, we are going to have Group Banana, they're going to share their idea.
Banana Student 1:	I just said you take the arc length, and you divide it by the circumference of like the respective circle. So, you either take the small s over the small circumference, or the big s over the big circumference.
Enzo:	So small s divided by small circumference and big s divided by big circumference [writes $\frac{little\ s}{little\ c}$ and $\frac{big\ S}{big\ C}$ on the board]. Why do you want to do that?
Banana Student 1:	Umm you should get the same angle.
Enzo:	You should get the same angle measure? We should get the same number?
Banana Student 1:	Yeah.

Enzo: [puts an equal sign between $\frac{little\ s}{little\ c}$ and $\frac{big\ S}{big\ C}$] So, the claim here is that those two numbers are going to be equal. I mean it sounds reasonable, and in fact it is correct. But I want you to try to come up with a justification for why that is true. So, let's take another minute. Let's figure it out. Is that true? It is, but why?

[students work in their groups for 2 minutes].

from Group Banana shared his thinking, Enzo prompted the class to devise a justification for why dividing the subtended arc length of each circle by the respective circle's circumference would give them one measure for the angle's openness. After allowing students to discuss this prompt for two minutes, Enzo asked students to determine how many times as large the circumference of the big circle is compared to the circumference of the smaller circle (see Figure 30).

This sequence of questioning that progressed from asking students to consider why an angle that cut off different length arcs of two different circles with the same center could have the same measure to his deciding to have the group that recognized the proportional relationship between the respective arcs subtended by an angle with its vertex at the circle's center and the two circles circumference suggest that Enzo (1) had an image of ways of thinking that would support students' learning of angle measure and (2) that he was attending to students' expressed thinking and taking actions to leverage it during his instruction (Decentering level 3). Moreover, Enzo's decision to ask students to determine the relative size of the two circle's circumferences also supports that he conceived of relative size reasoning as a way of supporting students' understanding that a circle's radius and circumference were appropriate units for measuring an angle's openness.

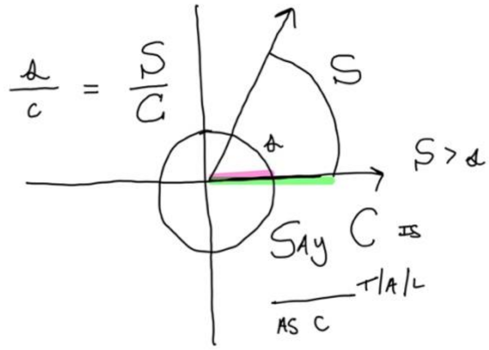


Figure 30. Enzo's Board Work as He Supports Students in Comparing Arc Lengths to the Circumference of Their Respective Circle

After some time, Enzo and the students concluded that the large circle was two times as large as the small circle because the radius of the large circle was twice as large as the radius of the small circle.[2] Enzo followed this by asking students to consider if they could replace the circumference in the ratios from Figure 30 with the radius of the respective circle. After giving students a few minutes to consider this question, Enzo asked students to think about how to make a circle bigger (see Excerpt 19, lines 7-8). He then told students that the circle's radius determines a circle's size. As a result, they could determine any angle's measure by multiplicatively comparing the arc by the angle's rays to either the circumference of the respective circles or its radius (see Excerpt 19, lines 12-15).

Excerpt 19

Enzo: So, if we say that the big circumference is twice as large as the little circumference, we're really saying that the big circle has a radius that is something like twice as large as the little circle. So, I guess what I am asking is why can I just… is it okay that I just replace circumference here with radius? Do I lose anything? Talk with your groups, go!
[students work for one minute]
Enzo: Okay, folks, million-dollar question is on the table. How do you make the circle bigger? So, we have a circle here, of radius little r. Which by the

[2] The students in Enzo's class were accustomed to thinking about the size of a quantity's value relative to the size of another quantity's value. In the post-teaching interview that directly followed this class, Enzo told me that he had supported the students in this way of reasoning since the start of the semester.

way, that measurement is the only thing we need to determine a circle. Anybody here used a compass before? Like a geometry compass. There is only one setting you get to mess with, how far the needle is from the pencil. That's it. Right? That is the only thing you adjust… All of this is just to say, that yes we can make this comparison of arc lengths to circumference and that's cool. But we can also make the same comparison with the radius. And we will get the same number [regardless of the size of the circle/radius we choose].

Following Enzo's teaching of this lesson, I interviewed him twice. Once immediately after his first lesson on angle measure, and for a second time, two days after the class. During the interview immediately following his teaching, I asked Enzo about his decision to draw the image in Figure 30. Enzo said he constructed the diagram "totally on the fly" because a student in group Banana was comparing the subtended arc lengths to their respective circumference. Enzo stated that he originally planned to introduce radian angle measure before generalizing to a multiplicative comparison between the subtended arc and the circumference. However, he said he wanted to honor this student's thinking by sharing it with the class, but he "needed to think of a way to get there."

Two days after this teaching episode, I showed Enzo a video clip of this segment of his teaching and asked Enzo about his decision to ask the student in group Banana to share his thoughts with the class. Enzo stated, "Sometimes I need to buy time to think myself about what I am gonna do next, how I am going to sequence, how I am thinking about their thinking." Enzo then said that he recognized that this student and other students in his class were multiplicatively comparing the subtended arc to the circle's circumference. However, he stated that his instructional goal for the day was for his students to recognize that they could also multiplicatively compare the subtended arc to the circle’s radius since his goal was to introduce radian angle measure by the end of the class. Enzo then indicated that he decided to ask students to consider how to scale a circle to support them in recognizing that the circumference of a circle is a constant number of times as large as the circle’s radius. He conveyed that he thought this task would support his students in multiplicatively comparing the subtended arc to the radius of the respective circle. This data also suggests that Enzo’s understanding of angle measure as a relative size comparison between an arc length, subtended by an angle’s rays and the circle’s radius, influenced his question.

In this example, Enzo attended to a student’s thinking, considered how the student was thinking and took actions to leverage and advance the student’s thinking in the moment of teaching. These interactions (in Excerpts 16-19) with students reveal that he adapted his questioning and interactions based on his second-order model of a student’s thinking. It also appears that Enzo’s way of thinking about teaching angle measure, which includes his MMT angle measure, his commitment to quantitative reasoning, and his image of students’ way of thinking about angle measure, supported him in decentering and ultimately interacting productively with students in the moment of teaching. As one example, Enzo’s conception of angle measure as the result of a relative size comparison supported him in recognizing that students were (1) thinking about determining an angle’s measure by measuring the length of the arc subtended by an angle’s rays in linear units (Excerpt 16, lines 1-3) and (2) posing a question that prompted students to consider the reasonableness of their thinking (Excerpt 16, lines 13-15). Thus, it appears as though Enzo’s conception of angle measure as the result of a multiplicative comparison between the subtended arc length and any unit proportional to the circumference of a circle centered at the angle's vertex supported him in (1) constructing a model of the student's thinking and (2) spontaneously posing a sequence of questions aimed at advancing his students' thinking.

Enzo's Preparation for His Next Class

The second post-teaching interview with Enzo occurred an hour before his next angle-measure lesson. In the second half of this interview (the next pre-teaching interview), I asked Enzo about his plan to teach his second lesson on angle measure that day. Enzo said he planned to start his lesson with the task shown in Figure 31. Enzo said he designed the task because it "highlights two ways of thinking that were present" in the last class. Enzo also noted that after giving students time to consider the task, he planned to ask them to revise Kyle's definition to correct it. In particular, Enzo expressed that he wanted to provide students with a second opportunity to reflect on (1) the different ways of thinking that were present in their last class and (2) how they could refine Kyle's definition so that they would generate only one measure for the angle's openness.

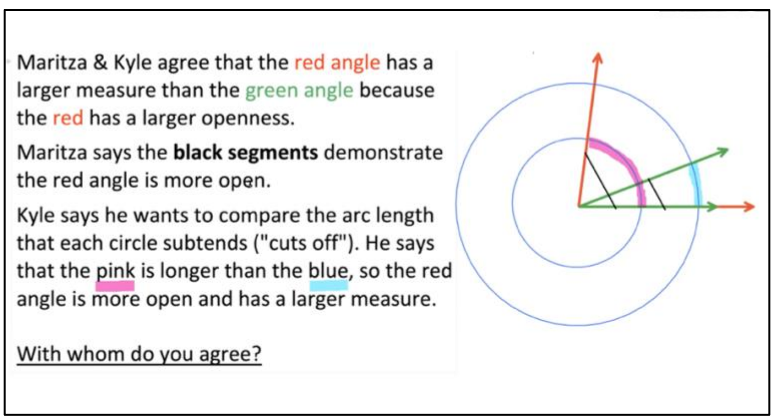


Figure 31. Enzo's Bellringer Task Used During His Second Lesson on Angle Measure

Enzo’s design of the task presented in Figure 31, together with his plan for engaging students in the task, provide an example of how a teacher’s actions to decenter supported him in designing a task that directly targets the nature of students’ conceptions that contributed to confusion and unproductive actions in the previous class. Additionally, this example illustrates how a teacher’s efforts to decenter supported him in fostering reflective discourse in his teaching. In particular, the task shown in Figure 31 is an example of a teacher using students’ meanings and expressions as an object of discussion in his teaching.

Following the bellringer task, Enzo said he planned to have students think about the relative size of a circle’s circumference to its radius length (see Figure 32). In particular, Enzo expressed that he wanted to build on the thinking he “tried to get at” toward the end of his previous class, and he felt that he executed his goal “very inelegantly.” While discussing his plan for using the task (see Figure 32), Enzo stated that he wanted to continue supporting his students in recognizing that they need to measure the length of the arc subtended by an angle’s rays with “something that changes with it.”

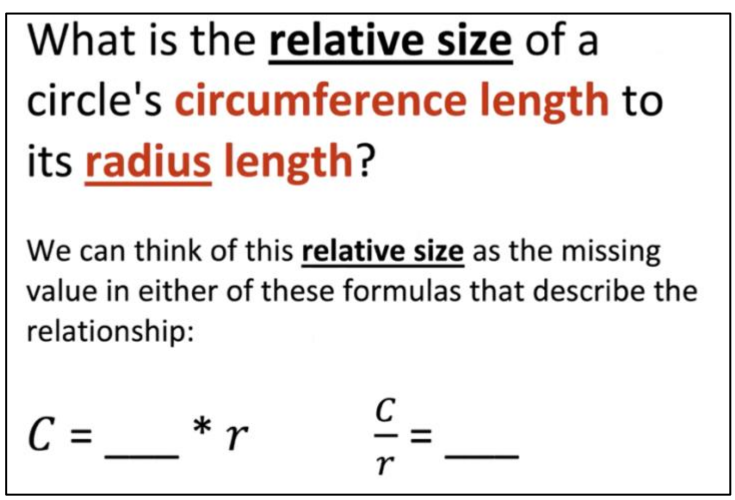


Figure 32. Enzo’s Second Task Used During His Second Lesson on Angle Measure

Enzo further conveyed that he planned to have students think about the relative size of the circumference and the radius so that they would recognize that these quantities are proportional, and as a result, the students could use either as a unit for measuring the length of the arc subtended by an angle's rays. Enzo's plan to use the circumference task (Figure 32) after the class discussions (see Excerpts 7 and 8) supports that his image of the understandings he intended his students to construct for angles and their measures had advanced to include a more refined image of ways of thinking students might engage in to develop and refine their understanding of angle measure.

CONCLUSIONS AND DISCUSSIONS

This paper characterizes a teacher's mathematical meanings for teaching angle measure and the teacher's interactions with students in the context of the teacher's classroom instruction when teaching the idea of angle measure. The findings support that there exists a symbiotic relationship between a teacher's mathematical meanings for teaching angle measure and the teacher's decentering actions when interacting with students while teaching. Enzo's strong mathematical understanding of angle measure, together with his commitment to quantitative reasoning as a critical way of thinking, appeared to influence (1) the nature and sequencing of the questions he posed to students and (2) his ability to select and pose questions that were responsive to students' thinking and effective in advancing his students' thinking. Enzo's actions to decenter also appeared to advance his mathematical meanings for teaching angle measure to include more refined images of how students might be thinking, and more insights into instructional moves that might advance students' thinking.

As one example, the second-order model he constructed of students' thinking led to him recognizing the conceptual affordances of supporting students in developing an understanding of the relationship between the circumference and the radius of a circle. After teaching his first lesson on angle measure, Enzo expressed that he had not previously thought about how students understanding of the relative size relationship between the circumference of a circle and the radius of the circle could support them in recognizing that the unit used to measure the length of the arc subtended by an angle's rays needed to be proportional to the circumference of the circle centered at the angle's vertex. Enzo further expressed that supporting students' understanding of the relative size relationship between the circumference of a circle and its radius (see Figure 32) had become an explicit learning goal moving forward.

This paper also provides an example of a teacher's ways of thinking about teaching angle measure. I classify a teacher as having a productive way of thinking about teaching an idea if they have an image of how an idea might be productively understood and learned and image(s) of the thinking a student might engage in when learning an idea. During clinical interviews, Enzo expressed meanings for angles and their measures that were grounded in quantitative reasoning. Enzo also expressed that engaging students in relative size reasoning supported their understanding that a circle's radius and circumference were appropriate units of measure. Enzo's expression of coherent meanings for angle measure and his images of thinking that supports students in learning and understanding the idea of angle measure are useful for characterizing Enzo's way of thinking about teaching angle measure.

This paper also provides an empirical example of how a teacher's mathematical meanings for teaching an idea can support their ability to engage students in reflective discourse. Enzo's robust meanings for angle measure supported him in recognizing and reflecting on student thinking, ultimately leading to his task design that prompted students to discuss two hypothetical students' thinking (see Figure 31). In particular, Enzo designed the *Hypothetical Student Thinking* task (see Figure 31) so that each student's thinking included in the task represented the thinking some students expressed during the prior class. In this example, Enzo's mathematical meanings and actions to decenter supported him in designing a task that he used to reveal students' meanings and subsequently elevate them to an object of discussion when using them to engage his students in the reasoning they expressed in their prior class.

Finally, this paper also provides empirical support for the productivity of quantitative reasoning as a valuable way of thinking for teachers to engage in. Enzo's commitment to quantitative reasoning as a way of thinking supported him in engaging in pedagogical actions that provided students with opportunities to (1) reason quantitatively themselves and (2) construct robust mathematical meanings for a foundational trigonometry concept. The results of this study also demonstrate that a teacher's commitment to quantitative reasoning as a coherent way of thinking consistently supported him in noticing, reflecting on, and effectively leveraging student thinking while teaching.

I hasten to note that Enzo's expression of coherent mathematical meanings for teaching angle measure grounded in quantitative reasoning, as expressed both in his teaching and clinical interviews, is atypical of first-year graduate student instructors.

Nevertheless, Enzo's expression of coherent meanings for angles and their measures illustrates the impact that sustained professional development (that provides teachers with repeated opportunities to construct coherent meanings for the ideas they are teaching) can have on advancing an instructor's mathematical meanings for teaching an idea and teaching practice while teaching that idea. Although Enzo was in his first year of teaching this course at the university level, he had approximately six years of experience (in his undergraduate studies and teaching in high schools) with the Pathways Pre-calculus curriculum before this study. Enzo had also attended approximately fifty hours of professional development led by the Project Pathways Professional Development team before this study.

LIMITATIONS

This study includes data collected from an instructor who was (1) teaching precalculus using a research-based conceptually oriented curriculum and (2) enrolled in a professional development seminar designed to support his teaching of the course's ideas with fidelity. The unique experiences of this teacher, together with the nature of this empirical investigation, necessitated my conducting a case study which calls into question issues of generalizability. The reader needs to keep in mind that case studies are generalizable to theoretical propositions, not specific populations (Yin, 2009). As such, this study aimed to characterize the interaction between a teacher's mathematical meanings for teaching an idea and their decentering actions. Therefore, the findings of this study cannot and should not be applied more generally to a population of mathematics instructors.

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CHAPTER 5

PAPER 3: A CROSS-CASE ANALYSIS OF TWO INSTRUCTORS' MMT FOR SINE FUNCTION AND THEIR TEACHING PRACTICES

INTRODUCTION AND LITERATURE REVIEW

Over the past thirty years, research on teachers' *mathematical knowledge for teaching* (*MKT*) has developed to improve teaching and students' learning. Ball and many of her colleagues have investigated the various types of knowledge teachers use when teaching (Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2005) and the relationship between teacher knowledge and student performance (Ball, Thames, & Phelps, 2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004, Hill et al., 2008). This area of research has primarily focused on investigating what teachers do in their teaching and the knowledge required to engage in these teaching actions (Ball & Bass, 2003). Although many mathematics educators have investigated the relationships among the mathematics that teachers know, their instruction, and students' learning, few researchers have studied the sources of teachers' pedagogical decisions and actions, teachers' *mathematical meanings for teaching* (*MMT*) (Thompson, 2016).

In 2016, Thompson introduced a new construct, *mathematical meanings for teaching* (*MMT*), to make explicit that he was using knowledge in the sense of Piaget and not in the sense of Ball and colleagues (e.g., Ball 1990, Ball & Bass, 2003, Ball, Hill, & Bass, 2005) (Thompson, 2013). To Piaget, knowledge and meaning were largely synonymous and grounded in the knowers' schemes (Montangero & Maurice-Naville, 1997). Thompson also proposed using the word meaning as it connotates something personal to readers rather than knowledge, which seems less personal and disjoint from the knower (Thompson, 2016). The construct of MMT also differs from Ball and

Colleagues' use of the construct of MKT in the nature of the phenomena they describe. Most researchers using the MKT construct have done so to describe a knowledge base for teachers to achieve. In contrast, Thompson and his colleagues use the MMT construct to describe the current state of teachers' mathematical understandings as they work within the dynamics of teaching.

Thompson and his colleagues have led most, if not all, investigations of teachers' MMT. In 2016, Thompson also proposed a means for gaining insight into teachers' mathematical meanings. Along with his research team (project ASPIRE), he designed a 43-item diagnostic instrument called the Mathematical Meanings for Teaching secondary mathematics (MMTsm). These scholars developed the MMTsm to assess teachers' expressed meanings for variation and covariation, functions (including function notation, models, and properties), proportionality, rate of change, frames of reference, magnitudes, and structure (Thompson, 2015). In addition, team ASPIRE leveraged prior research on students' understanding of various mathematical ideas to develop the rubrics to assess teachers' mathematical meanings. Thompson and colleagues administered the MMTsm to 250 US high school mathematics teachers and 364 South Korean teachers. These scholars considered teachers' meanings productive if they prepared students for future learning and lent coherence to the meanings students already had (Thompson, 2016).

The development of the MMTsm led to many investigations into teachers' mathematical meanings for teaching various secondary-level mathematics topics. For example, Yoon and Thompson (2020) and Thompson and Milner (2018) reported on MMTsm items designed to investigate teachers' meanings for function and function notation. These researchers found that most U.S. teachers they studied thought of the left side of a function definition, e.g., *c(v)*, as a name for the rule on the right side. In particular, when responding to different tasks, these researchers found that U.S. teachers often input the wrong variable into the left side of a function definition. Yoon and Thompson (2020) hypothesized that many teachers used letters other than *v* inside function notation because they thought *c(v)* was simply a replacement for *y*. Results from a second item included in the MMTsm showed that most of the sampled U.S. teachers also believed that every function had to be defined by a rule (Yoon & Thompson, 2020). These researchers further claimed that many of the U.S. teachers used function notation as a label to replace a word (i.e., *A(t)* to replace the word "area") rather than a representation of two quantities values (Yoon & Thompson, 2020; Thompson & Milner, 2018).

Yoon and Thompson (2020) and Thompson and Milner (2018) proposed that the responses to the two function notation items indicate that more than half of the sampled U.S. teachers held unproductive and sometimes incoherent meanings for function notation. The data that Yoon and Thompson (2020) presented also demonstrated a large discrepancy between the meanings held by U.S. teachers and their S.K. counterparts. The South Korean teachers' higher-level responses on both items indicated that most S.K. teachers had more productive meanings for function and function notation (Yoon & Thompson, 2020).

Thompson and his colleagues have also reported on the incoherent nature of teachers' meanings of average rate of change. For example, Yoon, Byerley, and Thompson (2015) found that the sampled South Korean teachers' meanings for average rate of change were significantly stronger than the sampled U.S. teachers' meanings. In particular, one-third of the U.S. teachers they studied revealed meanings for average rate of change as an arithmetic mean of rates. In contrast, almost all South Korean teachers expressed productive meanings for this idea (Yoon, Byerley, and Thompson, 2015). These researchers hypothesized that one reason for the disparity in meanings across the sampled teachers might be that South Korean teachers developed stronger meanings for an average rate of change while they were students (Yoon, Byerley, and Thompson, 2015). As such, these researchers proposed that researchers focus on improving the mathematical meanings students develop in elementary and high school.

If we wish to improve students' understanding of mathematics in the U.S., researchers and teacher educators must attend to teachers' mathematical meanings. Musgrave and Carlson (2017) have demonstrated that teachers' meanings for a foundational mathematical idea can become more productive if these teachers are engaged in interventions designed to support their development of coherent mathematical meanings. "A focus on [teachers] MMT would also foster the field's conceptualization of bridges among what teachers know (as a system of meanings), how they teach (their orientation to high-quality conversations), what they teach (meanings that an observer can reasonably imagine that students might construct, over time, from teachers' actions), and what students learn (the meanings they construct)" (Thompson, 2013, p. 82).

