Improving student success and supporting student meaning-making in large-lecture precalculus classes

T. H. McNicholl
Department of Mathematics
Iowa State University
Ames, Iowa 50011 USA
mcnichol@iastate.edu

K. Frank
Department of Mathematics
Towson University
Towson, Maryland 21252 USA
kfrank@towson.edu

K. Hogenson
Department of Mathematics and Computer Science
Colorado College
Colorado Springs, CO 80903 USA
khogenson@coloradocollege.edu

J. Roat
Mathematics Department
SUNY Cortland
Cortland, NY 13045 USA
jolie.roat@cortland.edu

M. P. Carlson
Department of Mathematics
Arizona State University
Tempe, Arizona 85287 USA
marilyn.carlson@asu.edu

Improving student success and supporting student meaning-making in large-lecture precalculus classes

Abstract: We discuss an implementation of the research based Pathways precalculus curriculum in a large-lecture format supported by recitations, clickers, and online homework. We describe our approach to adapting the implementation of Pathways precalculus from a small class to a large lecture instructional format. We report data on student learning and retention before and after this Pathways implementation which show improved student success in precalculus, improved student retention in calculus, and significant shifts in student understanding of precalculus ideas that are foundational for learning calculus.

Keywords: precalculus, large-lecture classes, clickers

1 Introduction

There is widespread interest in increasing the number of STEM graduates (for example, [28]). A logical starting point for achieving this goal is to increase student success in Calculus I since it is an early required course for nearly all STEM degrees. However, more than 30% of first-year students who arrive at a public 4-year university are unprepared for the challenges of a university calculus course and are therefore placed in a precalculus course or lower [31] . College-level precalculus courses are of particular concern to educators since there is evidence that many university-level precalculus courses do not adequately prepare students to learn or succeed in calculus (see, for example, [5], [35]). Hence, addressing student success and learning in university precalculus courses is vital to securing the national goal of increasing the number and quality of STEM majors.

Studies have shown that the Pathways precalculus curriculum developed by researchers at Arizona State University [4] address many of the problems with student learning in precalculus. These precalculus materials are rooted in a constructivist approach to learning and include questions and prompts intended to support students in engaging with and making connections among ideas. The materials focus on fundamental concepts that are needed for learning calculus, including ideas of function composition, average rate of change, constant rate of change, exponential growth, and trigonometric functions. Prior to 2013, the Pathways materials had been used only in small class sections at secondary and post-secondary institutions. However, due to resource constraints that are not likely to abate any time soon, if ever, many large public universities must offer precalculus instruction in large lectures supported by recitations and online homework. This is a challenging learning environment for students, and accordingly pass rates can be very low.

In Fall 2013, the mathematics department of Iowa State University undertook a study to see if the Pathways curriculum could be scaled-up to a large-lecture format. To do so, they introduced a new student-centric and constructivist framework for large-lecture classes that incorporated recitation sections, clicker questions, and online homework developed by a team of faculty and graduate students at Iowa State University and Arizona State University.

Before registering for a mathematics course, all students at Iowa State University are required to take a placement exam. Prior to Fall 2013, students who needed to take a first-semester calculus course but did not achieve a sufficiently high placement score (administered via ALEKS) were advised to take one of two courses depending on their score: either college algebra or a trigonometry and analytic geometry class. Both were taught with traditional procedure-focused materials in large lectures. We will refer to the former course by the abbreviation 'CA' and the latter by the abbreviation 'TAG'. In 2013, the department offered these courses and the new one-semester precalculus course based on the Pathways curriculum; we will refer to this course as 'PP'. Students whose placement test score was not high enough to enter calculus but was sufficiently high to avoid being advised into CA were advised to take either TAG or PP. At the end of the semester, the results of these courses were compared in terms of acquisition of fundamental precalculus concepts and pass rates; overall PP compared very well. We note that placement criteria for PP and TAG were the same. Additionally all students in PP and TAG took an initial pre-test, the Precalculus Concept Assessment (PCA); the mean scores for both populations were not significantly different.