Thompson, Carlson, and Silverman (2007) claim, "if a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students" (p.

416-417). However, although Thompson and his colleagues have widely investigated teachers' MMT, they have yet to examine how it informs their teaching practices. This paper addresses this gap in the literature on teachers' MMT by providing a model of two instructors' MMT for sine function and a description of their goals for student learning, their teaching of the sine function, and their rationale for their pedagogical actions.

THEORETICAL PERSPECTIVE

In this study, I leveraged radical constructivism as a background theory (Glasersfeld, 1995). Background theories "constrain the types of explanations we give, to frame our conceptions of what needs explaining, and to filter what may be taken as a legitimate problem" (Thompson, 2002, p.192). Radical constructivism is a model of knowing that describes how an individual comes to know an idea and the nature of an individual's knowledge. In particular, "constructivism says that whatever sense people make of their experience, they construct that sense themselves — regardless of what anyone else does to influence it. What a person actually ends up knowing can be influenced by what others do, but communication happens only through interpretation" (Thompson, 2002, p.193). Three central tenets of constructivism are (1) that knowledge resides in the mind of the knower, (2) knowledge is not passively received but constructed by the knower, and (3) "the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of an ontological world" (Glasersfeld, 1995, p. 114). Constructivism also holds that individuals cannot know an objective "Reality." Adopters of radical constructivism believe that an individual constructs their own reality and that the reality of each individual is idiosyncratic and the result of prior constructions and experiences.

Quantitative Reasoning

While radical constructivism constrains the kinds of explanations one gives, an explanation’s content comes from theories specific to what is being explained or described. These domain-specific theories address “ways of thinking, believing, imagining, and interacting that might be propitious for students’ and teachers’ mathematical development” (Thompson, 2002, p.194). Much of what I discuss and investigate in this study derives from a theory of quantitative reasoning (Thompson, 1990; 1993; 2011). Quantitative reasoning is the analysis of a situation into a quantitative structure- a network of quantities and quantitative relationships (Thompson, 1990, 1993, 2011).

Within the theory of quantitative reasoning, a quantity is a quality of something that one has conceived as admitting some measurement process (Thompson, 1990). Quantities exist in the mind of the individual conceiving them. To comprehend a quantity, an individual’s conception of “something” must be elaborated to the point that they “see” characteristics of the object that are admissible to the process of quantification (ibid). Quantification is a direct or indirect measurement process that results in a value. A quantity’s value is the numerical result of a quantification process. Numerical operations are used to calculate a quantity’s value, however, numerical operations differ from quantitative operations. A quantitative operation is the conception of two quantities taken to produce a new quantity. Put another way, a quantitative operation is a description of how intensive quantities come to exist (Thompson, 1990).

The discussion of two teachers’ different ways of thinking about the sine function presented in problem statement (see Figure 1) provides an example of a teacher’s reasoning about the output of the sine function as the result of a quantitative operation. In the example, Teacher B conceives of the output of the sine function as a relative size comparison or a multiplicative comparison of the terminal point's vertical distance above the center of a circle and the circle's radius. In contrast, Teacher A conceives of the output of the sine function as the result of a division of two numbers. These two teachers' thinking about the value of the sine function is qualitatively different as Teacher B conceived of division as measurement. In particular, Teacher B determined the value of sine by multiplicatively comparing two quantities' values. This teacher was then able to express the relative size of these two quantities' values by thinking about and expressing the magnitude of the terminal point's vertical distance above the horizontal diameter in terms of a multiple of the magnitude of the circle's radius. In contrast, Teacher A conceived of the output of the sine function as the result of a numerical operation or dividing the length of two sides of a triangle.

Conceptual Analysis

Moore (2014) described a meaning for sine grounded in quantitative reasoning. Namely, sine is a function that inputs the measure of an angle (measured from the 3 o'clock position) and outputs the vertical distance of a terminal point above the horizontal diameter in radii. As such, the value of $\sin(\theta)$ represents how many times as large the vertical distance of a terminal point above the horizontal diameter is compared to the circle's radius when the terminal ray of an angle is rotated to the point that is θ radians counterclockwise from the 3 o'clock position. For example (see Figure 33), the point $(0.8\,, sin(0.8)\ \approx 0.717)$ conveys that for an arc length of 0.8 radii counterclockwise from the 3 o'clock position on any circle, the terminal point's corresponding vertical distance above the circle's horizontal diameter is $sin(0.8)$ or 0.717 times as large as that circle's radius.

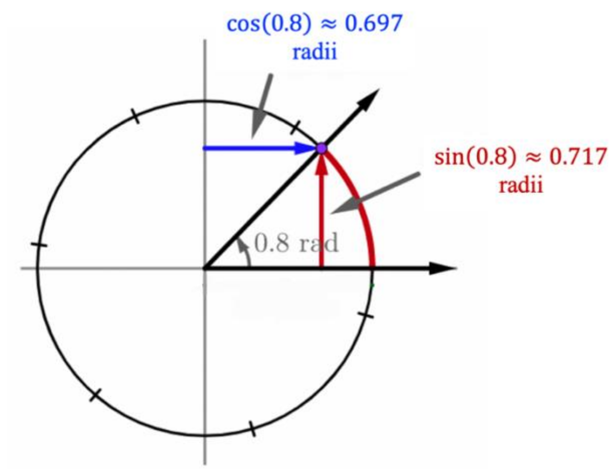


Figure 33. Sine and Cosine
Adapted from Carlson, Oehrtman, Moore, & O'Bryan 8th Edition

However, to conceive of the sine function in the way described above, a student must (1) recognize that for any angle measure (measured from the 3 o'clock position), there exists a vertical distance that the terminal point is above the horizontal diameter of any circle centered at the vertex of the angle, (2) conceive of the vertical distance as a quantity that one can measure-this necessarily requires the student to also conceives of a unit for measuring the vertical distance (radius of the circle) and a process to measure the vertical distance (multiplicative comparison), and (3) recognize that measuring the vertical distance using the radius makes the value of the sine function independent of the size of the circle for which the subtended arc is a part (Tallman, 2015).

A coherent meaning for sine function also involves covariational reasoning (Carlson et al., 2002) and robust connections between right triangle trigonometry and unit circle trigonometry. An individual engages in covariational reasoning when she coordinates two varying quantities while attending to how the quantities' values vary in tandem. Moreover, when a student or teacher attends to how the value of sin (θ) changes as the angle measure, θ varies, she is engaging in covariational reasoning. A teacher with a quantitative meaning for trigonometric functions may view the commonly used trigonometric ratios (SOHCAHTOA) as using the hypotenuse (radius) as a unit of measure for the legs of the right triangle. As one example, the output of the sine function can be viewed as the length of the triangle's leg opposite the angle whose measure is input to the sine function, measured in units of the hypotenuse (see Figure 34).

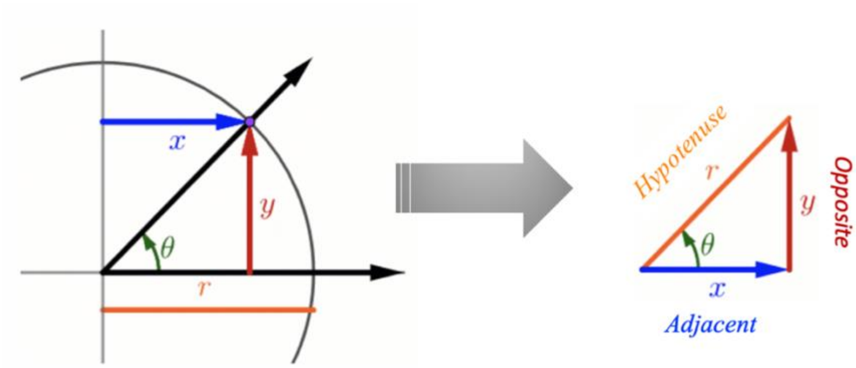


Figure 34. Relating Unit Circle and Triangle Trig
Adapted from Carlson, Oehrtman, Moore, & O'Bryan 8th Edition

Moreover, the sine function relates angle measures with lengths measured in units of the radius or hypotenuse regardless of context (right triangle or circle). As such, the values associated with the unit circle and the ratios associated with side lengths of right triangles emerge as multiplicative comparisons of quantities that hold across circles and triangles (Moore, 2014).

METHODS

Context

This paper presents a cross-case analysis of two cases from a multiple-case study (Stake, 1995; Yin, 2009). A case study is "an in-depth exploration of a bounded system (e.g., an activity, event, process, or individuals) based on extensive data collection" (Creswell, 2002, p.485). I chose this methodology because it was the most appropriate method for examining the relationship between teachers' MMT an idea and their enacted teaching practices. Furthermore, the exploratory case study methodology was most appropriate because it allowed me to (1) investigate phenomena over which I had very little or no control and (2) the data I had access to was limited to contemporary events (e.g., classroom observations and clinical interviews). More specifically, I conducted a multiple-case study to provide contrasting cases of two instructors' MMT angle measure and sine function and the relationship between their meanings and their practice. In addition, a cross-case analysis of two instructors' meanings and practice can allow for more robust interpretations and "perhaps better theorizing" (Stake, 2000, p. 46).

The cases for this study are two graduate student instructors (GSIs) who taught precalculus using the research-based Pathways Precalculus Curriculum (Carlson, Oehrtman, Moore, & O'Bryan, 2020) at a large research university in the southwest United States. Enzo was enrolled in a mathematics education Ph.D. program at the time of the study, and Shira was enrolled in an applied mathematics Ph.D. program. I selected Enzo and Shira as cases for this study because they were both in their first year of teaching precalculus with the Pathways Precalculus curriculum at the University level. I also selected Enzo and Shira because they had the following:

1. Different levels of teaching experience.
2. Different levels of experience using the Pathways Curricular Materials.
3. Varying levels of experience attending professional development seminars led by the leaders of the Pathways Precalculus research team.

Lastly, these instructors were selected because they were willing to have their classroom instruction video recorded.

I selected Enzo to participate in this study because he had many unique experiences with the Pathways Curriculum and professional development led by a Pathways Project research team member. Before entering the Ph.D. program, Enzo earned a bachelor's degree in mathematics education from the same university. Enzo completed Enzo completed two courses for preservice secondary mathematics teachers that used the Pathways pre-calculus curriculum to support preservice teachers in advancing their mathematical meanings for key ideas of secondary mathematics and to explore lesson design and methods for teaching secondary mathematics. Enzo also taught high school mathematics for four years after completing his undergraduate degree. In the last three of these four years, Enzo taught both pre-calculus and algebra II using the Pathways Curricular Materials. In addition, while teaching with the Pathways curriculum materials in the high school, Enzo participated in approximately ten (three to four each school year) 3-hour professional development seminars led by a Pathways Project research team member. Since teaching precalculus at the University level, Enzo had also attended a semester and a half worth of professional development seminars led by a Pathways Project research team member, for a total of 50 hours.

In contrast, at the time of the study, Shira had (1) only one semester of teaching experience before this study, (2) one semester of experience teaching with the Pathways Curricular Materials, and (3) only one semester of professional development experiences led by a Pathways Project research team member (approximately 15 hours).

Pathways Precalculus Curriculum and Professional Development

The Pathways Precalculus curriculum is a research-based curriculum that was designed as a result of many years of research on students' learning and instructors' teaching of precalculus ideas (Carlson, 1995; 1997; 1999; Carlson, Jacobs, Larsen, Coe, & Hsu, 2002; Carlson, Madison, & West, 2015; Carlson, Oehertman, & Engelke, 2010; Engelke, Carlson, & Oehertman, 2005; Kuper, 2018; Marfi, 2017; Moore, 2010; O'Bryan, 2019; Tallman, 2015). The Pathways curriculum was designed to (1) increase the retention and success of students in STEM and (2) shift instructors' practice to have a greater focus on student understanding (Carlson, 2019).

Concurrent with the study, Enzo and Shira were enrolled for a second semester in a required professional development seminar that met weekly for ninety minutes. The professional development seminar was led by a designer of the Pathways Curricular Materials and focused on (1) supporting GSIs in developing coherent meanings and ways of thinking about the ideas to be taught and (2) supporting the GSIs in clearly explaining their meanings for these ideas to others. During the seminar, GSIs and the seminar leader regularly discussed and engaged in the *Pathways Conventions* for supporting quantitative reasoning. These conventions included reinforcing patterns for speaking about quantities and relationships among quantities (speaking with meaning), conceptualizing a graph as a record of how two quantities' values vary together (quantity tracking tool), representing the quantitative structure of a problem context (quantitative drawing), and consistent expectations and methods for defining variables, constructing algebraic expressions and defining formulas (emergent symbolization) (Carlson, O'Bryan, & Rocha, 2023).

Experimental Methods

My experimental methods can be broken down into two phases. In the first phase, I engaged Shira and Enzo in a task-based clinical interview (Goldin, 1997) to construct a model of each instructor's (1) meanings for and ways of thinking about the sine function, (2) commitment to quantitative reasoning, and (3) image of how to support students in developing coherent meanings for the sine function. The second phase of my data collection involved semi-structured clinical interviews that occurred before, directly following, and two days after each instructor's lessons on the sine function. I also collected video data on each instructor's teaching during this phase. Tables 1 and 2 outline the timeline of the research activity conducted with each instructor during each phase of the study.

Date	Research Activity	Duration
	Phase 1	
3/31	Sine Function TBCI	64 min
	Phase 2	
4/1	Pre-Teaching Clinical Interview 5	65 min
4/1	Classroom Observation 5	50 min
4/1	Post-Teaching Clinical Interview 5	32 min
4/4	Pre-Teaching Clinical Interview 6	66 min
4/4	Classroom Observation 6	50 min
4/6	Post-Teaching Clinical Interview 6	21 min
	Video Analysis 6 & Final Clinical Interview	65 min

Table 7. Sine Function Data Collection Schedule with Enzo

Date	Research Activity	Duration
	Phase 1	
3/31	Sine Function TBCI	45 min
	Phase 2	
4/1	Pre-Teaching Clinical Interview 3	64 min
4/1	Classroom Observation 3	50 min
4/1	Post-Teaching Clinical Interview 3	18 min
4/4	Pre-Teaching Clinical Interview 4	57 min
4/4	Classroom Observation 4	50 min
4/4	Post-Teaching Clinical Interview 4	11 min

4/7	Video Analysis 4 & Final Clinical Interview	50 min

Table 8. Sine Function Data Collection Schedule with Shira

I engaged both instructors in pre-teaching clinical interviews to gain insight into the nature of the teacher's lesson-planning process. In particular, during the pre-teaching clinical interviews, I posed questions to each instructor to probe their meanings for the sine function, their image of the understandings they wanted students to construct, and their approach to planning lessons. During these interviews, I also asked the instructors to share their plans for the class, including any activities and/or tasks they planned to use. I then inquired about the activities and their rationale for the selection and/or design of the activities and conversations about the activities they anticipated having with students.

I observed and video-recorded each instructor's teaching after the pre-teaching interview. Immediately following their teaching, I conducted short semi-structured post-teaching clinical interviews. During the post-teaching clinical interviews, I probed the instructors' rationale for specific instructional actions I observed during their teaching. Two days after each instructor's lesson on the sine function, I conducted "video-analysis" clinical interviews with each instructor. These video-analysis clinical interviews were conducted immediately before the subsequent pre-teaching interview with each instructor. I showed each instructor short video segments from their previous lesson during the video-analysis clinical interview. While watching the video segments, I prompted the instructors to discuss their motives for specific comments, actions, and questions. I also posed questions to understand and characterize each instructor's mental actions when interacting with students. I selected each clip used in the video-analysis clinical interview according to the following, (1) the instructor interacted with a student or multiple students, (2) the instructor leveraged students' thinking in their teaching, (3) the instructor used an applet, table, or diagram to support their discussion of a mathematical idea, and (4) the instructor deviated from their original lesson plan. While watching the video segments with each instructor, I prompted them to explain their motive for specific comments, actions, and questions. During this time, I also posed questions to understand and characterize the mental actions the teacher engaged in when interacting with students.

Analytical Methods

Preliminary Analysis

The sine function TBCI constituted the only data for my preliminary analysis. During the TBCI, I took notes on the meanings each instructor expressed. At this analysis stage, I was alert to moments when the instructors (1) engaged in quantitative reasoning or (2) expressed a desire to support students' engagement in quantitative reasoning. After the interview, I took notes to record my initial thoughts and impressions of the instructors' meanings for the sine function. Within 24 hours of the TBCI, I rewatched the video recording of the interview and wrote detailed memos that described each instructor's responses to each task included in the TBCI. These memos constituted my initial model of each instructor's MMT for the sine function.

Ongoing Analysis

A large portion of my analysis occurred as I collected data. Throughout the pre-teaching, post-teaching, and video analysis clinical interviews, I wrote detailed notes on each instructor's meanings, engagement in quantitative reasoning, and interactions with students while teaching. I also wrote detailed notes on instances where the instructor discussed the meanings they wanted students to have for the sine function. During the post-teaching and video analysis interviews, I noted any instances in which the instructor expressed interest in student thinking or discussed their image of how students were thinking during their lessons on the sine function. Following each clinical interview, I wrote detailed memos on (1) how I hypothesized each instructor to be reasoning about the sine function, (2) their commitment to quantitative reasoning, and (3) their attention to and image of students' thinking.

My ongoing analysis of each instructor's teaching involved taking detailed notes as I observed their instruction. In particular, I noted moments in which the instructor (1) expressed a meaning for the sine function, (2) interacted with students in their groups, and (3) attended to or leveraged student thinking while teaching. Within twenty-four hours of each instructor's teaching, I watched the video recording of their instruction. During this time, I selected segments of their instruction according to the three criteria above. These selected video segments were later used during the video-analysis clinical interviews.

Retrospective Analysis

I primarily based my conclusions on the retrospective analysis of the data. My retrospective analysis can be broken down into three phases. The first phase involved my retrospective analysis of the TBCI, the pre-teaching, and the post-teaching clinical interviews. The procedures I used to analyze the TBCIs, and clinical interviews with the instructors are consistent with Simon's (2019) approach to analyzing qualitative data. The second phase of my retrospective analysis involved analyzing each instructor's teaching. Finally, the third phase of my analysis involved reviewing the clinical interviews in which the instructors watched short video segments of their teaching. Each phase described above was consistent with the methods described in the previous two papers.

Drawing Comparisons Between Enzo and Shira's Meanings and Instruction

I leveraged my models of Enzo and Shira's MMT sine function when comparing the instructor's meanings for this idea. Similarly, I leveraged my model of the instructors' practice and the nature of their interactions with students when comparing their teaching. To display the differences in each instructor's MMT sine function, I chose to present their responses to the same tasks included in the sine function TBCI. More specifically, I determined which tasks to include in the next section based on the degree to which they supported my conveying a model of each instructor's MMT for the sine function. Similarly, in the next section I present examples of their instruction that best model the nature of their instruction and illustrate the differences in each instructor's practice and interactions with students.

RESULTS

Shira and Enzo participated in a task-based clinical interview that I designed to elicit (1) their meanings for and ways of thinking about sine function, (2) their image of the meanings they want students to have for the sine function, and (3) how they planned to support students in constructing what they consider to be coherent understandings of this idea. The tasks included in the TBCI elicited these aspects of each teacher's thinking by prompting them to respond to open-ended tasks and describe what they might do in their teaching to support students in understanding the mathematical ideas needed to respond to the task. This section presents my models of Shira and Enzo's MMT for the sine function.

Shira's and Enzo's Response to Task 1

The first task included in the sine function TBCI is in Figure 35. I designed this task to uncover the degree to which the instructors (1) conceived of the sine function as a function that relates two quantities' whose values covary and (2) conceived of the output of the sine function as a relative size measurement. Shira's response to this task is in Excerpt 20. In particular, Shira expressed that she wanted students to recognize that each side of the table represents input and output values. She further said that she wanted students to realize that "sine of theta equals y or that it's the ratio of y with respect to the radius" (Excerpt 20, lines 4-5). Shira concluded her response to this task by stating that she wanted students to be able to use the table of values to plot points (Excerpt 20, lines 11-13 and Figure 36).

Imagine that you gave students the below table of values. How do you want students to reason about what the values in the table represent?

Angle Measure, θ (in degrees)	$\boldsymbol{sin(\theta)}$
42	0.669
95	0.996
163	0.292
210	-0.5
337	-0.391

Figure 35. The First Task Included in the Sine Function TBCI

Although Shira identified the angle measures on the left side of the table as input values and the values of sine on the right side of the table as output values, it is noteworthy that she did not describe sine as a function that relates two quantities' values. Furthermore, Shira's activity of plotting the points represented in the table on a coordinate plane supports that she may have been thinking about the value of the sine function as a y-value on a graph.

Excerpt 20

Shira: So, I would hope I guess initially, I would hope that they can see that we have input and output values. So, the input is the angle measure in degrees and then the output is what happens after we evaluate sine at those angle measures. I would also hope that they know that sine theta equals *y* or that it's the ratio of *y* with respect to the radius. I would hope that my students are able to create a plot using this table of values.

Int: So, can you show me kind of what you're thinking of? Like what you'd want them to be able to do?

Shira: Yeah. So, um, I would hope that they would be able to kind of create something that looks like this [draws two axes and labels them]. This is not in any way, shape or form drawn to scale, but yeah. So, I hope that they could see that this is theta in degrees and that then the output is sine of theta and then they would be able to, I don't know, do like [plots points-see figure 36].