The goals of this article are to describe the Pathways approach to precalculus as well as the development, implementation, and assessment of PP, a large lecture implementation of the Pathways materials. The article is organized as follows. In Section 2, we discuss the Pathways approach to precalculus education and its research basis. In Section 3, we describe the new format developed for large lecture classes. This format relies on carefully developed recitations; these recitations are detailed in Section 4. The structure of the lectures and our approach to developing the clicker questions are discussed in Section 5. In Section 6, we describe the development of the conceptually focused online homework. In Section 7, we discuss the assessment of the course. We conclude by summarizing our approach and results in Section 8.

2 The Pathways approach to precalculus education

The Pathways Precalculus project is based in research on student learning in precalculus (for example, [6], [7], [5]) and beginning calculus (for example, [2], [24], [32], [13], [37], [38], [39]). These investigations revealed widespread and severe weaknesses in students' understanding of fundamental concepts such as rate of change and function,[1] even after students successfully completed a college level precalculus course [5]. Further research into student learning in the context of beginning calculus revealed that students' weak understanding of these concepts were major obstacles to student learning in calculus (for example, [2], [19], [34], [15], [24]).

The findings from these studies led to and informed the design of the Pathways Precalculus materials. The work began with the Pathways research team designing hypothetical learning trajectories for key ideas that had been identified to be foundational for learning calculus. In addition to attending to the development of students' meaning for linear, exponential, and trigonometric functions, an overarching learning goal was to support students in acquiring a renewed confidence in their ability to make sense of ideas and problems, and to support students in developing meaningful formulas to represent patterns and relationships

[1] The findings revealed weaknesses in students' understanding of the idea of function including: i) function composition; ii) function inverse; iii) constant rate of change; iv) average rate of change; v) function notation; vi) quadratic and exponential growth; vii) a function's end-behavior; viii) attending to how two quantities in a function relationship change together; ix) angle measure and trigonometric functions, etcetera.

between covarying quantities.

The Pathways authors attempted other approaches to improve student learning and student success. They developed in-class worksheets to supplement traditional texts, used group work during class, and assigned homework every class meeting. These interventions led to greater student success in the course, but they did not lead to improved student learning of the course's key ideas; nor did they lead to more students continuing on and succeeding in calculus ([5]). These findings led to the realization that the wide-spread curricular focus on helping students work specific problem types, with little attention to the conceptions and reasoning abilities students needed for learning calculus and using ideas of precalculus was ineffective. The Pathways Precalculus instructor and student materials provide a research-based response to this problem. The material's initial design was guided by research on student learning in both precalculus and calculus. In addition, the materials are updated regularly as the researchers continue to engage in both qualitative and quantitative research into student learning. Qualitative studies (e.g., [7]; Carlson, Larsen and Jacobs, 2001; [2]; [5]; Carlson, Larsen and Lesh, 2003; Frank, 2017; Kuper, 2019; Moore, 2010; Moore and Carlson, 2012; O'Bryan, 2018; Strom, 2008) investigate student thinking as the student completes research-scaffold tasks. The data from these studies have produced novel insights about productive ways to support students' in developing strong and connected conceptions, and productive reasoning abilities. These findings inform the refinement of Pathways investigations, applets, and online homework. It was only after implementing the research developed Pathways student and instructor materials that significant gains were achieved in student learning [8], [3], [17], [21], [26], [25], [27], [36]).

Studies examining student learning in the context of Pathways materials have consistently revealed that the question scaffolding in Pathways materials helps students conceptualize the quantities in applied problem contexts, and reason about how two varying quantities in the situation are related and change together. These conceptualizations have been shown to be essential for students' development of meaningful function formulas and graphs, and for using various function types (linear, exponential, trigonometric) to solve novel problems (Moore, 2010; Moore and Carlson, 2012; O'Bryan, 2018). The Pathways materials are now in their 7th Edition and continue to be adapted as researchers produce new knowledge about how to support students in developing both competence and confidence in their understandings, reasoning abilities, and mathematical practices. The knowledge that is generated about student learning, including both productive and unproductive meanings, are discussed in the Pathways instructor materials and workshops, and are frequently revealed to the instructor as students confront Pathways conceptually scaffolded questions during class. The materials now consist of a student workbook, an online textbook, and instructor resources such as online videos, possible solutions of the workbook exercises, and discussions of common student misconceptions. These instructor resources support instructors in shifting their precalculus instruction to be effective in developing students' foundational meanings of precalculus mathematics. During recent iterations of the materials, online clicker questions and homework were added.