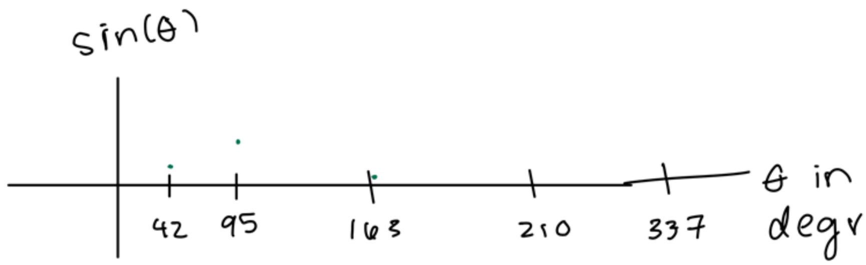


Figure 36. Shira's Work from the Task Shown in Figure 35

Enzo's response to this task is in Excerpts 21, 22, and 23. Enzo first responded to this task by stating, "I want them to see at the outset is that this is a functional relationship" and that S-I-N is a "three-letter word for a function" (Excerpt 21, lines 1-3). He then expressed that he wanted students to recognize that every input value (angle measure) has one output value (of the sine function) (Excerpt 21, lines 4-7). He also expressed that the values of sin (θ) represent "the height above the horizontal diameter of the point, which is at the intersection of the terminal ray of the circle centered at the vertex of the angle" (Excerpt 21, lines 8-10). It is also noteworthy that Enzo described the values in the table as snapshots of an animation (Excerpt 21, line 12). In particular, Enzo said he was thinking about the table of values as representing “instances in angle measure” and “the corresponding height above the horizontal diameter” (Excerpt 21, lines 14-15). Enzo’s description of the values in the table as “snapshots” of an animation implies that he might have been thinking about the values of angle measure and their corresponding values of sine function as covarying.

Excerpt 21

Enzo: Well, one level of thinking would be thinking about input and output that we have with this thing right here [underlines sin], which is a three-letter word for a function. So, what I want them to see at the outset is that this is a functional relationship. Clearly here, every input that’s listed has only one output. But thinking about variation of this guy [circles θ] across its domain, every angle measure is going to have an output…a sine value, and there’s only going to be one of them. I want them to think about this this quantity over here [underlines $\sin(\theta)$] as the height above the horizontal diameter of the point, which is at the intersection of the terminal ray of the circle centered at the vertex of the angle. So, we can sort of imagine this angle measure increasing and that corresponds with like imagining this terminal ray sweeping out and like this table of values is sort of like a snapshot like we reported the animation at these instances, not necessarily time, but like these instances in angle measure. And then this would be the corresponding height above the horizontal diameter

As Enzo responded to the task in Figure 35, he also explained what the values of $\sin(\theta)$ represented. For example, Enzo said that the value of -0.391 in the right side of the table represented the number of radius lengths the terminal point was above (below) the horizontal diameter (Excerpt 22, lines 4-7 and Figure 37). He further expressed that the vertical distance was “some 39%, some 40% of a radius length” (Excerpt 22, lines 8-10).

Excerpt 22

Int: So, you said something earlier about 0.39 above and .39 below. Can you dig into that a little bit more? Like, what do you mean by .39 above? Like what is that .39?

Enzo: So, this negative 0.391. So, I guess I can draw it, but something like this point right here, right, that distance is going to be 0.391. But in particular, it's below this horizontal diameter. Well, that's 0.391 what? It is measured in radius lengths. These guys are in radius lengths [writes "Radius l" under the right column of the table]. So yeah, I mean the diagram is not like super precise, but like this chunk right here [points to the vertical distance of the terminal point] is some, I don't know, like some 39%, some 40% of a radius length.

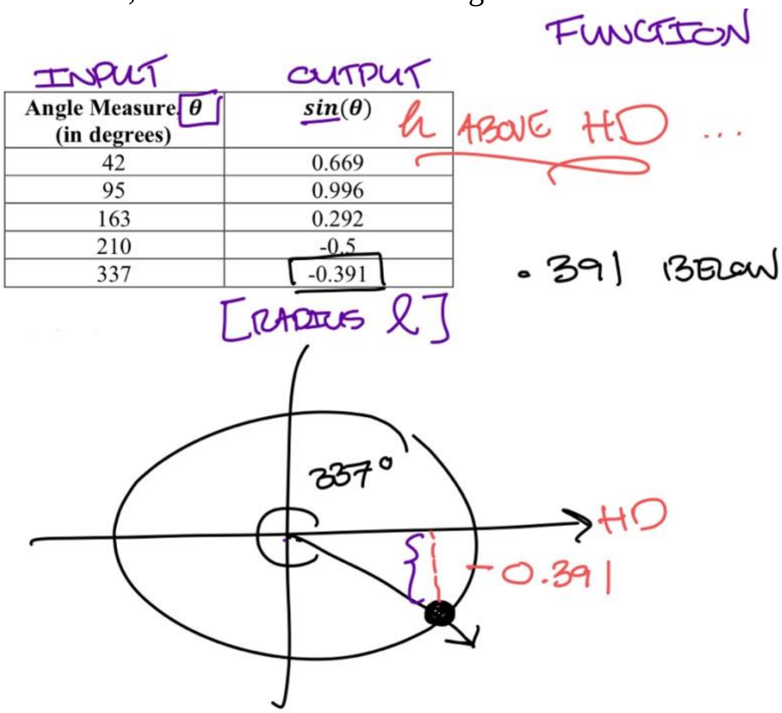


Figure 37. Enzo's Work from the Task Shown in Figure 35

Enzo's meanings for the value of the sine function are further evidenced by his description of what the output value of -0.5 represented. Enzo stated that he could "go back and forth between thinking of [the terminal point's vertical distance above the horizontal diameter] as like 50% of a radius length" and "a length that is 0.5 times as large as a radius" (see Excerpt 23 and Figure 38). Enzo also said that he imagines "the vertical distance above the horizontal diameter varying from like 0% extension to like 100% extension or from like -100% to 100% if [he] think[s] about the directionality" (Excerpt 23, lines 6-8). Enzo's description of the value of sine as (1) a number of radius lengths and (2) a percentage of a radius length support that he conceived of the value of the sine function as a relative size comparison between a terminal point's vertical distance above the horizontal diameter of a circle and the circle's radius. Enzo's description of the vertical distance above the horizontal diameter varying from 0% extension to 100% extension also supports his thinking about the values of the sine function varying.

Excerpt 23

Enzo: And so, if this [points to sin (θ)] outputs 0.5 and thinking about it as a length that is 0.5 times as large as a radius. I can go back and forth between thinking of it as like 50% of a radius length. But yeah, and the reason, the reason why I like the percent is because it just helps me imagine some of that variation. Like I can imagine the vertical distance above the horizontal diameter varying from like 0% extension to like 100% extension or from like -100% to 100% if we think about the directionality.

Figure 38. Enzo Show's That an Output of Sine Equal To .5 Means That the "length" Is .5 times as Large as the Radius or 50% of a Radius

Shira and Enzo's Responses to Task 2

In the next section, I present Shira and Enzo's response to a second question in the sine function TBCI. This task included three parts, shown in Figures 39, 40, and 41. I designed part (a) of this task (in Figure 39) to elicit the degree to which each instructor thought about sine as a function that relates an angle measured from the 3 o'clock position to a terminal point's vertical distance above the horizontal diameter measured in radius lengths. I also designed part (a) to elicit the degree to which the instructors engaged in quantitative reasoning when reasoning about the sine function.

I designed parts (b) (Figure 40) and (c) (Figure 41) of the task to elicit the degree to which the teacher conceives of the value of the sine function as a relative size measurement. Although part (b) of this task does not directly require the teacher to relate their answer to the value of sine, I was interested in discerning if the instructor made any connections between the thinking they engaged in when responding to part (b) and the meanings they want to support students in developing for the sine function.

Halloween is coming up and you and your friend decide to go to a circular corn maze (shown below). In the center of the corn maze is a haunted house. The entrance of the corn maze is 1.5 km south of the haunted house.

(a) While you are at the corn maze, you and your friend get separated. To find your friend, you begin walking around the circular edge of the corn maze. Write a function to determine your distance north of the haunted house (in kilometers) in terms of the distance you have walked (in kilometers), *m*, from the entrance.

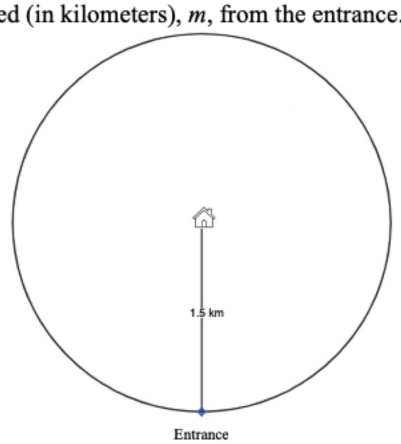


Figure 39. Part (a) of the Second Task Included in the Sine Function TBCI

(b) As you walk around the corn maze you realize you have your friend's location on the "find my friends" app in your phone. The application marks your friend's location with an X as shown below. Estimate your friend's distance (in kilometers) north of the haunted house. Estimate your friend's distance (in kilometers) east of the haunted house. Explain how you determined your estimates.

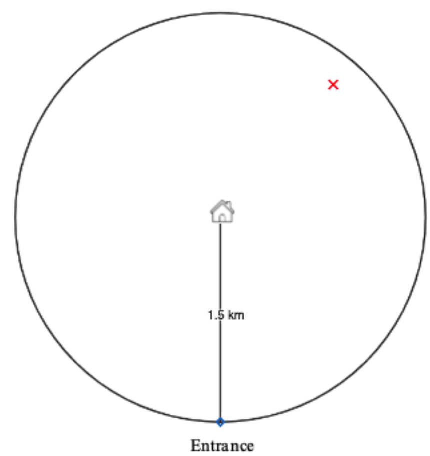
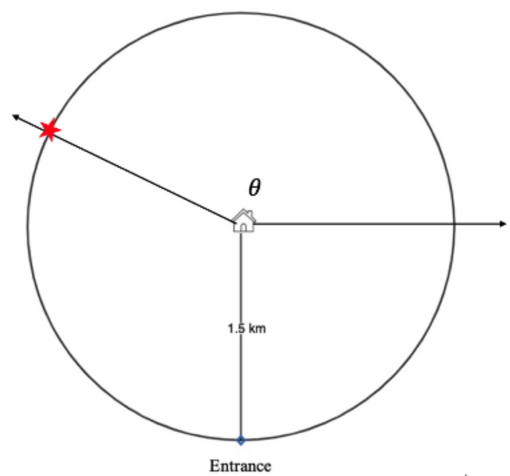


Figure 40. Part (b) of the Second Task Included in the Sine Function TBCI

(c) Let θ represent the measure of an angle (whose vertex lies at the center of the circle) from the 3 o'clock position (in radians). If the exit of the corn maze is located where the star is on the map, estimate the value of $\sin(\theta)$ and $\cos(\theta)$.



Part (c) of the Second Task Included in the Sine Function TBCI

Shira and Enzo's Responses to Task 2 Part A

Enzo's response to part (a) of task 2 (Figure 39) is in Figure 42. After Enzo responded to the task, I prompted him to describe what each part of the function he wrote represented. Enzo responded that $\frac{m}{1.5}$ represented the measure of the angle swept out from the 3 o'clock position. Enzo then said that he subtracted $\frac{\pi}{2}$ to account for starting at the entrance of the corn maze. As such, Enzo expressed that $\frac{m}{1.5} - \frac{\pi}{2}$ represented the measure of the angle swept out from the entrance of the corn maze to where he was at on the circle. Enzo then said that $\sin\left(\frac{m}{1.5} - \frac{\pi}{2}\right)$ represented his distance north of the haunted house measured in radius lengths. Following this, he concluded that $1.5\sin\left(\frac{m}{1.5} - \frac{\pi}{2}\right)$ represented his distance north of the haunted house in kilometers. Enzo's activity while responding to this task, and his ability to describe the

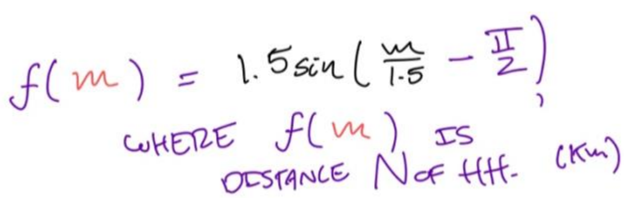


Figure 42. Enzo's Response to the Task Shown in Figure 39

quantity represented by each expression included in his answer support that he was reasoning quantitatively while responding to this task. In particular, he consistently identified an attribute of an object he was measuring, a measurement process, and a unit of measure. Enzo also consistently described the starting points of each of the quantity's measures.

Shira's response to part (a) of the second task (Figure 39) is in Figure 43. Unlike Enzo, Shira struggled quite a bit while responding to this task. After reading the task a few times, Shira said she needed to use the sine function because the "y-value is what will give the distance north of the *x*-axis." Shira further explained that she was unsure about her decision to input *m* into the sine function, but she did this because the problem said, “in terms of the distance, m,” which “[means] m has to be the input.” Shira expressed discomfort inputting m because “m is measured in kilometers, but angles can’t be measured in linear units.” When I asked Shira how she might reconcile what she wrote in green, Shira said that instead, she could put “sine of 1.5 m”. When I asked her why she changed her answer to $\sin(1.5m)$, Shira said she did this because “if you want to find the arc length, you multiply the angle by the radius”. However, after writing $\sin(1.5m)$, Shira said she wasn’t sure because “m is not the angle.”

It is noteworthy that Shira recognized (1) she needed to use the sine function to answer this question and (2) that the value she input into the sine function needed to be an angle measure. However, Shira struggled to answer this question because she had not conceptualized the angle measure as a relative size comparison between the distance she traveled from the entrance of the corn maze and the circle’s radius. Moreover, Shira’s decision to use the sine function because the “y-value is what will give the distance north of the *x*-axis” further supports that she was thinking about the value of the sine function as a *y*-value on a coordinate plane.

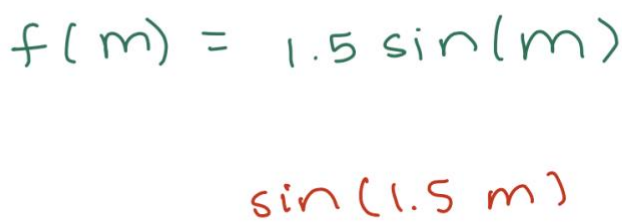


Figure 43. Shira’s Response to the Task Shown in Figure 39

Shira and Enzo’s Responses to Task 2 Part B

Recall that I designed part (b) of the second task included in the TBCI (Figure 40) to discern if the instructor made any connections between the thinking they engaged in when responding to part (b) and the meanings they wanted to support students in developing for the sine function. I also designed part (b) of this task to elicit whether the teachers recognized that measuring the vertical distance using a circle's radius makes the sine function's value independent of the circle's size for which the subtended arc is a part.

Enzo responded to part (b) of the task by estimating a point's (where the circle intersected a ray that started at the center and went through the red X) vertical distance above the haunted house and horizontal distance to the right of the haunted house (see Figure 44). In particular, he estimated the vertical distance of the point above the horizontal diameter by multiplicatively comparing its length to the length of the circle's radius. He then estimated the horizontal distance of the point to the right of the vertical diameter using the same thinking. Following this, Enzo concluded that the vertical and horizontal distances to the red X were the same as the vertical and horizontal distances to the point where the terminal ray intersected the circle because "the percentage of radius length, or like the sine and cosine values, like, they're the same" (Excerpt 24, lines 14-15). Enzo's response to this task supports that he (1) conceived of the vertical distance as a quantity that he could measure, (2) conceived of the radius of a circle as a unit for measuring the vertical (horizontal) distance, (3) conceived of a process to measure the vertical distance (multiplicative comparison), and (4) recognized that measuring the vertical distance using the radius makes the value of the sine function independent of the size of the circle for which the subtended arc is a part.

Excerpt 24

Enzo: Okay, so to the right, it's more than half, umm two-thirds, whatever. Let's say two-thirds [writes $(0.\overline{6}$].

Int: How did you get two-thirds?

Enzo: I just did that [draws two red segments on the horizontal diameter]. Umm cutting this guy[the circle's radius] into three pieces.

Int: Okay.

Enzo: So, it looks like two-thirds of a radius length. And then the height… uhh… the number I want to say is 80%. I don't know, looks like, let's say… looks like more than three-quarters [draws red segments on the vertical diameter and writes$(0.\overline{6}, 0.8)$]. It is no different than doing the exact same thing with a smaller radius [draws a smaller circle with the origin as its center]. Yeah so they're the same.

Int: What do you mean? What is the same?

Enzo: So, the percentage of radius length, or like the sine and cosine values, like, they're the same. So, I can make this little change [labels the larger circle's radius r_1 and changes $(0.\overline{6}, 0.8)$ to $(0.\overline{6}\, r_1, 0.8r_1)$ then labels the smaller circle's radius r_2]. So then, I could say, my friend is at $(0.\overline{6}\, r_2, 0.8r_2)$ where r_2 is less than 1.5, which if I just estimate it from the picture, we could say that r_2 is approximately six-sevenths of r_1.

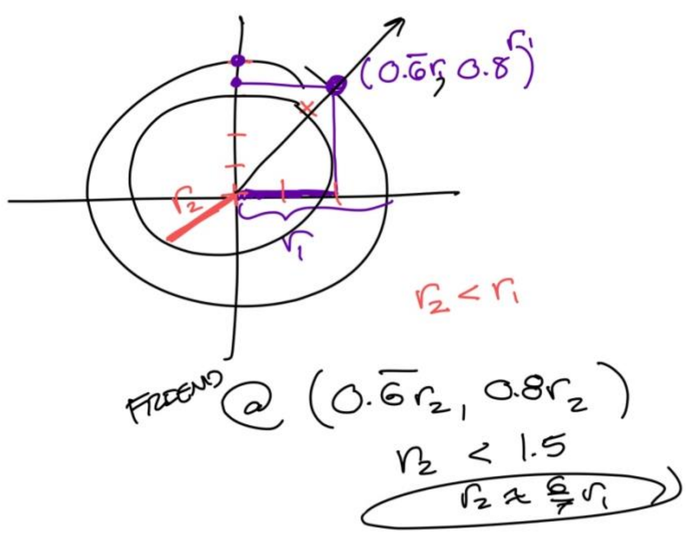


Figure 44. Enzo's Response to Part (B) of the Corn Maze Task Shown in Figure 40

Shira responded to part (b) of the task (Figure 40) by using Pythagorean Theorem. Like Enzo, Shira initially determined the vertical distance of a point where the circle intersected a ray that started at the center and went through the red X (Figure 45). Shira then estimated the horizontal leg of the triangle in Figure 45 to be one. Following this, she used Pythagorean's Theorem to determine the y-value or the length of the vertical leg of the triangle in Figure 45.

Excerpt 25:

Shira: So initially I'm thinking Pythagorean theorem. So, I know that this distance right here is 1.5 kilometers [draws horizontal radius and labels it]. You can imagine the thing going through here [draws a line from the center through the red X] or through her location. And then we can find that vertical distance [draws red line from red X to the horizontal radius]. I would say that this, if I was approximating I would say that this distance is one. And then we also know that this distance is also 1.5. So, then we would only be left with to find that [works through calculation and solves for y]. So, her vertical distance from the house would be this many kilometers [writes $y \approx \sqrt{\frac{5}{4}}$ km] and I am approximating that she is about 1 kilometer east of the house.

Int: So, would that give the friend's vertical distance?

Shira: Oh yeah, that would give this point [marks on circle where the terminal ray intersects the circle].

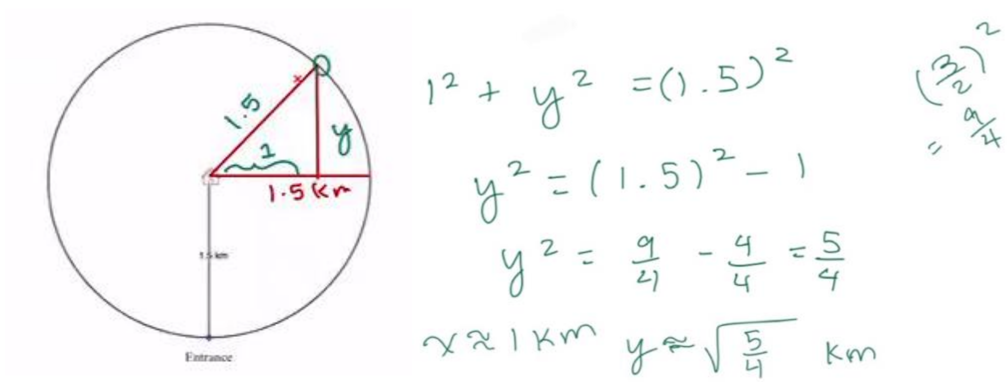


Figure 45. Shira's Response to Part (B) of the Corn Maze Task Shown in Figure 40

Once Shira realized that she had approximated the wrong vertical distance, she changed her answer by re-estimating the distance from the haunted house to the red X. She then used Pythagorean's theorem again to determine the red X's vertical distance above the horizontal diameter in kilometers. Although Shira correctly estimated the red X's vertical distance above the horizontal diameter, it is noteworthy that she did not attempt to measure the vertical distance using the circle's radius. Nor did she relate her activity while responding to this task to the meanings she wanted students to develop for the sine function. I hypothesize that Shira did not relate her activity while responding to this task to the meanings she wanted students to develop for the sine function because, as is argued later, she conceived of the value of the sine function as the ratio of y to r, as opposed to a measure of how many times as large a vertical distance is compared to the radius of a circle.

Shira and Enzo's Response to Task 2 Part C

I designed the last part of the second task included in the sine function TBCI to elicit the degree to which the teachers conceived of the output of the sine function as a relative size measurement. After reading the task, Enzo immediately identified that he needed to determine the star's distance above the horizontal diameter and the distance east of the vertical diameter. As such, Enzo drew a melon-colored vertical segment from the horizontal diameter to the star and a purple horizontal segment from the vertical diameter to the star (Figure 46). Enzo then determined the measure of these segments by multiplicatively comparing them to the circle's radius (Excerpt 26 and Figure 46). In particular, Enzo estimated that the horizontal purple segment was 0.8 times as large as the circle's radius or "80% of the way there".

Similarly, Enzo estimated that the vertical melon segment was 0.45 times as large as the circle's radius. Therefore, Enzo concluded that $\cos(\theta)$ was -0.8 since the star was west of the vertical diameter. Consequently, he also concluded that the value of $\sin(\theta)$ was 0.45. Enzo's activity while responding to this task further supports that he had a meaning for sine function grounded in quantitative reasoning as he consistently identified an attribute to measure, a unit of measure, and a measurement process (multiplicatively comparing the vertical distance to the radius).

Excerpt 26:

Enzo: Okay, cosine is really big. Hmm… I don't know [writes (-0.8,]. That's looking to be really close to half, but smaller [writes (-0.8, 0.45)]. Okay, so what I am thinking here is, so, I've got these two segments and this melon colored one there and I am trying to see how long it is compared to the radius length [labels the radius r]. I am trying to think about relative size. What I am doing is measuring this melon segment using this guy [points to vertical radius length] as my measuring stick. Yeah, a little less than half up and almost all the way left.

Int: So, you mentioned relative size, so in what way do you see relative size reasoning involved or as a way of thinking that you can use to answer this task?

Enzo: Well, so we're measuring this guy [draws melon and black vertical segments to the right-labels them d_n and r] and we're also measuring this [draws purple vertical segment and labels it $d_{to\ E}$]. So, this is the distance to the west [labels purple segment] and this is the distance north [labels melon segment] both measured in radius lengths. So, if I am saying distance to the west, maybe that is a little imprecise since it's negative so maybe I want to call it distance to the east, so, I have a frame of reference.

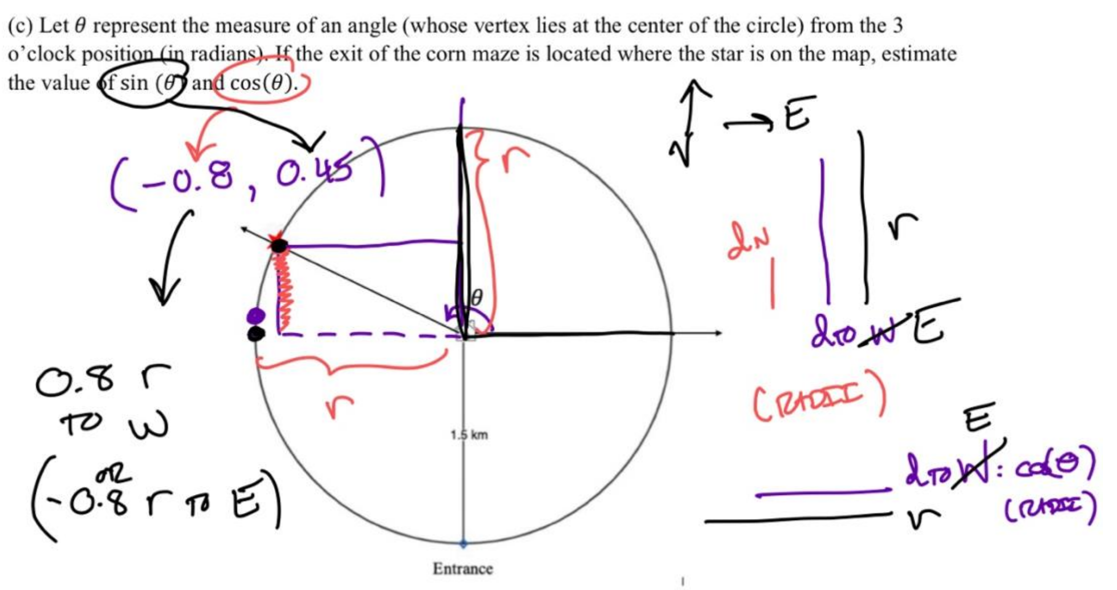


Figure 46. Enzo's Response to Part (C) of the Corn Maze Task Shown in Figure 41

Shira initially responded to this task by estimating how many radians the angle measured. In particular, she determined that the angle measured 2.25 radians by laying 2-

and-a-quarter radius lengths along the arc subtended by the angle's rays. Following this, she paused for a long time and then wrote $\sin(\theta) = \frac{y}{r}$. She then approximated *x* and *y* to be 1.3 and 0.7 kilometers, respectively. When I prompted Shira to explain how she determined that *y* was 0.7 kilometers, Shira said, "I know that halfway [up the radius] would be 0.75 [kilometers], so I am just approximating it's just about 0.7 [kilometers]" (Excerpt 21, lines 20-21). Shira further explained that she knew "halfway" up the radius was 0.75 kilometers. As such, she concluded that the star was approximately 0.7 kilometers above the horizontal diameter because "it just seem[ed] like slightly below the halfway point." Shira then determined the values of sin (θ) and cos (θ) by dividing 0.7 and 1.3 by the radius length of the circle centered at the angle's vertex.

Excerpt 27

Shira: I am thinking about using Pythagoreans theorem again to find *x* and *y*.
Int: Okay
Shira: So, that distance in total is 1.5 [draws horizontal line to the left of the haunted house]. So, this distance is like 1.3 [labels this on the diagram]. Umm that distance [draws vertical line from the star to the horizontal diameter] is like… [pauses and draws a vertical radius length from the center]
Int: Talk to me about what you're thinking.
Shira: Umm I am just trying to see how many times the smaller line goes into this bigger line [moves vertical radius length next to the vertical line she drew from the horizontal diameter to the star]. Umm I'd say it is a little less than halfway.
Int: Yeah, and why do you want to know how many times the smaller line goes into the bigger line?
Shira: Umm… because then I would be able to find the length of the smaller line.
Int: Okay.
Shira: Yeah. So, then it's about 0.7. So, then I would say…
Int: How did you get 0.7? Sorry could you just say it one more time?
Shira: Yeah so, if I put this like back here [moves the vertical radius back to the center] this is like the halfway point [marks this with a red dash] and when I bring it back over here [moves it back close to the vertical line to the star], it [the vertical distance of the star] just seems like slightly below the halfway point. So, I know that halfway would be 0.75 so I am just approximating it's just about 0.7.

Int: So, 0.7 what?
Shira: Kilometers. So, then sine of theta would be 0.7 divided by 1.5 and then cosine of theta would be negative 1.3 divided by 1.5.

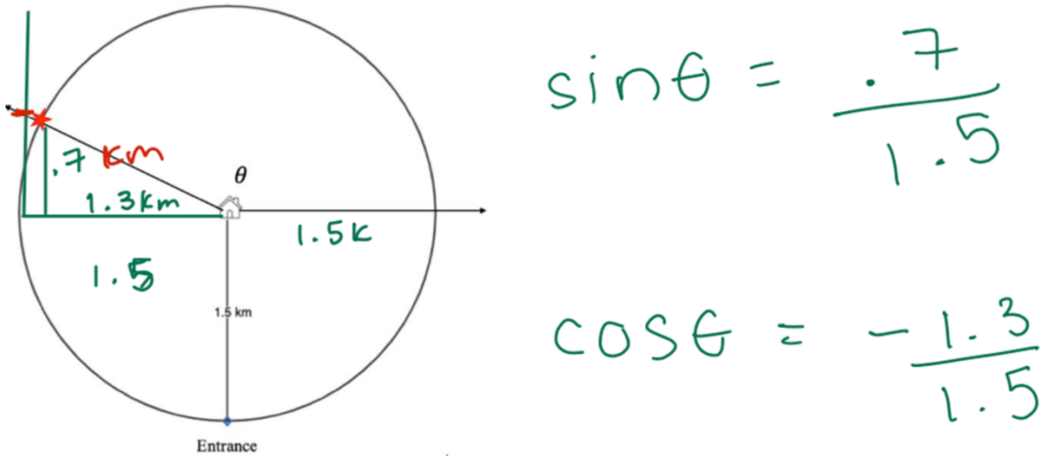


Figure 47. Shira's Response to the Task Shown in Figure 41

Although Shira correctly determined the values of sine and cosine functions, it is noteworthy that she did not recognize that she could multiplicatively compare the length of the star's vertical distance above the horizontal diameter and horizontal distance to the right of the vertical diameter to the radius of the circle to determine the values of the sine and cosine functions respectively. As such, Shira's response to this task supports that she also conceived of the value of the sine function as the result of a numerical operation. This is evidenced by her need to approximate the vertical distance of the star above the horizontal diameter in kilometers *before* dividing it by the radius length in kilometers. In particular, when Shira determined that the star was 0.7 kilometers above the horizontal diameter, she did not recognize that her activity of determining that the star was "like slightly below the halfway point" of the radius was a method for determining the star's vertical distance above the horizontal diameter in radius lengths or the value of the sine function. As such, it appears Shira was thinking about the value of the sine function as the result of comparing two numerical values rather than the relative size of the star's vertical distance above the horizontal diameter and the circle's radius.

Shira and Enzo’s Learning Goals for their Lesson on Sine Function

I also interviewed Shira and Enzo before their lessons on Sine and Cosine functions. During these pre-teaching clinical interviews, I posed questions to gain insight into Shira and Enzo’s meanings for the sine function, their lesson planning process/practices, their image of the understandings they want students to have, and the teacher’s image of activities and conversations when using those activities, they anticipate having when teaching angle measure and the sine function.

Before teaching two lessons on sine and cosine functions, Enzo repeatedly expressed that he wanted to support students in recognizing that sine and cosine are functions that relate two quantities’ values (see Excerpt 28). In particular, Enzo said he wanted to support students reasoning about how angle measure and the values of sine and cosine functions vary together. Enzo also said that he wanted to support students in recognizing that “S-I-N and C-O-S are just names for two really important functions.”

Excerpt 28: Enzo’s Goals

Enzo: We are we are looking at circular motion. And I’m going to try to be really disciplined about having the quantities we care about emerge from the class. And I’ve developed an applet to try to facilitate that. And the goal for today or Friday is to be able to write down definitions for sine and cosine. That’s the mile marker. But it’s going to take a while to get there. I know it. So, I have one crazy idea. The crazy idea is to symbolically like name sine and cosine with like more involved weird names so that once we realize, oh, this is the sine function, it [the name of the function] collapses into something smaller. So, what I’m playing with is like “d sub right of theta” and “d sub up” because I can do a lot of work there to be like, okay, this is the name of our function.

Int: Yeah, what are you hoping to draw students’ attention to by doing that?

Enzo: I’m trying to draw students’ attention to the fact that, you know, this relationship between these distances and the input of the angle measure is a functional relationship and that what do we do when we have functions? Well, we give them names, usually f, g and h. Sometimes there’s weird names like “L-O-G” [spells it out] or like “S-I-N” [spells it out]. But underneath it all, I really need them to understand this functional

> relationship. That for every angle measure there exists one of these distances and in particular there exists only one…because otherwise the discussion about inverse trigonometry, is going to be like silly or like limited if they don't have that, they don't have the functional relationship or an understanding of the functional relationship. So, I am really still undecided how to like, open up class today. I might have some bellringer I might not. But at some point we're gonna get to this [applet]. And, okay, the reason why I don't have like a plan written out is because a lot of the scaffolding is implicit in these views [of the applet]. And I really want to be disciplined about having discussions where the quantities emerge and really responding to student thinking.

When discussing his plan for class, Enzo said he designed an applet to support the emergence of the definitions of sine and cosine functions during class (Excerpt 28 lines 24-27). In particular, Enzo expressed that he planned to use an applet to support students in reasoning about how the values of angle measure vary with either the terminal point's vertical distance above the horizontal diameter or the terminal point's horizontal distance to the right of the vertical diameter. This supports that Enzo conceived of sine as a function that relates two covarying quantities' values and that he wanted to support students in developing this understanding as well.

Enzo also expressed that he programmed specific points on the applet so that students could estimate a point's vertical and horizontal distance on the terminal ray of an angle using the circle's radius. Although Enzo planned to ask students to estimate various point's vertical and horizontal distances, he said he did not have a rigid plan for the class because he "really want[ed] to be disciplined about having discussions where the quantities emerge and really responding to student thinking" (Excerpt 28, lines 24-27). Enzo's programming of specific points on the applet and plan to prompt students to estimate the point's vertical distance above the horizontal diameter using the circle's radius supports that he (1) conceived of the value of the sine function as the result of a multiplicative comparison and (2) aimed to support students' development of this understanding. Similarly, Enzo's contention that he did not have a rigid plan for class because he "really want[ed] to be disciplined about having discussions where the quantities emerge and really responding to student thinking" supports that he was interested in students' thinking and anticipated using his observations of students' thinking while teaching.

It is also noteworthy that Enzo mentioned that he wanted students to understand that for any angle measure, "there exists one of these distances, and in particular, there exists only one" because otherwise his "discussion about inverse trigonometry, [would] be like silly or like limited if they don't have that, they don't have the functional relationship or an understanding of the functional relationship" (Excerpt 28, lines 17-21). This supports that Enzo's teaching preparation was informed by his understanding of the material he would cover later in the chapter. In particular, Enzo expressed a desire to support students' understanding of sine as the name of a function that relates two quantities' values so that his later teaching of inverse trigonometric functions would not be "limited."

Shira's goals for student learning are in Excerpt 29. When discussing her goals for students' learning, Shira said that she wanted students to "see that sine of theta is the ratio of the *y*-value to the radius. But then also like if the radius is one, then the *y*-value is equal to the sine value" (Excerpt 29, lines 1-3). This further supports that Shira conceived of the value of the sine function as (1) a y-value on a coordinate plane and (2) the result of a numerical operation.

Excerpt 29

Shira: I would hope that they can see that sine of theta is the ratio of the *y*-value to the radius. But then also like if the radius is one, then the *y*-value is equal to the sine value. Aside from that, I would hope they could see the same with the *x*-value for cosine.

Furthermore, when Shira discussed her plan for teaching two lessons on sine and cosine, she said she planned to select problems from the workbook investigations covering these ideas. When I prompted Shira to explain how she chose the problems to discuss in class, she stated that she typically looks at the instructor’s solutions provided by the curriculum and decides on problems based on “the answers that make sense” to her. When discussing her plan for the two lessons on sine and cosine functions, Shira also said she was extremely comfortable with the material covered in the first lesson (module 7 investigation 3) because it “just covered the definitions.” However, Shira conveyed that she was less comfortable with the second investigation that included material on these ideas (Excerpt 30, lines 1-2). In particular, Shira said that she wasn’t “exactly sure if [she was] understanding the point of module 7.4 or what it’s trying to get at” (Excerpt 30, line 1-2). As such, Shira said she planned to “[choose] the problems that kind of embed a lot of information” with the hope that students would “captivate what module 7.4 is trying to captivate.” This supports that Shira’s conception of sine function may have constrained her understanding of the curriculum’s goals and intentions. It is also noteworthy that Shira did not attempt to use her reasoning to complete the problems. Instead, it appears that she relied on her prior approach to justify the answer provided in the instructor notes of the student workbook.

Excerpt 30

Shira: If I’m being completely honest, I’m not exactly sure if I’m understanding the point of module 7.4 or what it’s trying to get at. From what I can tell it

> is trying to get students more comfortable with using sine and cosine and I guess kind of relating that to the graphs of sine and cosine. But again, I’m not completely sure… I don’t feel like I could just tell you 7.4 is trying to teach students this. I don’t know what that would be. So in, in choosing problems for this, I’m just kind of choosing the problems that kind of embed a lot of information and then hopefully students just… I don’t know… captivate what module 7.4 is trying to captivate.

At this point, providing the reader with some background on the material and ideas included in investigation 7.4 in the Pathways Precalculus curriculum seems imperative. Module 7 Investigation 4 is titled *Using Sine and Cosine Functions to Track Circular Motion*. The instructor manual for this investigation includes the following introduction:

> This investigation continues our exploration of the sine and cosine functions grounded in circular motion. The students are asked to evaluate the sine and cosine functions for various input values, while maintaining a focus on the quantities related by these functions and the units of measure that are used for their input and output values. At all times students should explain their solutions in terms of the quantities and the units used to measure the quantities. Also, as students mention using sine and cosine functions, prompt them to explain the output quantities and unit *before* they determine the output value by evaluating the function (Carlson, Ohertman, Moore, & O’Bryan, 2020).

Figures 48 and 49 show two tasks from investigation 7.4 of the Pathways Pre-Calculus curriculum. In the first task shown in Figure 48, students are prompted to explain what the values of a coordinate point on a circular ski trail represent. This task also prompts students to illustrate both measures on a diagram. In the second task shown in Figure 49, students are prompted to determine the location of a skier when they have skied a certain distance around a circular ski trail. Both of these tasks included in the curriculum prompt students to reason about the measures of quantities’ values. As such, it is not surprising that Shira was confused about the goals and purpose of investigation 7.4, given that (1) she did not conceive of the values of these functions as the result of a relative size comparison of two quantities' values and (2) the investigation aimed to support students in making relative size comparisons between two quantities' values.

An arctic village skiing trail has a 1-kilometer radius. A skier started at the position $(1, 0)$ on the coordinate axes and skied counter-clockwise. After skiing counter-clockwise, the skier paused for a rest at the point $(0.5, 0.87)$.

a. Describe what the value 0.5 represents and illustrate this measurement (value) on the axes from Exercise #1.

b. Describe what the value 0.87 represents and illustrate this measurement (value) on the axes from Exercise #1.

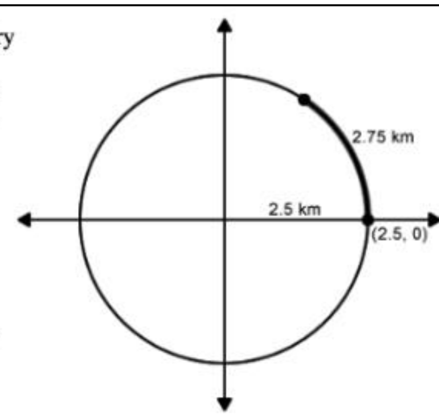
Figure 48. A Task Included in Investigation 7.4 from the Pathways Pre-calculus Curriculum

*4. A second arctic village maintains a circular cross-country ski trail that has a 2.5-kilometer radius.

a. A skier started skiing from the position $(2.5, 0)$ and skied counter-clockwise for 2.75 kilometers before stopping for a rest.

i. Explain what $\frac{2.75}{2.5}$ represents in this context.

ii. Determine the ordered pair (measured in radius lengths) on the coordinate axes that identifies the location where the skier rested.



iii. Determine the ordered pair (measured in kilometers) on the coordinate axes that identifies the location where the skier rested.

b. The skier then continued along the trail, traveling counter-clockwise until he had traveled for a *total* of 10 kilometers before stopping for another rest.

i. Determine the ordered pair (measured in radius lengths) on the coordinate axes that identifies this location where the skier rested.

ii. Determine the ordered pair (measured in kilometers) on the coordinate axes that identifies this location where the skier rested.

Figure 49. A Second Task Included in Investigation 7.4 from the Pathways Pre-calculus Curriculum

It is noteworthy that Shira and Enzo's goals for student learning were highly aligned with their meanings for these ideas. In the TBCI, Enzo expressed a meaning for sine and cosine functions grounded in quantitative and covariational reasoning. When discussing his goals for student learning, Enzo said that he had designed an applet to support students' engagement in both ways of thinking. Similarly, during the TBCI, Shira expressed a meaning for the value of the sine function as (1) a y-value and (2) the result of a numerical operation. When discussing her goals for student learning, Shira said she wanted students to understand that $\sin(\theta) = \frac{y}{r}$ and $\cos(\theta) = \frac{x}{r}$ when the radius isn't one. Shira further said that she wants students to understanding that $\sin(\theta) = y$ and $\cos(\theta) = x$ when the radius is one. As such, it appears that Enzo and Shira's mathematical meanings these ideas greatly informed the meanings they wanted students to construct.

It is also noteworthy that Enzo designed an applet to support students in conceiving of sine as a function that relates two covarying quantities values. Enzo's development of an applet to support students' learning of the sine function suggests that he not only had an image of the understandings he wanted students to construct for this idea, but he also had an image of an activity and conversations about that activity that would support students in constructing a coherent meaning for sine function. This suggests that he conceived of the applet as a didactic object and had constructed a model for engaging students in reflective discourse while using the applet (didactic model) (Thompson, 2002).

In contrast, Shira's conception of the value of the sine function as both a y-value and the result of a numerical comparison constrained her ability to understand the goals of the curriculum she was using and her selection of tasks to use in her teaching. In particular, Shira's plan to "[choose] the problems that kind of embed a lot of information" with the hope that students would "captivate what module 7.4 is trying to captivate" together with her contention that she selects tasks to use in class based on "the answers that make sense" suggests that she planned her lessons with little regard for the reasoning that was explained in the curriculum materials that could be used to justify the answer.