In order to facilitate and support the most effective use of the course materials, members of the Pathways research and development team provide 2–3 days of professional development to faculty and graduate students who are preparing to use Pathways materials in their lecture and recitation sections. At these sessions, the workshop leaders model inquiry instruction by engaging the participants in tasks that require the use of key reasoning abilities and mathematical understandings, such as quantitative and covariational reasoning, that are needed for learning calculus. Workshop participants form small groups where they discuss their approach and the thinking they used to complete tasks. They are then asked to verbalize their approach, with workshop leaders who pose questions as needed to support participants in explaining what quantities they visualized when reading and making sense of a problem, explaining the rationale for their approach (rather than just how they arrived at their answer), describing what student difficulties they anticipate, etcetera. Later, workshop leaders engage the participants in discussions of quantitative data that reveal student misconceptions and weaknesses regarding relevant precalculus concepts. Workshop participants broadly report experiencing a profound shift in their image of good teaching and their awareness of what is entailed in learning and understanding ideas in precalculus. This is reflected in the participants choosing to recreate many of the discussions and activities that they experienced in the workshops both in their classrooms and at their weekly course planning meetings.

3 New learning sequence for the large lecture

The remedial math courses at Iowa State University, CA and TAG, are taught in classes of 200 or more students with a topic being introduced during lectures delivered in an auditorium. Students then attended a smaller recitation (about 30 students) led by a graduate student or instructor. Recitation sections typically involved the instructor providing a review lecture on some of the topics covered in the prior large lectures. The recitation instructor might also work through additional examples and answer students' questions about the homework.

There are several obstacles to using this format in the PP course with the Pathways materials. First of all, the primary aim of Pathways is to build conceptual understanding, and the overwhelming evidence shows that lecturing is not effective in developing students' understandings of concepts (for example, [29], [18]). While there have been some implementations of small group work in large lecture settings, these typically require facilities that are not widely available, such as rotating chairs in auditorium style lecture halls.

Accordingly, we decided that in PP we would invert the traditional large lecture format described above. The resulting learning flow is shown in Figure 1. Unlike the traditional large lecture learning process, our revised delivery format introduces each topic in the recitation by having students work collaboratively to complete an engaging set of questions. This is in keeping with the learning goals discussed in Section 2 so that students have opportunities to construct their own understanding of the topic. These recitations are described in more detail in Section 4. The day following the recitation, the same topic is discussed again by the professor leading the 200-student large lecture. This lecture begins with a review of the just-completed group project followed by a sequence of clicker questions designed to engage the students in exploring and improving their understanding of each topic. The format of these lectures and development of the clicker questions is discussed in more detail in Section 5. Following the large lecture, students complete an online homework assignment (via WeBWork) that is designed to assess and advance student learning of the key ideas of the lecture and also provide students practice that we hypothesized would be needed for students to solidify their understanding of the material. More information regarding the process of developing these assignments is presented in Section 6.

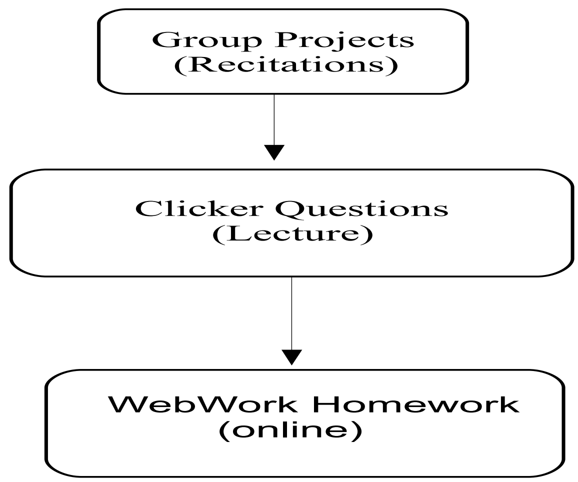


Figure 1. The Pathways Large Lecture Learning Process

4 Structure of the recitations

In PP, students spend two of the class's four weekly contact hours in recitation sections. Each recitation is led by either a graduate teaching assistant or a lecturer and contains at most 32 students from the large lecture. The recitations are further broken down into groups of 4 to 6 students. Each small group works as a team to complete investigations that explore precalculus topics not yet addressed in lecture. For recitation instructors, there are three major components to the recitations: preparation, class participation, and evaluation.

Each week, all recitation instructors attend a meeting to prepare for the week's classes. During this time, problems from the Pathways workbook are discussed and selected to provide PP students with a solid foundation of the upcoming concepts. Within this discussion, recitation instructors anticipate student questions or misconceptions and develop appropriate responses, often in the form of guiding questions. For example, when discussing the concept of a function, instructors might expect some students to rely on the "vertical line test" that they learned previously, often to a fault. If this happens, the instructor can direct them back to the definition of a function and ask questions such as "If y is not a function of x, is x a function of y? Why or why not?" and "When does the vertical line test work, and when does it fail?"

The actual recitation classes start out in a variety of ways. Some instructors start with a warm-up question. These questions might revisit difficult material from a previous class, or they might serve to motivate the current day's lesson. Other instructors begin with a brief overview of new terminology and a discussion of the purpose of the investigation for that day. No matter how the class begins, most of the recitation session is spent with students working on the day's group project, which consists of problems selected from the Pathways workbook at the pre-class instructors' meeting. Instructors encourage students to work through the problems as a group.

The main goal of the group projects is to provide students an opportunity to explore a concept before it is formally introduced or defined in the lecture. We hypothesized this would support students in constructing an understanding that would be more robust than rote memorization of a definition. For example, when learning about angle measure, group members work together to construct angles measuring 1 radian using only a compass, string, and straight edge. This gives students a hands-on experience to learn about the relationship between the radius length of a circle and the arc length cut off by an angle with a measure of 1 radian. In the next lecture class, angle measure is more formally defined and the understanding students developed during the recitation activity is reinforced and applied to new situations. As students work through the group project, the instructor walks around the classroom asking probing questions about students' understanding of the question and their approach to answering the question, while encouraging discussions among group members and guiding students toward meaningful solutions. Whenever possible, the last few minutes of recitation are used to have representatives of each group present their group's solutions to the class. The problems students present are chosen in advance by the recitation instructor either to address a common misconception with the entire class or to facilitate a class discussion about a key idea. This gives students the chance to share the insights discussed within their groups and helps emphasize the concepts that they need to take away from the day's lesson. As students are selected by the instructor, rather than on a volunteer basis, it has the added bonus of holding every student accountable for the new material. This encourages students to participate actively within their groups throughout the recitation.

At the end of every recitation period, each group turns in a folder or binder that contains the group's completed projects. The recitation instructor then grades the problems that the team of instructors have selected for grading during the instructors' meeting. Initially, a group grade is assigned to each project, which is then scanned and emailed to group members so that they have a copy of their evaluated work. Around the time of each exam, this group grade is converted into individual grades. These individual grades are largely derived from confidential student peer evaluations using a variation of the Michaelson method [23], though they are occasionally influenced by the instructor's observations of group dynamics. In the peer evaluations, students are asked to distribute points to their group members so that the average number of points given out per person is 10. Students do not rate themselves and are encouraged to include comments to support their evaluations. These comments range from "We all worked well together as a group" to “This team member consistently took on a leadership role” to “This team member was often late and didn’t contribute to the discussion.” After each round of peer evaluations is completed, the recitation instructors reconfigure the assigned small groups for the next course unit. This serves as a great opportunity to encourage students to interact with new classmates and it allows instructors to break up unproductive student pairings that sometimes interfere with a group’s learning.