Enzo's Teaching of Sine Function

This section presents data from Enzo's first lesson on sine and cosine functions. During this lesson, Enzo used an applet that he designed to support students in understanding the sine and cosine functions as covarying relationships of an angle measure and length. Figure 50 shows the different viewing windows of the applet Enzo designed for this lesson. The applet could be played like an animation or manipulated using the various toggles manually. Enzo began his lesson on sine and cosine functions by projecting the applet, as shown in the top left view of Figure 50. He then animated the applet so that the bug rotated around the circle. As the bug rotated, the applet highlighted the arc subtended by the angle's rays in green. While showing this applet view (see top left Figure 50), Enzo prompted students to use the applet to identify as many fixed and varying quantities as possible. Following this, Enzo clicked the "Show$_{HD}$" button on the applet to show the horizontal diameter of the circle (see the top right view of Figure 50). He then asked students, "How can we use this line to figure out where the bug is?" After a few minutes, Enzo stopped animating the bug's position and rotated it above the horizontal diameter. Next, he asked students, "Is the bug above or below the horizontal diameter?" Enzo repeated this question after he rotated the bug below the horizontal diameter. Following this, he turned on the "$Show_1$" button which placed a blue segment from the bug to the horizontal diameter (see the left middle view of Figure 50). After doing this, Enzo animated the applet so that the bug rotated around the circle. As the bug rotated, the length of the blue line from the horizontal diameter to the bug varied accordingly. During this time, Enzo asked students, "Is this distance [points to the blue segment] enough to figure out where the bug is?" He followed this by asking students to "come up with another quantity that we can track to determine the position of the bug."

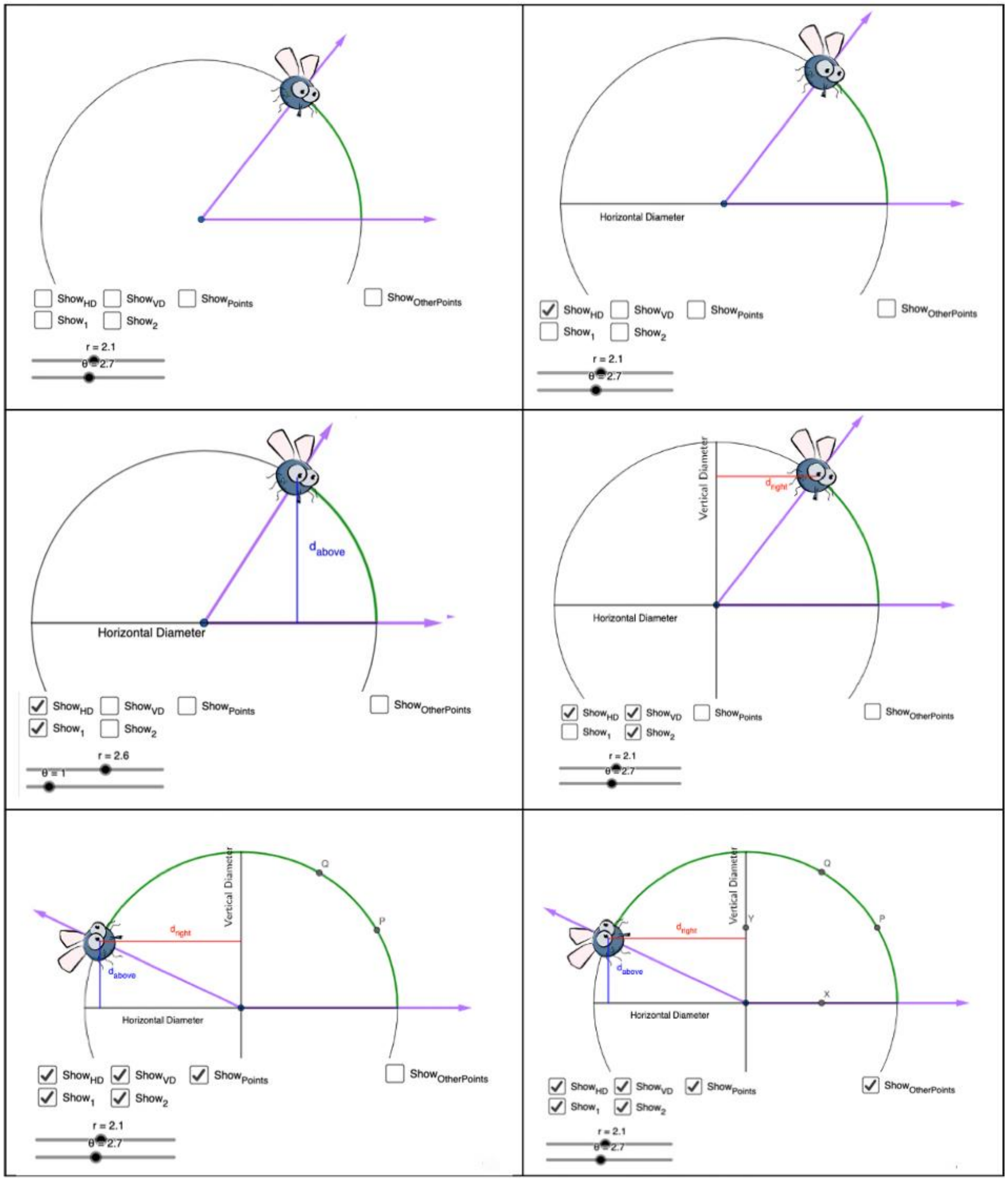


Figure 50. The Applet Enzo Designed for His First Lesson on Sine and Cosine Functions

Enzo repeated a similar process that introduced students to tracking the bug's horizontal distance to the right of the vertical diameter and vertical distance above the horizontal diameter, as a means for tracking the bug's location. Once students agreed that they could track the bug's position by determining these distances, Enzo moved the bug to point P shown in the bottom two views of the applet in Figure 50 and in Figure 51. Following this, he told students, “I am convinced that something interesting happens here at point P. What is interesting about point P?” Excerpt 31 shows the class discussion that ensued after Enzo allowed students to discuss this question for a few minutes.

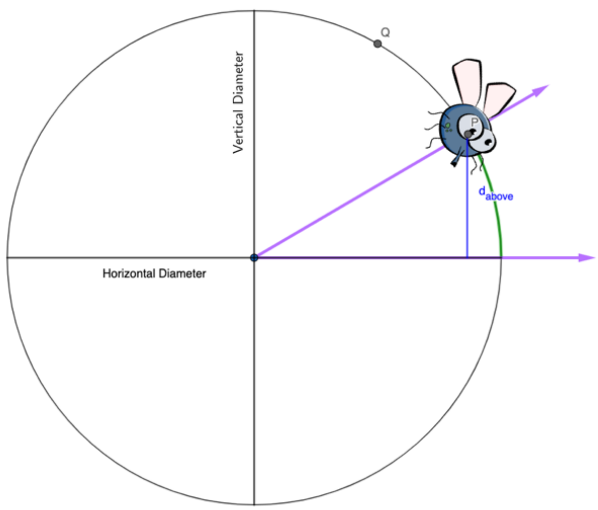


Figure 51. Enzo Moves the Bug to Point P

Excerpt 31

Enzo: Okay, folks, let’s do this real quick. Let’s compare how many times as large this blue length is to the radius length. Anybody have a result? How many times as large?

Students: Half

Enzo: Half? I heard half from some people, yeah? Okay, so what would we say the bug’s distance above is? We would say one half, right. But one half what? That’s what I want to clarify before we leave. What is the unit? And I want to remind you of what we just did. We took this length [points to the bug’s vertical distance above the horizontal diameter (blue segment)] and compared it to this length [points to the vertical radius length]. This one is the radius length. So, we used the radius length as a measuring stick, as a unit of measure, to measure this guy [points to the bug’s vertical distance above the horizontal diameter (blue segment)], this distance above. Okay, so we’re gonna write down “the distance above equals 0.5”. You answer 0.5 what.

After allowing students to discuss his question for a few minutes, Enzo told students to determine how many times as large the length of the blue segment in Figure 51 was to the length of the circle’s radius (see Excerpt 31, lines 1-3). Multiple students in the class agreed that the blue vertical segment was one-half times as large as the circle's radius (see Excerpt 31, line 4). Following this, Enzo prompted students to think about a unit for their measure of 0.5. At this time, Enzo reminded students that they determined the measure of 0.5 by comparing the length of the blue segment to the circle's radius length. Enzo also expressed that they could think about the radius length as their "measuring stick" (see Excerpt 31, line 11-12).

Enzo's discussion in Excerpt 31 exemplifies a teacher supporting students' quantitative reasoning. In particular, Enzo supported students in conceiving (1) an attribute of an object to measure (the length of the blue segment), (2) a unit of measure (the circle's radius), and (3) a measurement process (multiplicatively comparing the length of the blue segment to the radius length). Enzo also supported students in conceiving of division as a measurement in this example when he expressed that he was using the radius length as a "measuring stick." Enzo then allowed students to discuss the unit for 0.5 in their groups for approximately three minutes. After this, he brought the class back together and prompted students to explain what they discussed in their groups. Excerpt 32 includes the class discussion that ensued.

Excerpt 32:

Enzo: Okay, so, let's talk about this. Thoughts? 0.5 what?
Students: *class response inaudible*
Enzo: Okay, so I heard three things, all from the same group. I heard 0.5 r, 0.5 radius, and I heard 0.5 feet. All of those are reasonable. Let's consider the feet. Well, we're measuring something linear, so why not? But… what we actually did, is we compared this length [points to the bug's vertical distance above the horizontal diameter (blue segment)], the distance above, to the radius length [points to vertical radius length]. And we said that the distance above was how many times as large as the radius length? We said 0.5. Okay, so, we can kinda consolidate these two [circles 0.5r and 0.5 radius on the board] and say that the unit of measure is radius lengths. Okay, so folks, [moves the

bug to the 12 o'clock position of the circle (so that $\theta = \frac{\pi}{2}$)] how many radius lengths above the horizontal diameter is the bug here?
Class: One radius length
Enzo: One, okay, because the bug's distance above is one times as large as the radius length. Okay, so that's what we mean by radius length. Now, let's talk about feet. What if I told you that this radius length was 2.6 feet. Can we report this distance above [moves the bug back to point P] in a new unit now? Can we report it in feet? What would it be?
Student 2: 1.3 feet.
Enzo: Okay I don't know how you got that number. Walk me through it.
Student 2: It's gonna be half the size of a radius so half of 2.6.
Enzo: Okay, so I know this distance above is 0.5 radius lengths so I am going to use your way of thinking right here. You said it's going to be half the size of a radius length. So, I want a distance, I want a length that is one half times as large as 2.6 because each radius length is 2.6 feet [writes 0.5(2.6 ft) on the board] (see Figure 52). Oh, that's how you got 1.3! Oh okay, gotcha.

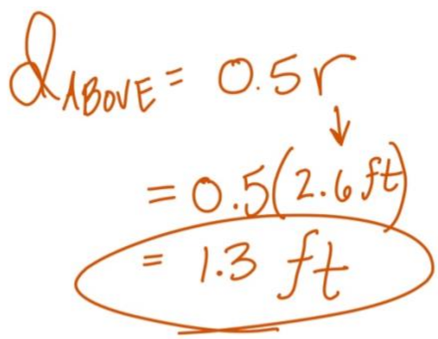


Figure 52. Enzo's Board Work During Discussion in Excerpt 32

After Enzo prompted students to explain what they discussed in their groups, he told them that he heard three responses: 0.5r, 0.5 radius, and 0.5 feet (see Excerpt 32, lines 3-4). Enzo then told students that their answer of 0.5 feet was reasonable since they were measuring something linear. He then followed this by reminding students for a second time that they determined the measure of 0.5 by comparing the bug's vertical distance above the horizontal diameter to the circle's radius (see Excerpt 32, lines 7-8). To support students in recognizing that the unit was *radius lengths* or *radii*, Enzo moved the bug to the top of the circle (12 o'clock position) and asked students how many radius lengths above the horizontal diameter the bug was at this position. Students responded that the bug was one radius length above the horizontal diameter. Enzo then concluded that the bug was one radius length above the horizontal diameter at the 12 o'clock position because "the bug's distance above is one times as large as the radius length" (see Excerpt 32, lines 16-17).

Following this, Enzo prompted students also to give the measure of the bug's vertical distance above the horizontal diameter in feet. A student in the front of the class immediately responded, "1.3," to which Enzo responded, "I don't know how you got that number. Walk me through it" (Excerpt 32, line 23). The student replied, "It's gonna be half the size of a radius, so half of 2.6." Following this, Enzo repeated what the student said and added that he "wanted a length that is one-half times as large as 2.6 because each radius length is 2.6 feet" (Excerpt 32, lines 27-28).

Enzo's explanations in Excerpt 32 provided students with the opportunity to construct a meaning for the output of sine function as a relative size comparison. Namely that the value of sin (θ) gives a measure that describes how many times as large the vertical distance of the angle's terminal point is above the horizontal diameter, as compared to the circle's radius. Enzo's explanations in Excerpt 32 also provided students with an opportunity to construct the understanding that they could determine the measure of the bug's vertical distance above the horizontal diameter in feet by multiplying the measure of the bug's vertical distance in radius lengths by the measure of the circle's radius in feet. His question scaffolding also provided insights about Enzo's MMT for the sine function. In particular, Enzo appeared to value having his students (1) identify quantities to measure, (2) describe a measurement process, and (3) make explicit a unit of measure. His interactions with students using the applet further reveals that he valued having his students conceptualize the quantitative relationships representing the output of the sine function.

Directly following Enzo's lesson, I interviewed him and asked him about his decision to ask students to determine the bug's height above the horizontal diameter when the bug was at point P. Enzo said that he chose to move the bug to point P because "[he thinks] this one was really easy to confidently estimate." Enzo further stated that he prompted students to determine the bug's height above the horizontal diameter at point P because "it forces us to be direct and explicit about as long as what, half as long as what." Enzo further said he chose this task because "you can't make estimates if you don't know what the quantity is… and at this point, we haven't really written [the definition of sine function] down, but I want them to be able to think about and make those estimates. So [this task is] just reinforcing the definition [of sine function], even as it emerges in the class."

In this example, Enzo prompted students to estimate the vertical distance of a point above the horizontal diameter as a relative size measurement. He described his reason for posing this question by saying, "you can't make estimates if you don't know what the quantity is." As such, it appears that Enzo was aware of the difficulty students might confront in conceptualizing the quantities to represent the output of the sine function, providing further insight into his MMT for the sine function. Moreover, Enzo's decision to prompt students to estimate the point's vertical distance using the circle's radius supports that he valued having his students conceptualize the output of the sine function as the ratio of the terminal point's vertical distance above the horizontal diameter of a circle and the circle's radius, and recognized that conceptualizing this quantity might be fostered by engaging them in visualizing these quantitative relationships and how they vary together using an applet he designed support students in thinking about measuring a vertical distance using the radius of a circle as a unit of measure.

Shira's Teaching of Sine Function

In this section, I present data from the start of Shira's second lesson on sine and cosine functions. Shira began this class by reviewing a problem she did not have time to finish during the previous class session (see Figure 53). While discussing this task, Shira called on students to provide answers to each part of the question in Figure 53.

Shira began her discussion of the task by telling students that the angle measure, θ, was 2 radians because it subtended an arc that was "2 radius lengths." Following this, Shira asked students how they could approximate the value of $3\sin(\theta)$. After a student said they didn't know, Shira asked, "So, on Monday, $\sin(\theta)$ was kinda attributed to what variable?" to which a student responded, "the *y*-value". Shira then responded, "the *y*-value of what?" Once Shira and the student agreed that they were talking about the y-value of a point on the terminal ray of the angle whose vertex was at the center of the circle, Shira prompted students for a second time to give a solution to part (b). At this time a student raised their hand and said, "wouldn't you just set it equal to 2.7?" To which Shira agreed. This student then said that they could determine the value of sin(θ) by dividing 2.7 by 3. Following this, Shira concluded her discussion of this task by stating, “what I want you to know is that $\sin(\theta)$ is the ratio of the *y*-value versus the radius” (Excerpt 33, lines 25-27).

In this example, it is noteworthy that Shira asked students, “on Monday, $\sin(\theta)$ was kinda attributed to what variable?” This further supports that she conceived of the value of the sine function as a y-value on a coordinate plane. It is also noteworthy that Shira did not prompt students to explain what $\sin(\theta) = 0.9$ represents. This further supports that she also conceived of the value of $\sin\,(\theta)$ as the results of a numerical operation.

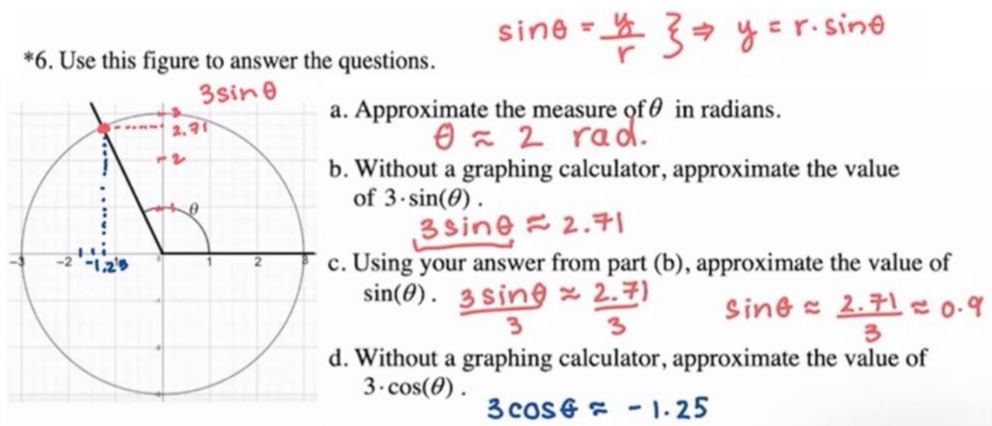


Figure 53. A Task Shira Used During Her Lesson on Sine Function

Excerpt 33

Shira: Student 1, would you be able to tell me without a graphing calculator, how it is that we could approximate the value of $3\sin\,(\theta)$?
Student 1: Umm… I do not know.
Shira: Okay, that’s totally okay. So, on Monday, $\sin\,(\theta)$ was kinda attributed to what variable?
Student 2: The y-value.
Shira: The y-value. Okay, the y-value of what?
Student 2: The coordinate of like the point that the terminal ray intersects the unit circle at.
Shira: Okay, perfect, I love that. So, the point at which the terminal ray intersects the… you said the unit circle. What is a unit circle?
Student 2: Umm… Well, that’s just kinda the one we were doing with the one radius.

Shira: Okay, so I think what you mean is using a circle with a radius of one. Okay, so, in this case right, I have something other than one, right. So, Student 3 tell me what do we think the value of 3 sin (θ) would be?

Student 3: I'm not sure.

Shira: You're not sure, okay, is there anyone who could figure out the value of 3 sin (θ) from this graph?

Student 4: I have a question, wouldn't you just set it equal to 2.7?

Shira Uhh huh…

Student 4: Like $3\sin(\theta) = 2.7$ then divide by 3 to get sin (θ).

Shira: Yes exactly. So, 2.71 is just an estimate of where this y-value hit, right. So, the value of 3 sin (θ) is about 2.71. So then, if we just wanted to find sin (θ) then going back to what Student 4 said, we could take 3 sin (θ) and just divide by 3. So sin (θ) is about 0.9. So that being said, what I want you to know is that sin (θ) is the ratio of the y-value versus the radius. So the reason why the 3 sin (θ) was equal to the point is because if we look at this [points to $\sin(\theta) = \frac{y}{r}$] if I multiply both sides by r, this would then provide me that y is r times sine of theta [writes $\sin(\theta) = \frac{y}{r} \Rightarrow y = rsin(\theta)$].

The segment of Shira's teaching shown in Excerpt 33 also provides an example of the ways in which a teacher's explanations may be constrained by their meanings for an idea. In Excerpt 33, Shira's explanations were constrained by her meanings for the value of the sine function as a *y*-value or the ratio of "*y* divided by *r*". Shira's teaching in Excerpt 33 also provides an example of a teacher expressing conflicting meanings to students. In this example, Shira told students that sin (θ) is attributed to a y-value. Shira also conveyed that the value of the sine function can be determined by dividing a y-value by the length of the radius. As such, it seems reasonable that a student from this class would construct two different meanings for the sine function. Namely, sin (θ) is a y-value and sin (θ) is the result of dividing the *y*-value by the radius.

Following Shira's class discussion of the task in Figure 53, a student asked, "so when is the coordinate $(\cos(\theta), \sin(\theta))$ instead of $\left(r\cos(\theta), r\, sin(\theta)\right)$?" (see Excerpt 34, lines 2 and 3). Shira responded, "you would use the coordinate $(\cos(\theta), \sin(\theta))$

when the radius is equal to one (see Excerpt 34, lines 4 and 5). Shira further explained that the coordinates would be $(\cos(\theta), \sin(\theta))$ when $r = 1$ because “$\sin(\theta) = \frac{y}{r}$ and $\cos(\theta) = \frac{x}{r}$, and so *y* divided by one and *x* divided by 1 are just *y* and *x*” (Excerpt 34 lines 11-13).

Excerpt 34

Shira: Any questions before we move on?
Student 2: I just have a question, so when is the coordinate $(\cos(\theta), \sin(\theta))$
instead of $\left(r\cos(\theta), r\, sin(\theta)\right)$?
Shira: So, umm… you would use the coordinate $(\cos(\theta), \sin(\theta))$ when the
radius is equal to one. And then, or like if… it would have to be like
one unit of radius length. And that kinda draws back to the definition
of sine of theta and cosine of theta. So, I said that sine of theta, right,
was y over r [writes $\sin(\theta) = \frac{y}{r}$] and cosine of theta is *x* over *r*
[writes $\cos(\theta) = \frac{x}{r}$]. So, the coordinate points would be sine of theta
cosine of theta if *r* is one, because then at that point, *y* is equal to sine
of theta [writes $y = \sin(\theta)$] and *x* is equal to cosine of theta [writes
$x = \cos(\theta)$]. Right, because this was equal to one [writes $r = 1$ in
$\sin(\theta) = \frac{y}{r}$ and $\cos(\theta) = \frac{x}{r}$], and so *y* divided by one and *x* divided
by 1 are just *y* and *x*.

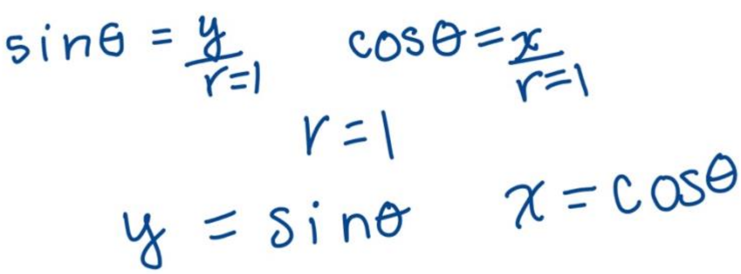


Figure 54. Shira’s Board Work as She Responded to the Student’s Question Shown in Excerpt 34

It is noteworthy that Shira did not mention units when responding to the student in Excerpt 34. Shira’s calculational orientation and inattention to units further support her thinking about the values of sine and cosine functions as the result of performing numerical operations instead of the relative size of two quantities’ values as they vary together. Shira's interaction with the student in Excerpt 34 also provides an example of how her explanations were constrained by her conception of the value of the sine function as a y-value or "*y* divided by *r*."

Following Shira's lesson on sine and cosine functions, I interviewed her about her teaching. During this interview, I asked Shira about her interactions with students (see Excerpt 35). In particular, I noticed Shira was inconsistent in the degree to which she probed student thinking while teaching. For example, when prompted to explain how she decided whether or not to prompt student thinking further, Shira said that she asks students more questions when "looking for a specific answer." In particular, she said, "if there's something that they haven't quite said, then that's when I will ask them more questions to help and kind of direct them into…whatever I'm thinking." Shira also expressed that she struggles to ask students questions when teaching and doesn't typically ask more questions if a student "said exactly what [she] was thinking."

Excerpt 35: Shira talks about her interactions with students.

Int: So, I've noticed that when you teach, after asking students a question, sometimes you will follow up with more questions and other times you will just say "okay" and walk to the next group. So how do you decided when to ask more questions and when to just say okay and move to another group?

Shira: So, I will tend to ask more questions if I'm looking. I guess you could say if I'm looking for a specific answer. And so, if they [the students] haven't, there's something that they haven't quite said, then that's when I will ask them more questions to help and kind of direct them into that, that I guess that whatever I'm thinking. But then if they say like exactly what I was thinking or something along the lines of what I was thinking, I don't usually pose more questions. But then because they said exactly what I was thinking, I just can't think of something else to ask them. There might be times when I can think of something that would actually like extend their understanding, and then I would try to be able to ask a question. But if it's just like spot on and like I can't, like in the moment, I just can't ask another question.

Shira's description of how she decides whether to probe student thinking provides an empirical example of a teacher interacting with students unreflectively. A teacher interacts with a student unreflectively if they are constrained to using their meanings of an idea (first-order model) when interacting with students (Teuscher et al., 2016; Bas Ader & Carlson, 2021). In this example, Shira's meanings not only created the space for the meanings her students could construct, but they also constrained the nature of her interactions with students.

CONCLUSION AND DISCUSSION

This paper presents two precalculus instructors' MMT for the sine function, their goals for students learning of the sine function, a description of their teaching, and their rationale for their instructional choices. The findings support that an instructor's mathematical meanings for teaching an idea influence the nature of the instructor's goals for students' learning, their explanations while teaching, their image of ways to support students' learning of an idea, and ultimately the meanings that students have the opportunity to construct. Although both instructors taught using the same research-based precalculus curriculum, they expressed different meanings for the sine function. In particular, Enzo repeatedly expressed meanings for the sine function that were grounded in quantitative and covariational reasoning. Namely, Enzo described sine as the name of a function that relates the measure of an angle (measured from the 3 o'clock position of a circle) and a terminal point's vertical distance above the horizontal diameter of the circle in radius lengths. In contrast, Shira was inconsistent in describing the value of the sine function as either a "y-value" or as the ratio of "*y* over *r*."

The meanings Shira and Enzo expressed for the sine function largely influenced the meanings they expressed while teaching and, as a result, the meanings that students had the opportunity to construct. Table 9 provides examples of each instructor's meanings for the sine function, their expressed meanings while teaching, and a description of the meanings students likely constructed as a result of the meanings that each instructor conveyed to students. It is noteworthy that Enzo consistently supported students in conceptualizing and relating quantities and, as a result, conveyed meanings that supported students in reasoning about quantities, how they covary, and measuring their values. As such, a student in Enzo's class likely conceived of sine as a function that relates two covarying quantities' values. In addition, students in Enzo's class likely conceived of the output of the sine function as the result of a multiplicative comparison between the vertical distance a point was above the horizontal diameter of a circle and the circle's radius. In contrast, Shira expressed meanings for the sine function that were inconsistent from a student's perspective. For example, Shira was inconsistent in describing the value of sine as a y-value or the ratio of a y-value and the radius of a circle. As such, students in Shira's class likely constructed meanings for sine that were grounded in a list of rules. Namely, when the radius is one, sine is a y-value; otherwise, sine is the ratio of "y over r".

It is also noteworthy that each instructor's meanings likely supported very different types of thinking in students. For example, Enzo's instruction supported students in reasoning about quantities and their measures. In contrast, Shira's instruction focused students' attention on applying rules and performing numerical operations. Thus, it is likely that students in Shira's class conceived of the three letters S-I-N as a directive to perform an operation. In contrast, students in Enzo's class likely conceived of the three letters S-I-N as the name of a function that relates two covarying quantities' values.

The results presented in this paper reveal that both instructors' meanings for sine function were germane to (1) their goals for student learning, (2) their selection/creation of tasks to use while teaching, (3) and their explanations and interactions with students when teaching. For example, Enzo expressed a meaning for sine as a function that relates two quantities' values that covary. Enzo also conceived of the output of the sine function as a relative size comparison of a terminal point's vertical distance above the horizontal diameter of a circle and the circle's radius. In alignment with his meanings, Enzo said that he wanted to support students in recognizing that sine was the name of a function that related two quantities' values (Excerpt 28). Enzo also expressed that he wanted to support students in conceiving of the output of the sine function as a relative size measurement (Excerpt 28). As such, he designed an applet to support students in reasoning about how a terminal point's vertical distance above the horizontal diameter covaries with the measure of an angle swept out from the 3 o'clock position of a circle. Enzo's design of the applet and his scaffolding of questions to advance students' understanding support that he (1) understood the goals of the curriculum and (2) had an image of how to support students' learning of these ideas.

Subject	Meanings Expressed During TBCI	Meanings Expressed to Students	Meanings Students May Have Constructed
Enzo	"I want them to think about this quantity over here [underlines $\sin(\theta)$] as the height above the horizontal diameter of the point, which is at the intersection of the terminal ray of the circle centered at the vertex of the angle." [Excerpt 2]	"Is this distance [points to the bug's vertical distance above the horizontal diameter] enough to figure out where the bug is? Come up with another quantity that we can track to determine the position of the bug." [page 153]	We can determine the location of points on a circle by tracking quantities' values.
	"Okay, so what I am thinking here is, so, I've got these two segments… and I am trying to see how long it is compared to the radius length. I am trying to think about relative size." [Excerpt 7]	"Let's compare how many times as large this blue length [points to bug's vertical distance above the horizontal diameter]is to the radius length." [Excerpt 12]	We can multiplicatively compare the vertical distance a point is above the horizontal diameter of a circle and the circle's radius.
	"And so, if this [points to $\sin(\theta)$] outputs 0.5 and thinking about it as a length that is 0.5 times as large as a radius. I can go back and forth between thinking of it as like 50% of a radius length." [Excerpt 4]	"So, we used the radius length as a measuring stick, as a unit of measure, to measure this guy, this distance above." [Excerpt 12, lines 11-12]	The radius of a circle can be used as a unit of measure for measuring the vertical distance of a point above the horizontal diameter.
Shira	"Sine of theta equals *y* or that it's the ratio of *y* with respect to the radius." [Excerpt 1]	"So, on Monday, $\sin(\theta)$ was kinda attributed to what variable?" [Excerpt 14]	$\sin(\theta)$ means the same thing as *y*
		"What I want you to know is that $\sin(\theta)$ is the ratio of the y-value versus the radius." [Excerpt 14]	$\sin(\theta) = \frac{y}{r}$
		"You would use the coordinate $(\cos(\theta), \sin(\theta))$ when the radius is one." [Excerpt 15]	The coordinate point changes depending on the length of the radius.

Table 9. A Comparison of Enzo and Shira's Meanings, Explanations, and the Meanings Students Might Have Constructed as a Result of Their Instruction

Shira's meanings for sine function also greatly informed her instructional decisions, explanations, and interactions with students. Prior to her teaching Shira expressed goals for student learning that aligned with her meanings for sine function. In particular, Shira said she wanted students to understand that $\sin(\theta) = \frac{y}{r}$ and $\cos(\theta) = \frac{x}{r}$ (Excerpt 29). Before teaching, Shira also selected a task that aligned with this goal. Unlike Enzo, Shira struggled to understand the goals of the curriculum. I hypothesize that Shira's conception of the values of sine and cosine functions as the result of a division of two values as opposed to the relative size of two quantities' values impacted her ability to understand the intention of an investigation included in the curriculum. As one example, Shira did not recognize the affordances of an investigation focused on circular motion because she did not conceive of sine and cosine functions as a relationship between two covarying quantities' values (see pages 150-151).

As such, this paper provides empirical support for Thompson's (2013) claim that a teacher's mathematical meanings for an idea constitute their image of the mathematics they teach, their pedagogical decisions, and the language they use to cultivate similar images in students' thinking. This paper also corroborates Tallman's (2015) claim that a teacher's teaching actions are strongly related to their mathematical meanings. Additionally, this paper provides empirical evidence of the affordances of a teacher's engagement in quantitative reasoning. Enzo regularly expressed meanings for the sine function grounded in quantitative reasoning during the TBCIs and in his teaching. Enzo's commitment to quantitative reasoning as a critical way of thinking supported him in (1) stating goals for student learning in terms of the thinking he wanted students to engage in, (2) designing an applet to support students' quantitative reasoning and learning of sine and cosine functions, and (3) posing questions to students that support their quantitative reasoning.

Lastly, this paper provides empirical evidence of the impact sustained professional development focused on providing teachers with repeated opportunities to construct coherent meanings for the ideas they are teaching can have on an instructor's thinking and practice. Although Enzo and Shira were both in their first year of teaching this course at the University level, Enzo had been teaching with the Pathways Precalculus and Algebra II Curriculum for approximately three years prior to this study. Enzo had also participated in approximately fifty hours of professional development led by the Project Pathways Professional Development team prior to this study. In contrast, at the time of the study, Shira had been teaching with the Pathways curriculum for a little over eight months. At this time, Shira had only participated in approximately fifteen hours of weekly professional development led by the Project Pathways Professional Development team that focused on preparing GSIs to teach the ideas in the investigations for the upcoming week. It is also noteworthy that Shira did not attend a 2-3 day pre-course workshop that engaged future Pathways instructors in completing and explaining the thinking they used to complete select tasks in the Pathways workbook investigations.

LIMITATIONS

The reader must bear in mind the unique experiences of both Enzo and Shira. These instructors were teaching using a research-based precalculus curriculum. Concurrent with this study, both instructors were also enrolled in a professional development seminar that focused on supporting instructors' (1) development of coherent meanings and ways of thinking about the ideas to be taught and (2) in explaining their meanings for these ideas to others. Because of the unique experiences of these instructors, the reader should not generalize these findings to a general population of secondary or post-secondary instructors. Instead, the purpose of this paper is to theorize about the ways in which an instructor's mathematical meanings for teaching an idea inform their practice.

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CHAPTER 6

CONCLUSIONS

Research on teachers' knowledge base has primarily focused on identifying what teachers need to "know" to teach mathematics and the relationships between what teachers "know" and student performance (Thompson, 2013; 2016; Tallman, 2015, 2021). Too often, this research examines teachers' ability to answer questions correctly instead of clarifying the meanings teachers have that enable them to perform as effective teachers (Thompson, 2013). This dissertation contributes to research focused on teachers' mathematical meanings by providing nuanced characterizations of two instructors' mathematical meanings for teaching angle measure and the sine function and illustrating the impact of the instructors' meanings on their instructional practices and interactions with students. My epistemological stance that an individual's meanings are constructed and idiosyncratic and my focus on mathematical meaning and coherence enabled me to provide detailed characterizations of teachers' mathematical meanings for teaching angle measure and the sine function.

In this study, I have chosen to use the construct *mathematical meanings for teaching* (MMT) as opposed to *mathematical knowledge for teaching* (MKT) purposefully to convey that I am using the construct MMT to describe a teacher's cognitive mechanisms relative to teaching. As such, the reader needs to recognize that my use of the construct MMT instead of MKT is not an issue of semantics. Instead, it demonstrates a commitment to modeling the nature of teachers' mathematical schemes, mental operations, and ways of thinking that enable teachers' behaviors while teaching.

This document is organized into three papers detailing a multiple case study that constitutes my dissertation. The first paper presents a model of the first subject's MMT for angle measure and illustrates the impact of a teacher's meanings on her interactions with students. This paper also identifies the implications of a teacher's meanings on students' learning opportunities. The second paper presents a model of the second subject's MMT for angle measure and provides an empirical example of the symbiotic relationship between a teacher's MMT for an idea and their decentering actions. This study's third and final paper builds on the first two papers in content and data. In particular, paper 3 presents a cross-case analysis of the two instructors' MMT for sine function and their enacted teaching practices. The data included in this paper illustrates the impact an instructor's MMT for an idea can have on an instructor's (1) task choice, (2) interactions with students, (3) explanations, and (4), ultimately, the meanings students have the opportunity to construct. Together these three papers illuminate how a teacher's mathematical meanings for teaching an idea impact their pedagogical decisions.

In this next section, I present my research questions and describe how each question is addressed within my dissertation study. The three research questions motivating this study were:

1. What mathematical meanings for teaching angle measure and the sine function do teachers construct when using a research-based curriculum?
2. What is the relationship between a teacher's mathematical meanings for teaching angle measure and the sine function, and the teacher's instructional practices, including their instructional decisions, explanations, and interactions with students when teaching?

3. What is the relationship between a teacher's mathematical meanings for teaching an idea and their decentering actions?

Paper 1 describes Shira's meanings for angle measure and discusses how her MMT for angle measure contributed to student confusion and ultimately limited the meanings students could reasonably construct. As one example, when describing what it means for an angle to measure ten degrees, Shira expressed that an angle measures ten degrees if it subtends ten little arcs that each measured one three-hundred sixtieth of the circumference of a circle centered at the angle's vertex. However, in another instance, Shira described ten degrees as "ten slivers or slices." As such, paper 1 presents data that illustrates how Shira's inattention to the attribute she measured when quantifying an angle's openness led to her giving confusing and inconsistent explanations to students.

Paper 2 describes Enzo's meanings for angle measure and discusses how his MMT for angle measure influenced his decentering actions and how the results of his decentering actions influenced his MMT for angle measure. The results support my claim of a symbiotic relationship between an instructor's MMT and their decentering actions. As one example, Enzo's robust mathematical meanings for angle measure, together with his commitment to quantitative reasoning, provided a lens for evaluating and responding productively to his students' thinking. Moreover, this paper presents data that illustrates how Enzo's MMT supported him in selecting and posing tasks that were responsive to some of his students' thinking and effective in advancing some of his students' thinking. The second paper also presents data that provides a compelling example of how Enzo's decentering actions engendered accommodations to his image of ways to support students learning of angle measure. In particular, Enzo's decentering actions appeared to advance his MMT for angle measure to include more refined images of how students might think about angle measure and more insights into instructional moves that may advance students' thinking in a productive direction.

Lastly, Paper 3 addresses research questions 1 and 2. This paper addresses research question 1 by providing models of Enzo and Shira's MMT for the sine function. The data included in paper 3 also addresses research question 2 by presenting data that illustrates the ways in which the teachers' MMT for sine function influence the nature of their (1) goals for students' learning, (2) explanations while teaching, (3) image of ways to support students learning of an idea, (4) choice of tasks, and (5) their understanding of the goals of a curriculum they are using. This paper also presents data that illustrates and compares how Shira and Enzo's MMT for sine function impacted the meanings that students in their classes had the opportunity to construct.

Overall, this dissertation contributes to the field of mathematics education in four key ways. First, this study describes and compares the two primary frameworks/perspectives of teacher knowledge. Second, this study provides detailed models of two instructors' mathematics and illustrates the implications of these teachers' meanings for their students' learning opportunities. Third, each paper included in this study illuminates the affordances of a teacher's engagement in quantitative reasoning. As one example, Enzo's consistent engagement in quantitative reasoning supported him in engaging in pedagogical actions that provided students with opportunities to (1) reason quantitatively themselves and (2) construct robust mathematical meanings for a foundational trigonometry concept. The data presented in paper 2 also demonstrate that a teacher's commitment to quantitative reasoning as a coherent way of thinking consistently supported them in noticing, reflecting on, and effectively leveraging student thinking while teaching. Lastly, this study illustrates the usefulness of the construct of decentering for characterizing the nature of a teacher's interactions with students while teaching.

Finally, this study has implications for mathematics educators. Most importantly, the results of this study illuminate the need for instructors and pre-service teachers to be engaged in professional development that provides them with repeated opportunities to construct coherent meanings for the ideas they are teaching. The two subjects included in this study had varying levels of experience attending professional development seminars led by the designers of the research-based curriculum they were teaching with. Although the relationship between time spent in professional development and the instructors' meanings for and teaching of angle measure and sine function was not directly investigated in this study, it seems reasonable that Enzo's extensive and sustained professional development experience may have supported him in developing coherent mathematical meanings and ultimately conveying these meanings to students.

This study also raises questions about the affordances of a teacher's use of a research-based, cognitively scaffolded curriculum and its impact on their mathematical meanings for teaching. Recall that Shira and Enzo taught using the Pathways precalculus research-based curriculum (Carlson, Oehrtman, Moore, & O'Bryan, 2020). Although not an explicit focus of this study, the curriculum, including its carefully scaffolded tasks, instructor solution guides, and the general ways of thinking it promoted, might have played a central role in the instructor's instructional decisions. As one example, Enzo regularly enacted the Pathways Conventions described by Carlson, O'Bryan, and Rocha

(2023) that were an explicit focus for the Pathways professional development. These conventions include expectations and actions for supporting students in (1) conceptualizing and speaking about quantities and how their values vary together, (2) representing how two quantities change together using a graph, and (3) representing quantitative relationships with expressions and formulas (Carlson, O'Bryan, & Rocha, 2023). Although not a focus of this study, it is possible that Enzo's use of the Pathways Curriculum and his enactment of the Pathways Conventions for quantitative reasoning influenced his choice of tasks, the nature of his questions, and his approach to engaging his students in quantitative reasoning. As such, it would be useful for future research to investigate how an instructor's use of a research-based, cognitively scaffolded curriculum influences instructors' MMT for teaching specific ideas.

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APPENDIX A

STUDENT CONSENT FORM TO PARTICIPATE IN CLASSROOM RECORDING

Title of research study: Characterizing Teacher's Mathematical Meanings for Teaching, Commitment to Quantitative Reasoning, and Decentering Actions in the Context of Research-based Curriculum

Principle Investigator: *Marilyn P. Carlson Co-Investigator: Abby Rocha*

Why am I being invited to take part in a research study?

We invite you to take part in a research study because you are a student enrolled in precalculus or calculus. Participants of this study must be 18 or older and the choice to participate or not participate will not impact your grades or standing with the university in any way.

Why is this research being done?

A research team at Arizona State University developed a research-based curriculum and instructional materials for learning key ideas of precalculus that are foundational for calculus. We aim to document the instructors' teaching actions during the implementation of these materials, particularly focusing on his or her explanations, questions, and interactions with students.

How long will the research last?

Data collection will involve video-and audiotaping at most 10 classes during the semester.

How many people will be studied?

About 5 instructors and 200 students will be involved in this research.

What happens if I say yes, I want to be in this research?

By participating in this study, you agree to video-and audiotape each class during the semester.

What happens if I say yes, but I change my mind later?

You can withdraw consent to be videotaped at any time without penalty.

What happens if I say no, I do not want to be in this research?

Any students who do not consent to being videotaped will not be videotaped or audio taped. These students will sit in an area of the classroom that will never be in the viewing/audio window of the camera.

Will being in this study help me in any way?

We cannot promise any benefits to you or others from your taking part in this research. However, possible benefits include improved mathematical learning and shifts in your understandings.

Is there any way being in this study could be bad for me?

There is minimal risk associated with your involvement in this study. Participation in this study will not impact your grades or standing with the university in any way.

What happens to the information collected for the research?

Efforts will be made to limit the use and disclosure of your personal information, including research study records, to people who have a need to review this information. We cannot promise complete secrecy.