Though there was some variation in how each recitation instructor ran his/her class, this general framework provides a reasonably consistent experience for each student attending the large lecture. Above all, the goal is always to get students to think critically about the main ideas by working together on key questions in that day’s investigation. Based on the substantive conversations that occur in these recitations, the authors are optimistic that this goal has been achieved.

5 Lecture structure and clicker questions, development process for clicker questions

The term “clickers” (also known as personal response systems) refers to a wide variety of electronic devices by which students communicate their answers to questions posed by the instructor. These devices may be hand-held RF transmitters, an internet-connected mobile phone, or laptop application with appropriate software. Depending on the sophistication of these devices, these questions might be multiple choice, short answer, or numerical. A common use of clickers is to assess student comprehension after the delivery of some unit of instruction. For example, after lecturing on the product rule for differentiation, an instructor might give one or more clicker questions on the product rule and, based on the responses, decide if additional coverage is necessary. Usually, students are first asked to answer the question on their own, and then spend

1–2 minutes explaining their work to their neighbors before answering again.

Put another way, clickers are merely a class of devices for implementing peer instruction. The essence of peer instruction is for students to engage in activities that require them to use the core concepts that were presented, and then explain their reasoning to their peers. The use of peer instruction and clickers to improve conceptual learning began in physics education (see for example, [22]), but has now spread to almost every scientific discipline. In all domains, peer instruction via clickers has been shown to improve conceptual learning dramatically; see, for example, [12],[30], [33]. Two useful guides to the use of clickers and peer instruction in mathematics are [1] and [9].

Peer instruction and clickers were a key component of the large lecture format developed for Pathways implementation at Iowa State University. A number of formats have been tried for the class sessions in the large-lecture halls. The usual format now is to begin with a short review and synthesis of the just-completed group project. The point of this review is to elucidate the key concepts and principles of the topic under investigation. For example, a key concept for function composition is the meaning of the notation $f(g(x))$ as turning the output of one function into the input of another function. Students were then given a clicker question and told to work on it independently for about 2 minutes and submit their answer via their clicker device. Afterward, students were asked to discuss their solutions with their neighbors and submit answers to the question a second time. Students were told their second response could be different from the first and, in order to encourage participation, that they would receive 1/2 credit on each response even if their answer was incorrect. These questions were multiple choice questions. The instructor would then show the students the distribution of their answers when they worked on their own and after they discussed their work. Finally, the instructor would reveal the correct response and discuss one or more correct methods of solution as well as possibly some common incorrect methods. During these discussions, care was taken to point out the connections to the recently completed group project. Sometimes students were given the opportunity to earn extra credit by presenting their solution in front of the class. Grades on clicker questions comprised 10% of each student's semester grade.

Initially, clicker questions were designed based on the research team's previous interactions with precalculus students and perception of common misconceptions; that is, initially they were based on educated guesswork. Over time, the knowledge base for the clicker questions was expanded by studying solutions to the workbook exercises and WeBWork problems for common incorrect reasoning patterns. The research team also examined the distribution of responses to each question. Incorrect options, also known as distractors, which were not frequently chosen were revised. In addition, questions which did not stimulate learning were discarded. That is, if a large fraction of the class answered the question correctly before and after discussion with their peers, then the team concluded very little learning was taking place as a result and the question was either significantly revised or discarded altogether. As an example, consider the clicker question in Figure 2.