The data collected will be anonymized and stored separately from individual students' names. All data collected will remain entirely confidential and will be stored so that it cannot be associated with individual names. Videotapes will be stored in a secure location at Arizona State University and digital copies will be stored on a secure server. We might show segments of videotapes during presentations of our research results at professional conferences, and we might use segments of video in professional development of instructors. Transcribed excerpts from class instruction or individual interviews might be included in published reports of the project. Students in them will be depicted anonymously. Similarly, while data about students' class performance or academic background might be reported in research publications, all students will remain anonymous.

Who can I talk to?

If you have questions, concerns, or complaints, talk to the research team by emailing Dr. Marilyn P. Carlson at marilyn.carlson@asu.edu and Abby Rocha at aerocha@asu.edu, or contact the Social Behavioral IRB office at (480) 965-6788 or by email at research.integrity@asu.edu if:

- The research team is not answering your questions, concerns, or complaints.
- You cannot reach the research team.
- You want to talk to someone besides the research team.
- You have questions about your rights as a research participant.
- You want to get information or provide input about this research.

Please initial to indicate your preference for the following statements, then sign below.

I am 18 years or older and give permission to be videotaped during classes about my mathematical ideas.

Yes_________ No _________

I am 18 years or older and give permission to show excerpts of videotapes containing myself at scientific research meetings and in relevant university courses. I understand that I will not be identified by name whenever this occurs.

Yes_________ No _________

I am 18 years or older and give permission to use anonymous excerpts of my words in papers published in reports and research journals.

Yes_________ No _________

Signature: ____ Date: _______________

Printed Name: ____

APPENDIX B

TEACHER CONSENT FORM TO PARTICIPATE IN CLASSROOM RECORDING

Title of research study: Characterizing Teacher's Mathematical Meanings for Teaching, Commitment to Quantitative Reasoning, and Decentering Actions in the Context of Research-based Curriculum

Principle Investigator: Marilyn P. Carlson **Co-Investigator:** Abby Rocha

Why am I being invited to take part in a research study?

We invite you to take part in a research study because you are currently an instructor of record for a precalculus or calculus course. Participants of this study must be 18 or older and the choice to participate or not participate will not impact your grades or standing with the university in any way.

Why is this research being done?

A research team at Arizona State University developed a research-based curriculum and instructional materials for learning key ideas of precalculus that are foundational for calculus. We aim to document the instructors' teaching actions during the implementation of these materials, particularly focusing on his or her explanations, questions, and interactions with students.

How long will the research last?

Data collection will involve video-and audiotaping at most 10 classes during the semester.

How many people will be studied?

About 5 instructors and 200 students will be involved in this research.

What happens if I say yes, I want to be in this research?

By participating in this study, you agree to video-and audiotape up to four classes during the semester.

What happens if I say yes, but I change my mind later?

You can withdraw consent to be videotaped at any time without penalty.

What happens if I say no, I do not want to be in this research?

Any individual who do not consent to being videotaped will not be videotaped or audio taped.

Will being in this study help me in any way?

We cannot promise any benefits to you or others from your taking part in this research. However, possible benefits include improved mathematical learning and shifts in your understandings.

Is there any way being in this study could be bad for me?

There is minimal risk associated with your involvement in this study. Participation in this study will not impact your standing with the university in any way.

What happens to the information collected for the research?

Efforts will be made to limit the use and disclosure of your personal information, including research study records, to people who have a need to review this information. We cannot promise complete secrecy.

The data collected will be anonymized and stored separately from individual students' or teachers' names. All data collected will remain entirely confidential and will be stored so that it cannot be associated with individual names. Videotapes will be stored in a secure location at Arizona State University and digital copies will be stored on a secure server. We might show segments of videotapes during presentations of our research results at professional conferences, and we might use segments of video in professional development of instructors. Transcribed excerpts from class instruction or individual interviews might be included in published reports of the project. Students in them will be depicted anonymously. Similarly, while data about students' class performance or academic background might be reported in research publications, all students will remain anonymous.

Who can I talk to?

If you have questions, concerns, or complaints, talk to the research team by emailing Dr. Marilyn P. Carlson at marilyn.carlson@asu.edu and Abby Rocha at aerocha@asu.edu, or contact the Social Behavioral IRB office at (480) 965-6788 or by email at research.integrity@asu.edu if:

- The research team is not answering your questions, concerns, or complaints.
  - You cannot reach the research team.
  - You want to talk to someone besides the research team.
  - You have questions about your rights as a research participant.
  - You want to get information or provide input about this research.

Please initial to indicate your preference for the following statements, then sign below.

I am 18 years or older and give permission to be videotaped during classes about my mathematical ideas.

Yes_________ No _________

I am 18 years or older and give permission to show excerpts of videotapes containing myself at scientific research meetings and in relevant university courses. I understand that I will not be identified by name whenever this occurs.

Yes_________ No _________

I am 18 years or older and give permission to use anonymous excerpts of my words in papers published in reports and research journals.

Yes_________ No _________

Signature: ____ Date: __________________

Printed Name: ____

APPENDIX C

185 TEACHER CONSENT FORM TO PARTICIPATE IN CLINICAL INTERVIEW

Characterizing Teacher's Mathematical Meanings for Teaching, Commitment to Quantitative Reasoning, and Decentering Actions in the Context of Research-based Curriculum

I am a professor in the School of Mathematical and Statistical Sciences at Arizona State University. I am conducting a research study to characterize the relationship between teachers' mathematical meanings for teaching and decentering actions when teaching.

I am inviting your participation, which will involve at most four, 1.5-hour interviews about your experience teaching the research-based Pathways Pre-calculus course. During this interview you will be asked about various mathematics topics in the curriculum and your experience teaching them. You will be encouraged to think aloud, draw pictures, and write down any calculations. You have the right not to answer any question, and to stop participation at any time.

To participate in this study you must be 18 or older. Your participation in this study is voluntary. If you choose not to participate or to withdraw from the study at any time, there will be no penalty. You will be compensated $30/hour for participating in the interviews.

There is no direct benefit from participating in this research study, other than possibly gaining further insight into mathematical concepts covered in the Pathways curriculum. There are no foreseeable risks or discomforts to your participation.

Your responses will be anonymous. The results of this study may be used in reports, presentations, or publications, but your name will not be used. If any transcriptions are used in presentations or writing, a pseudonym will be used.

We are also asking your permission to video record the interview. Only the research team will have access to the recordings. The recordings will be deleted immediately after being transcribed and any published quotes will be anonymous. Your face and your hand gestures will be recorded. The interview will not be recorded without your permission.

If you have any questions concerning the research study, please contact the research team at: (aerocha@asu.edu or by phone at (515) 664-6770). If you wish to contact the PI directly, please email Dr. Marilyn Carlson at MARILYN.CARLSON@asu.edu. If you have any questions about your rights as a subject/participant in this research, or if you feel you have been placed at risk, you can contact the Chair of the Human Subjects Institutional Review Board, through the ASU Office of Research Integrity and Assurance, at (480) 965-6788. Please sign below if you wish to be part of the study, and agree to the above terms.

By signing below you are agreeing to be part of the study and that you are 18 years of age or older.

Signature: ____

Name: ____

Date: ____

APPENDIX D
INTERVIEW PROTOCOLS

Angle Measure Task-Based Clinical Interview Protocol

Review consent forms with teacher and make sure they have a signed copy saved.

Interviewer: “Thank you for taking the time to meet with me today. Before we get started with the interview, I want to review the consent form and tell you a little about what to expect.”

Review consent forms with Teacher and make sure they have copy.

1. Do you consent that your 18 years or older and agree to participate in research project?

2. Do you consent to being video recorded during this interview?

3.Your responses are anonymous. If they are used in a research report they will be used under a pseudonym.

4. Please remember that your participation is voluntary. If you choose to withdraw or not participate in the study, there will be no penalty.

Answer questions, if any arise.

Interviewer: “The purpose of this interview is to gain insight into your thinking about the trigonometry unit you will be teaching very soon. I may ask you to clarify your verbal and written responses, or request that you explain how you determined an answer. This does not mean that you are wrong. For example, if I asked you what 2 times 3 was and you said 6, I may ask you how you got that answer or request that you describe how you would explain your approach to a student. Do you have any questions before we begin?”

Time of Interview: This interview will be conducted with teachers prior to their lessons (7.1 & 7.2) on angle and angle measure.

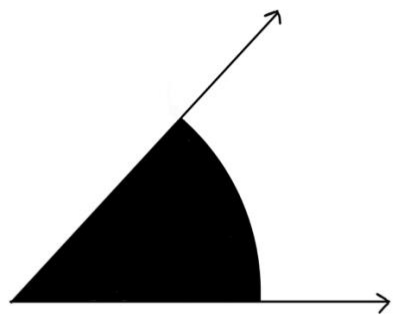
Research Question 1 Being Addressed

Goals of Interview:

- Gain insight into the teacher's mathematical meanings for angle and angle measure and their commitment to quantitative reasoning.
- Investigate the teachers' image of how to support students in developing meanings for angle and angle measure that they consider to be coherent.

Tasks:

Question 1: Suppose a student draws the image below and tells you that the shaded area between the two rays is an angle. How will you respond to this student?



- If the teacher says the student is correct: ask the teacher how the student knew where to draw the arc to stop shading.
    - If the teacher says the student is incorrect, ask the teacher to explain their thinking.
- *Follow up:* How would you respond to a student who said, "the angle is 40 degrees."?
    - When the student says the angle is 40 degrees, what attribute do you think the student is imagine measuring?
    - What might the student think 40 degrees is a measure of?

Rationale: The purpose of this question is to investigate a teacher's meanings for angle and angle measure. In particular, a teacher's response to this question may reveal whether the teacher distinguishes between an angle as an object and the measure of an angle as an amount of openness of an angle's two rays. The teacher's response to this question may also provide me with insight into the teacher's meaning for angle measure, including what angle measure is a measure of, the teacher's process for measuring angles, and what the teacher considers to be an appropriate unit for measuring an angle's openness.

Question 2: What does it mean for an angle to measure 3.5 radians?

- *Follow up:* Ask the teacher to draw an angle that measures 3.5 radians and explain how they made their drawing.

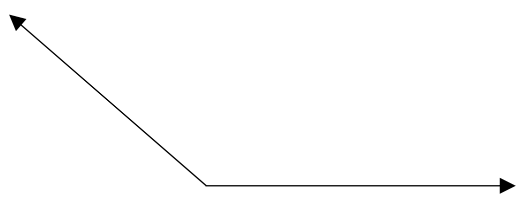
Rationale: The purpose of this question is to investigate a teacher's meaning for radian angle measure. In particular, the teacher's response to this question may reveal the degree to which the instructor thinks about angle measure quantitatively. The teacher's response to this question may reveal their meaning for radian including the attribute of the angle that the teacher measures and the process by which they determine the measure of an angle. It might also provide insights into the teacher's meaning for and use of relative size reasoning in comparing an arc length and the circle's radius.

Question 3: A student in your class says that if they see an angle measure with π in it, they know it's measured in radians. Otherwise, it's measured in degrees. Comment on this student's understanding about angle measure. [*Task borrowed from Alan O'Bryan*].

- *Follow up question:* What might you do in your teaching to support this student in developing a coherent understanding of angle measure?

Rationale: The purpose of this question is to investigate the teacher's meaning for radian angle measure. In particular, the teacher's response to this question may reveal the degree to which the instructor thinks about angle measure quantitatively. The teacher's response to this question may also provide me with insight into the teacher's image of the meanings they want students to develop, including the teacher's image of how to support students in developing these images.

Question 4a: Imagine that you were sent back in time before radians and degrees were invented. Create your own unit of measure and determine the measure of the angle below.



- If the teacher draws a circle or mentions using a circle, ask the teacher, "Why do we use circles to measure angles?"
- If the teacher proposes to measure the length of the radius on the angles' two rays, then measure the straight distance across. Ask the instructor, "Would the measure of the angle change if the length of the radius changed?"
- If the teacher struggles with this question and is unable to come up with their own unit of measure, skip to question 5.
- *Follow up:* Ask the teacher to explain why their unit of measure works.
    - Ask the teacher if the measure of the angle would still be the same if they varied the length of the radius of the circle they drew to measure the angle.
        - If the teacher says yes, ask them to explain why.

        - If the teacher says no, ask them if this is problematic, and to explain why or why not.

Rationale: The purpose of this question is to investigate what the teacher believes to be an appropriate unit for angle measure. This task is designed to uncover whether the teacher recognizes that the unit they create to measure the angle must be proportional to the circumference of any circle centered at the angle's vertex. This question is also designed to uncover whether the teacher understands why the unit of measure they create must be proportional to the circumference of any circle centered at the angle's vertex.

Question 4b: In Question 3a, you determine that the measure of the angle was {insert the measure that the teacher previously determined}. What is {measure that the teacher determined i.e. 120 degrees} a measure of?

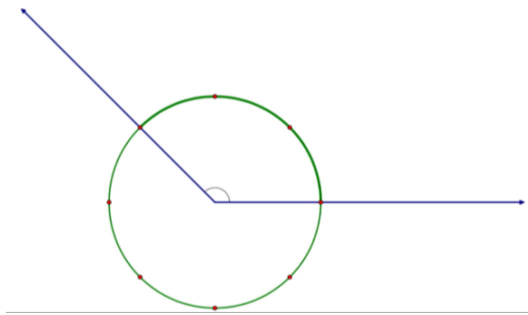
Rationale: The teacher's response to this question may uncover the attribute that they are imagining measuring when measuring an angle.

Question 4c: Suppose you asked students to respond to Question 4 and Marley, a student in your class chose to measure the angle in "pinky fingernail lengths". Marley then proceeded to place the vertex of the angle at the center of a circle and measure the length of the subtended arc by marking off how many pinky fingernail lengths fit into the arc. How would you respond to Marley?

- Follow up: Ask the teacher, "are pinky fingernail lengths an appropriate unit for measuring angles?" Ask the teacher to explain why or why not.
    - Ask the teacher if the measure of the angle would still be the same if they varied the length of the radius of the circle they draw to measure the angle.
        - If the teacher says yes, ask them to explain why.
        - If the teacher says no, ask them if this is problematic, and to explain why or why not.

Rationale: The purpose of this question is to investigate what the teacher believes to be an appropriate unit for angle measure. This task is designed to uncover whether the teacher recognizes that the unit they create to measure the angle must be proportional to the circumference of any circle centered at the angle's vertex. This question is also designed to uncover whether the teacher understands why the unit of measure they create must be proportional to the circumference of any circle centered at the angle's vertex. The teacher's response to this question may also provide me with insight into their image of the meanings they want to support students in developing and their image of how to support students in developing these images.

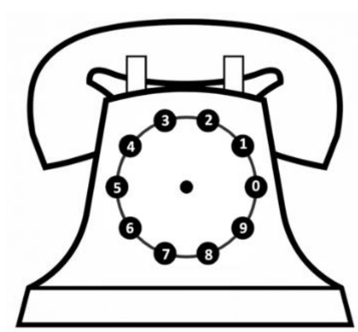
Question 5: Suppose you give students the angle below. Lamonte, a student in your class claims that the measure of the angle is 3/8$^{\text{ths}}$. Arlyse, another student in your class, claims that the measure of the angle shown below is 3. How is Lamonte thinking about measuring the angle? How is Arlyse thinking about measuring the angle? Are they both correct?
Task adopted from Tallman (2015)



- *Follow Up Question:*
  - If the teacher says both students are correct, ask them "What might you do in your teaching to support students in recognizing that both Arlyse and Lamonte's answers are correct?".
  - If the teacher says that both students are not correct, ask them to explain if either student is correct and why or why not.
    - Ask the teacher, "what might you do in your teaching to support students in understanding why{Arlyse or Lamonte's} answer is correct?".

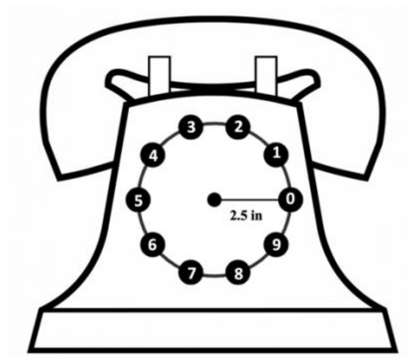
Rationale: The purpose of this question is to see if the teacher recognizes that 1/8th of the circles circumference and the circle's circumference are both valid units for measuring an angle. The teacher's response to this question will also provide me with insight into how they might explain to students why both units of measure can be used to measure the angle. The teacher's response to this question will also provide me with insight into the degree to which the teacher attends to the students thinking and leverages their thinking in their response.

Question 6a: Imagine that you want to use a rotary phone to call a friend. Assume that the number 0 starts at the 3 o'clock position and that each number on the phone is equally spaced around the circle as shown below. To dial a number, you must rotate the number (in the counterclockwise direction) to the 3 o'clock position. Determine the measure of the angle (whose vertex lies at the center of the clock) needed to dial the number 4.



Rationale: The teacher's response to this question will provide me with further insight into the teacher's meanings for angle measure. In particular, does the teacher attempt to measure this angle using the magnitude of the radius of the circle. Does the teacher leverage their understanding of relative size and proportions?

Question 6b: Continuing with the context from Question 6. If the center of the phone is 2.5 inches from the number 0, how many inches did you rotate the number 4 to dial it?



Rationale: The teacher's response to this question may provide me with further insight into the degree to which the instructor "sees" the relationship between the subtended arc length, the measure of the angle swept out, and the radius of the circle. The teacher's response to this question may also provide me with insight into the degree to which the instructor engages in relative size reasoning when thinking about angle measure.

Question 7: Suppose a colleague suggests to you, "To convert an angle measure from 2.7 radians to degrees, all you have to do is divide 2.7 by 2π and then multiply by 360 degrees" but is unsure why this works. How would you explain to your colleague why her method gives the correct answer?

- Follow up:
    - Ask the teacher, "What does $2.7/2\pi$ represent?".
    - Ask the teacher, "What does $(2.7/2\pi)$*360 represent?".
    - Ask the teacher, "In many curricula they teach students to multiply by $\frac{180}{\pi}$ to convert angles measured in radians to degrees. Why might using these numbers make it difficult to justify this process?"

Rationale: The teacher's response to this question will provide me with insight into the degree to which the instructor is committed to quantitative reasoning and leverages quantitative reasoning in their explanation. It will also provide me with insight into the degree to which the instructor's explanations are conceptually focus or grounded in calculations.

Sine Function Task-Based Interview Protocol

Review consent forms with teacher and make sure they have a signed copy saved.

Interviewer: "Thank you for taking the time to meet with me today. Before we get started with the interview, I want to review the consent form and tell you a little about what to expect."

Review consent forms with Teacher and make sure they have copy.

1. Do you consent that your 18 years or older and agree to participate in research project?

2. Do you consent to being video recorded during this interview?

3.Your responses are anonymous. If they are used in a research report they will be used under a pseudonym.

4. Please remember that your participation is voluntary. If you choose to withdraw or not participate in the study, there will be no penalty.

Answer questions, if any arise.

Interviewer: "The purpose of this interview is to gain insight into your thinking about the trigonometry unit you will be teaching very soon. I may ask you to clarify your verbal and written responses, or request that you explain how you determined an answer. This does not mean that you are wrong. For example, if I asked you what 2 times 3 was and you said 6, I may ask you how you got that answer or request that you describe how you would explain your approach to a student. Do you have any questions before we begin?"

Time of Interview: This interview will be conducted with teachers prior to their lessons (7.3 & 7.4) on sine function.

Research Question 1 Being Addressed

Goals of Interview:

- Gain insight into the teacher's mathematical meanings for sine function.
- Gain insight into the degree to which the instructor engages in quantitative and covariational reasoning when thinking about sine function.
- Investigate the teachers' image of how to support students in developing meanings for sine function that they consider to be coherent.

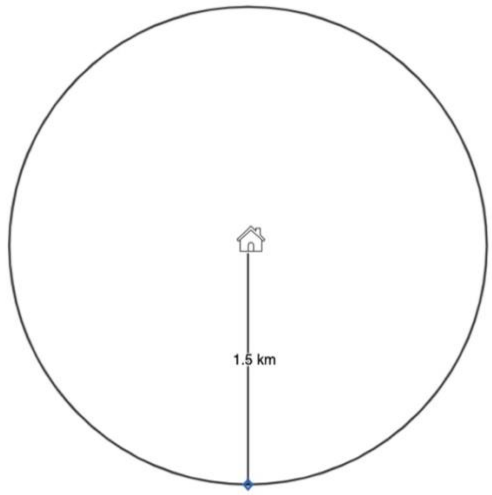
Tasks:

Question 1: Imagine that you gave students the below table of values. How do you want students to reason about what the values in the table represent?

Angle Measure, $\boldsymbol{\theta}$ (in degrees)	$\boldsymbol{sin(\theta)}$
42	0.669
95	0.996
163	0.292
210	-0.5
337	-0.391

Rationale: This task was designed to elicit the instructor's meaning for sine function. The instructor's response to this task will provide insight into the degree to which the instructor conceives of sine as a function that relates two quantities whose values vary. The instructor's response to this question will also provide insight into the degree to which they conceive of the value of sine as a relative size measurement between the terminal point's vertical distance above the horizontal diameter and the length of the radius of any circle for which θ is centered.

Question 2: Halloween is coming up and you and your friend decide to go to a circular corn maze (shown below). In the center of the corn maze is a haunted house. The entrance of the corn maze is 1.5 km south of the haunted house.