These options were developed based on common incorrect answers to similar questions during a recitation activity designed to review basic algebra. However, the results were disappointing. In Fall 2014, 86% of the students answered correctly on their own, and 98% answered correctly after discussion. It may have been that many assumed the longest answered had to be correct or that during their just-completed group project they had been thoroughly disabused of the sort of thinking

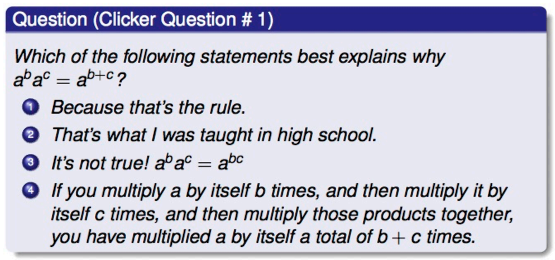


Figure 2. A clicker question on basic algebra.

illustrated by the first three responses. Whatever the reason, it was clear that no learning resulted from this question and so it was not used again in the future. By contrast, consider the clicker question in Figure 3 which was designed to engage students in using constant rate of change.

The responses to this question before and after discussion are shown in Figure 4.

In this case, slightly less than a majority answered the question correctly when the students worked on their own, whereas almost the entire class answered correctly after discussion. In addition, all distractors initially received at least 10% of the responses. These characteristics were taken as indication that student learning had occurred.

6 Development of WeBWork homework

Online homework is now a staple of large-lecture STEM courses and is very likely to remain so. While it has the disadvantage of providing considerably less feedback than a human grader, it also has the advantage of providing instantaneous feedback and greatly reducing the personnel-hours required to administer a large-lecture class.

To provide students with immediate feedback on their homework, the research team developed online homework problems using WeBWork, an

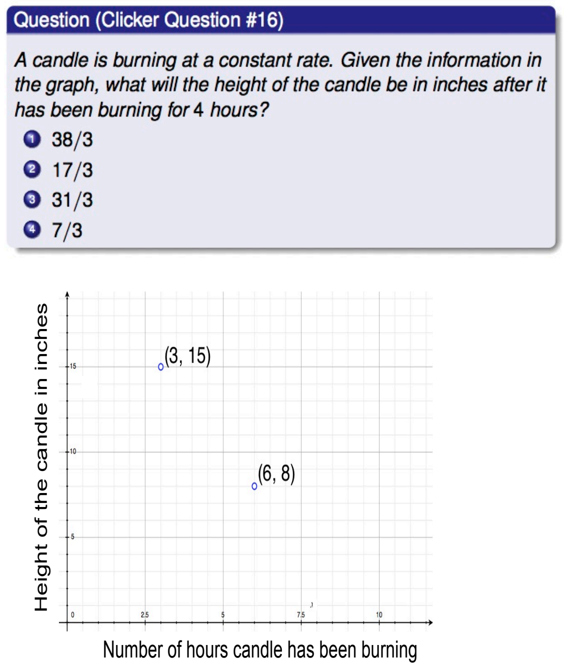


Figure 3. A clicker question on constant rate of change

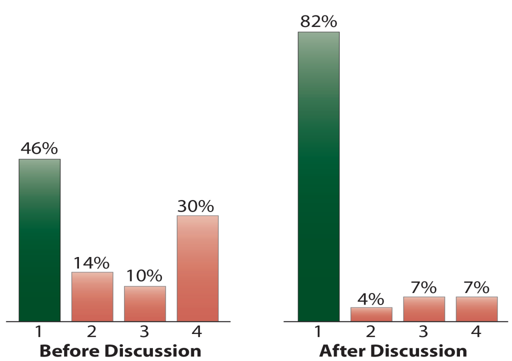


Figure 4. Responses to the question in Figure 3 before and after discussion. Answer Choice 1 is the correct answer.