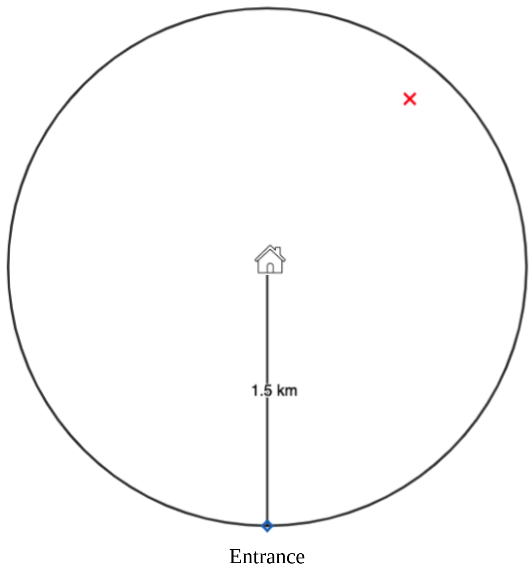
(a) While you are at the corn maze, you and your friend get separated. To find your friend, you begin walking around the circular edge of the corn maze. Write a function to determine your distance north of the haunted house (in kilometers) in terms of the distance you have walked (in kilometers), *m*, from the entrance.

Follow Up:

- Ask the instructor to explain the thinking they used when determining their answer.
- Ask the instructor to explain what each part of their formula represents.
  - "What does $\frac{m}{1.5}$ represent?"
  - "What does $\frac{m}{1.5} - \frac{\pi}{2}$ represent?"
  - "Why did you subtract $\frac{\pi}{2}$ as opposed to adding, multiplying, or dividing?"
  - "What does $\sin\left(\frac{m}{1.5} - \frac{\pi}{2}\right)$ represent?"
  - What does $1.5\sin\left(\frac{m}{1.5} - \frac{\pi}{2}\right)$ represent?"

Rationale: This task was designed to elicit the instructor's meaning for sine function. In particular, does the instructor think about sine as a function that relates an angle measured from the 3 o'clock position to the terminal point's vertical distance above the horizontal diameter. The instructor's response to this task may provide insight into the degree to which they engage in quantitative and covariational reasoning when thinking about sine function. Namely, does the teacher conceive of each part of the function that they write as representing the value of a quantity? Does the teacher think about how their distance north of the haunted house varies as their distance from the entrance varies?

(b) As you walk around the corn maze you realize you have your friend's location on the "find my friends" app in your phone. The application marks your friend's location with an X as shown below. Estimate your friend's distance (in kilometers) north of the haunted house. Estimate your friend's distance (in kilometers) east of the haunted house. Explain how you determined your estimates.

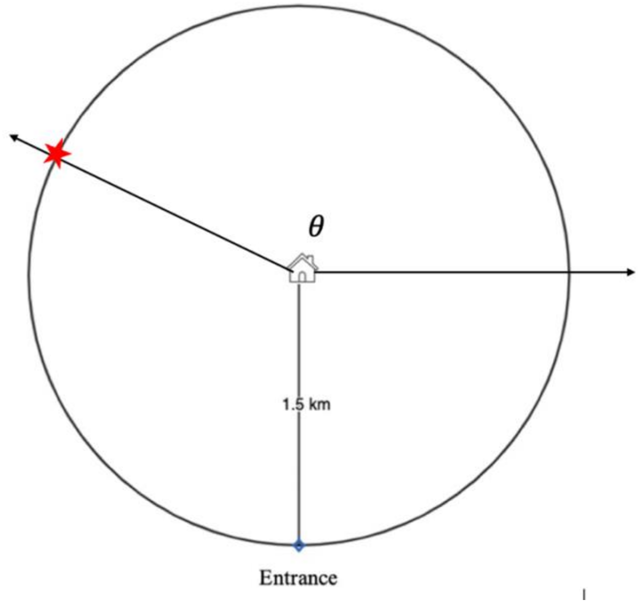


Follow Up: how might you leverage the reasoning that you used in your teaching?

- If the instructor estimates their friend's distance using relative size reasoning (i.e. my friend's distance is about 0.7 or 70% of the radius), ask the instructor, "do you see your activity in this task related to your teaching of trigonometric functions?".
- If the instructor estimates their friend's distance by writing a trig function (i.e. $1.3 \sin\left(\frac{\pi}{4}\right)$) ask the instructor to explain what each part of their formula represents.

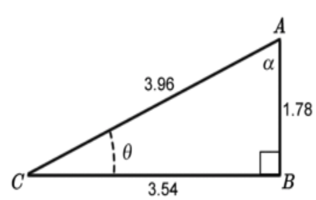
Rationale: The teacher's response to this task may provide insight into their use of relative size reasoning and/or quantitative reasoning. The instructor's response to this task may also provide insight into the degree to which the instructor relates their activity of determining a relative size measurement to the value of the sine function.

(c) Let θ represent the measure of an angle (whose vertex lies at the center of the circle) from the 3 o'clock position (in radians). If the exit of the corn maze is located where the star is on the map, estimate the value of $\sin(\theta)$ and $\cos(\theta)$.



Rationale: The teacher's response to this task may provide insight into the degree to which they conceive of the value of the sine and cosine function as a relative size measurement. Namely, does the instructor conceive of 0.2 as a measure of vertical distance in units of the radius of the circle? Does the teacher conceive of 0.2 as being determined by multiplicatively comparing a vertical distance of a terminal point above the center of the circle to the radius of the circle? The teacher's response to this task may also provide insight into the degree to which they leverage quantitative reasoning to explain why they multiply 0.2 by 1.5 to determine the exit's distance north of the haunted house in kilometers.

Question 3: Suppose you asked students to determine the value of $sin(\theta)$ using the image below. Pranjal, a student in your class tells you that sine is opposite over hypotenuse and therefore determines that sine = $\frac{1.78}{3.96}$. Comment on this student's understanding of sine.



1. What understanding would you want students to use when explaining why "sine = opposite over hypotenuse"?
2.What would be a more productive meaning?

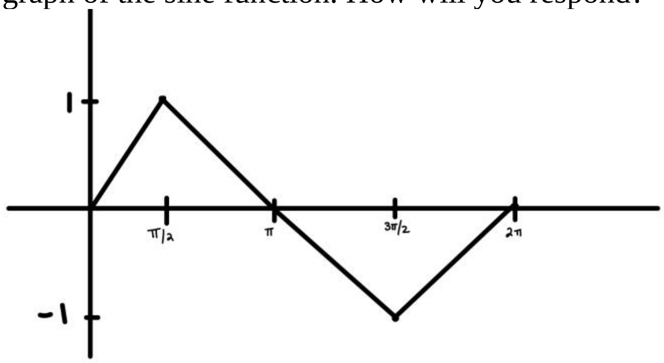
3. What would you do in class to support students in developing this meaning?

Rationale: The instructor's response to this question may provide insight into the degree to which the instruc of sine as a function. In particular, does the instructor recognize that this student's expressed meaning for sine entails a ratio of sides of a triangle rather than a function that relates two quantities whose values vary? This instructor's response to this task may also provide insight into the degree to which they attend to the thinking that may have led to this student's answer as opposed to their solution.

Question 4: Suppose a second student in your class makes the following claim, "when the radius of the circle is one, then sine is just the y-value. But when the radius of the circle is something other than one, then sine is the y-value divided by the length of the radius." Comment on this student's understanding of sine.

Rationale: The instructor's response to this question may provide insight into the degree to which the instructor conceptualizes the value of the sine function as a relative size measurement. The instructor's response to this question may also provide insight into if and how the instructor leverages quantitative reasoning when reasoning about sine function and the degree to which the instructor attends to the thinking that may have led to this student's answer as opposed to their solution.

Question 5: Imagine that a student in your class draws the following to represent the graph of the sine function. How will you respond?



Follow Up

- If the instructor says the graph needs to be a smooth curve, ask why.
- Ask the teacher, "what is concavity of the graph as theta increases from 0 to π?".
- Ask the teacher, "why is there a maximum at 1 and a minimum at -1?".
- Ask the teacher to "describe how you will support this student in constructing the correct graph for the sine function".

Rationale: The instructor's response to this question may provide insight into the degree to which the instructor values covariational reasoning and quantitative reasoning as keyways of thinking necessary for developing a coherent understanding of the sine function in students' thinking. The instructor's response may also provide insight into their image of how to support students in constructing the graph of sine function.

Dissertation Pre-Lesson and Video Analysis Interview Protocol

Review consent forms with teacher and make sure they have a signed copy saved.

Interviewer: “Thank you for taking the time to meet with me today. Before we get started with the interview, I want to review the consent form and tell you a little about what to expect.”

Review consent forms with Teacher and make sure they have copy.

1. Do you consent that your 18 years or older and agree to participate in research project?

2. Do you consent to being video recorded during this interview?

3.Your responses are anonymous. If they are used in a research report they will be used under a pseudonym.

4. Please remember that your participation is voluntary. If you choose to withdraw or not participate in the study, there will be no penalty.

Answer questions, if any arise.

Interviewer: “The purpose of this interview is to gain insight into your preparation for teaching. During this interview I will be asking you questions about how you prepare to teach and the thinking you engage in as you do so. During this interview, I may ask you to go into detail about some aspect of your preparation. This does not mean that you are doing anything wrong in your preparation. Do you have any questions before we begin?”.

Research Question Being Addressed:

- RQ1, RQ2, RQ3

Goals of Interview:

- To gain insight into the meanings the teacher wants students to develop for angle measure and sine function.
- To gain insight into the thinking the teacher engages in as they prepare to teach.
- To gain insight into the degree to which the teacher thinks about student thinking, and leverages student thinking in their preparation to teach.

Interview Duration: 20 minutes

Interview Questions:

Question 1: Ask the teacher, "Today, you will be teaching a lesson on {insert lesson topic}. What do you think about when preparing to teach? Describe the process you use as you plan to teach the course investigations."

- If teacher mentions how students may think about the idea before going to class, ask the teacher "what meanings do you think students will have for {insert lesson topic} prior to attending today's lesson?"
- If the teacher mentions having goals for students' learning, ask the teacher, "What are your goals for students' learning during your lesson today?".

Rationale: The purpose of this question is to gain insight into the nature of the teacher's preparation activity. In particular, does the teacher leverage their image of epistemic students? Does the teacher engage in decentering actions during their preparation? The teacher's answer to this question will provide me with insight into the degree to which the teacher's preparation is informed by their image of student thinking.

Question 2: Ask the teacher the following, "Today you will be teaching a lesson on {insert lesson topic e.g., angle and angle measure}.What meanings/understandings do you want your students to construct for {insert lesson topic}?"

Rationale: The purpose of this question is to gain insight into the meanings the teacher wants to support students in developing. The teacher's response to this question may also provide me with insight into the degree to which the teacher is committed to quantitative reasoning/covariational reasoning and "sees" these ways of thinking as important for students to engage in.

Question 3: Ask the instructor, "What do you plan to do in your teaching to support students in developing the understandings you just described?".

Rationale: The teacher's response to this question may provide me with insight into the teacher's image of how to support students in constructing a meaning for angle measure or sine function that they consider to be coherent. The teacher's response to this question may also provide me with insight into the ways of thinking the teacher believes are important for students to engage in.

Question 4: Ask the teacher the following, "Walk me through your plan for class today."

- If the teacher describes tasks from the curriculum, ask the teacher, "How did you choose the tasks you want to discuss during class?".
- If the teacher describes tasks that they designed, ask the teacher, "Why did you design your own task to use during class?".
  - Ask the teacher, "What thinking led to the design of this task?".
  - If the teacher mentions student thinking, ask "What thinking are you hoping to engage students in while using this task during class?"
- Ask the teacher, "How do you plan to use each of these tasks during class? Do you plan to lecture, engage students in group work, etc.?"

Rationale: The teacher's response to this question may provide me with insight into the teacher's image of how to support students in constructing a meaning for angle measure or sine function that they consider to be coherent. The teacher's response to this question may also provide me with insight into the ways of thinking the teacher believes are important for students to engage in. The teacher's response to this question may also provide insight into the degree to which the teacher is thinking about student thinking in their preparation, and/or leveraging this thinking in the design of their lesson.

Question 5: Ask the teacher, "What are some key problems or activities you plan to have students work on today? Why did you select these problems? What solution and thinking do you hope your students produce when completing/responding to these problems and activities?"

Rationale: The teacher's response to this question may provide me with insight into the teacher's image of how to support students in constructing a meaning for angle measure or sine function that they consider to be coherent. The teacher's response to this question may also provide me with insight into the ways of thinking the teacher believes are important for students to engage in. The teacher's response to this question may also provide insight into the degree to which the teacher is thinking about student thinking in their preparation, and/or leveraging this thinking in the design of their lesson.

Question 6: Ask the teacher, "During class how do you plan to assess whether students are engaging in the thinking and developing the understandings you desire?"

Rationale: The purpose of this question is to see how the teacher plans to assess student understanding. The teacher's response to the question will provide me with insight into the degree to which the teacher is anticipating interactions with students. The teacher's response to this question may also provide me with insight into the degree to which the teacher plans to attend to student thinking and what the teacher considers to be evidence of student understanding.

Question 7: Ask the teacher, "Prior to teaching do you work through the problems you plan to use in class?".

- If the teacher says yes, ask the teacher, "What do you think about as your work through the problems?"
- If the teacher says yes, ask the teacher, "How does your working through the problems inform your thinking about your teaching?".

Rationale: The purpose of this question is to gain insight into the nature of the teacher's preparation and the degree to which the teacher is thinking about student thinking in their preparation.

Question 8: Ask the teacher, "Is there anything else that you do during your preparation that you think I should know about?".

Rationale: The purpose of this question is to provide the instructor with an opportunity to express anything that they feel is important that I have not explicitly asked about.

Question 9: Ask the teacher the following, "You taught this lesson last semester. Has your prior teaching of this topic informed your preparation for today's lesson? If so, in what ways?"

Rationale: The teacher's response to this question may provide me with insight into the degree to which the teacher leverages their image of epistemic students in their preparation for teaching. The teacher's response to this question may also provide me with insight into the ways of thinking the teacher believes are important for students to engage in. The teacher's response to this question may also provide insight into the degree to which the teacher is thinking about student thinking in their preparation, and/or leveraging this thinking in the design of their lesson.

***** Video Analysis Clinical Interview Questions *******

*Show videos of the teacher's instruction from the day before- *** Tell interviewee the following: I am going to show you a clip of your teaching during your last class. While you watch the clip, I want you to think about your teaching. After watching the clip, I will ask you a few questions.*

Select a clip that includes an interaction with students, or a student expressing his/her thinking.

- Ask clip specific questions here (TBD after analyzing classroom teaching).
- General questions to ask about the clip:
  - What do you notice about your teaching in this clip?

  - During this clip you asked students {insert question here}. Why did you pose this question to students?
  - During this clip a student said {insert student statement}, how do you think this student was thinking?
  - During this clip a student said {insert student statement} and you responded with {insert teacher statement}. Why did you respond in this way?
  - If given a second chance to respond to this student, would you respond in the same way or differently? Why? How would your response change?

Select a second clip from the teachers' instruction.

- Ask clip specific questions here (TBD after analyzing classroom teaching).
- General questions to ask about the clip:
  - To what degree do you feel you were successful in conveying your meaning for [insert idea here] to students coherently?
  - Will you make any changes to your approach when using this task with students next semester? If yes, what changes will you make? If no, why not?
  - To what degree do you feel that you were attentive to and leveraged your knowledge of students' ways of thinking when discussing the task shown in this clip?
  - Is your teaching of this task an example of what you would consider to be effective teaching? Why or why not?

Rationale: The instructor's response to these questions may provide insight into the degree to which the teacher reflects on his/her teaching. The teacher's response to these questions may also provide insight into the teacher's meanings for angle measure and trig functions, the degree to which the teacher's instruction is student centered, and the degree to which the teacher values quantitative and covariational reasoning. The teacher's response to these questions may also provide insight into the degree to which the teacher is attending to and leveraging student thinking in their teaching (i.e., forming a second order model of students' thinking).

Post Lesson Interview

Review consent forms with teacher and make sure they have a signed copy saved.

Interviewer: “Thank you for taking the time to meet with me today. Before we get started with the interview, I want to review the consent form and tell you a little about what to expect.”

Review consent forms with Teacher and make sure they have copy.

1. Do you consent that your 18 years or older and agree to participate in research project?

2. Do you consent to being video recorded during this interview?

3.Your responses are anonymous. If they are used in a research report they will be used under a pseudonym.

4. Please remember that your participation is voluntary. If you choose to withdraw or not participate in the study, there will be no penalty.

Answer questions, if any arise.

Interviewer: “The purpose of this interview is to discuss the lesson you just taught and your feelings about the lesson. During this interview, I may ask you to go into detail about some aspect of your lesson and/or teaching. This does not mean that you did anything wrong, rather I am interested in understanding your thinking/the reason for making the choices you made. Do you have any questions before we begin?”.

Goals of Interview:

- To gain insight into the ways in which the instructor thinks about their teaching after their lesson.
- To gain insight into the teacher's image of students' thinking about {insert lesson topic}.
- To ask any clarifying questions about anything the teacher did during their lesson that I am interested in knowing more about.

Interview Duration: 20 minutes

Interview Questions:

Question 1: Ask the teacher, "How do you feel about your lesson today? What is your assessment of how your lesson went?".

Rationale: The purpose of this question is to gain insight into how the teacher thinks about their teaching after their lesson. Does the teacher's description focus on their actions or the thinking that they engaged in during their teaching? Does the teacher think about students' thinking after class?

Question 2: Ask the teacher, "Did anything stand out to you from your lesson today?"

Rationale: The purpose of this question is to gain insight into if the teacher thinks about student thinking while teaching and/or after teaching. Has the teacher's image of students' thinking or image of how to support students' development of a coherent understanding changed? If so, in what ways?

Question 3: Ask the teacher, "Did you deviate at all from your lesson plan? If so, how and why?"

Rationale: The purpose of this question is to identify any moments of instructional deviation.

Question 4: Ask the teacher, "Did you accomplish what you wanted to while teaching today? If not, what kept you from doing so?"

Question 5: Ask the teacher, "After teaching today, do you have any new insights about students' thinking?"

Rationale: The purpose of this question is to gain insight into the degree to which the teacher thinks about their teaching and student thinking following their lesson.

Question 6: To what degree do you think your students understood the main ideas of the investigation? Explain.

Rationale: The purpose of this question is to gain insight into if the teacher thinks about student thinking while teaching and/or after teaching. Has the teacher's image of students' thinking or image of how to support students' development of a coherent understanding changed? If so, in what ways?

Question 7: Ask the teacher, "In future semesters when you teach this lesson will you do anything different? Why or why not?".

Rationale: The purpose of this question is to discern how or if the instructor refines their epistemic students. This question will also provide insight into the thinking the teacher engages in to make changes to their instruction.

*****End interview with any questions specific to what the teacher did in their lesson that I am curious to know more about.**

Ex: Ask the teacher, "During your lesson I noticed that you walked around the room while students were working. Why did you do this? What were you thinking about as you did this?".

Ex: Ask the teacher, "During your lesson you asked Matea to come to the board and explain her solution to problem 3. Why did you do this? What were you thinking about as you did this?".

Ex: Ask the teacher, "During your lesson you asked Student 1 the following, {insert question} and they responded by saying {insert student response}. Then you responded with {insert teacher response}. Why did you respond in this way?

- If the teacher mentions something about Student 1's thinking, ask the teacher, "How do you think Student 1 was thinking when they said {insert student explanation}?".

APPENDIX E

HUMAN SUBJECTS APPROVAL LETTER

EXEMPTION GRANTED

Marilyn Carlson
CLAS-NS: Mathematical and Statistical Sciences, School of (SMSS)
480/452-7435
MARILYN.CARLSON@asu.edu

Dear Marilyn Carlson:

On 3/1/2022 the ASU IRB reviewed the following protocol:

Type of Review:	Initial Study
Title:	Characterizing Teacher's Mathematical Meanings for Teaching, Commitment to Quantitative Reasoning, and Decentering Actions in the Context of Research-based Curriculum
Investigator:	Marilyn Carlson
IRB ID:	STUDY00015483
Funding:	None
Grant Title:	None
Grant ID:	None
Documents Reviewed:	• Formal IRB Response , Category: Other; • Interview Protocol, Category: Measures (Survey questions/Interview questions /interview guides/focus group questions); • Spring 2022 Protocol, Category: IRB Protocol; • Student Consent Form, Category: Consent Form; • Teacher Interview Consent Form, Category: Consent Form; • Teacher Payment Form, Category: Other; • Teacher Recruiting Form, Category: Recruitment Materials;

The IRB determined that the protocol is considered exempt pursuant to Federal Regulations 45CFR46 (1) Educational settings, (2) Tests, surveys, interviews, or observation on 2/23/2022.

In conducting this protocol you are required to follow the requirements listed in the INVESTIGATOR MANUAL (HRP-103).

If any changes are made to the study, the IRB must be notified at research.integrity@asu.edu to determine if additional reviews/approvals are required. Changes may include but not limited to revisions to data collection, survey and/or interview questions, and vulnerable populations, etc.

REMINDER - - Effective January 12, 2022, in-person interactions with human subjects require adherence to all current policies for ASU faculty, staff, students and visitors. Up-to-date information regarding ASU's COVID-19 Management Strategy can be found here. IRB approval is related to the research activity involving human subjects, all other protocols related to COVID-19 management including face coverings, health checks, facility access, etc. are governed by current ASU policy.

Sincerely,

IRB Administrator

cc: Abby Rocha