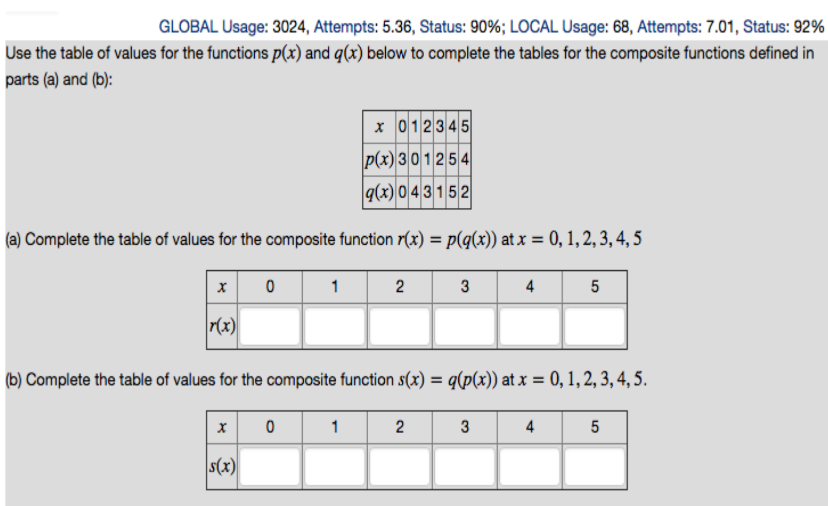
open-source online homework system supported by the MAA and NSF.[2] While there are over 35,000 problems[3] in WeBWork's Open Problem Library (OPL), it was essential for the research team to develop problems aligned with the Pathways materials. Specifically, it was necessary to develop questions that went beyond procedural practice and instead supported students in developing meaningful formulas and graphs to represent how two quantities change together in a word problem setting. Since research (for example, [16], [26]) revealed that students have difficulty conceptualizing the quantities in a word problem and using variables to represent varying values of some quantity in a problem context, we developed items to assess these abilities. To illustrate the need to design new WeBWork questions, consider the 192 problems in the OPL classified as precalculus function composition problems. Of these problems only two (from [10]) were situated in a context. Additionally, these two problems were the only problems that asked students to explain the meaning of their calculation. The remaining 190 problems can be classified as procedural practice; the problem asks students to determine a composed function given a decontextualized function expressed as a graph, table, or formula. One such representative problem is shown in Figure 7; this problem asks students to evaluate a function composition given tables of values representing the input-output pairs of two functions.

After determining that the existing WeBWork questions were not aligned with the learning goals of the Pathways materials, the Pathways research team developed conceptual items that aligned with the Pathways online text and workbook problems. The research team concurrently developed procedural homework tasks to reinforce skills such

[2]Since then the Pathways project has developed online homework in the IMathAS format as well.

[3]As of April 15, 2019

as factoring and simplifying. Thirty questions were written for each investigation including at least 10 conceptually oriented questions and 10 procedurally oriented questions. Some of these questions focused on interpreting a mathematical expression and others prompted students to interpret the meaning of a calculation. For example, of the sixty problems written by the Pathways team about function composition, 40 asked students to interpret either their answer or some expression/notation in the context of a situation. Sample function composition problems are given in Figures 5 through 7.



GLOBAL Usage: 3024, Attempts: 5.36, Status: 90%; LOCAL Usage: 68, Attempts: 7.01, Status: 92%

Use the table of values for the functions $p(x)$ and $q(x)$ below to complete the tables for the composite functions defined in parts (a) and (b):

x	0	1	2	3	4	5
$p(x)$	3	0	1	2	5	4
$q(x)$	0	4	3	1	5	2

(a) Complete the table of values for the composite function $r(x) = p(q(x))$ at $x = 0, 1, 2, 3, 4, 5$

x	0	1	2	3	4	5
$r(x)$						

(b) Complete the table of values for the composite function $s(x) = q(p(x))$ at $x = 0, 1, 2, 3, 4, 5$.

x	0	1	2	3	4	5
$s(x)$						

Figure 5. A representative function composition problem from WeBWork OPL; 190 of 192 problems tagged as Precalculus Function Composition can be classified as procedural practice.

Each WeBWork assignment consisted of about 10–20 problems. Each assignment was due the morning after the corresponding lecture. Students were allowed 5 attempts on each problem, and only the answer on

A ball is thrown into a lake, creating a circular ripple whose radius travels outward at a speed of 7 cm per second.

a) Complete the following table. Round your answers to the nearest hundredth.

Number of seconds elapsed since pebble hit the water, t	Radius of the ripple in centimeters, r	Area of the ripple in square centimeters, A
0		
1		
2.5		
5		

b) What is the radius of the ripple, r, in centimeters, after t seconds have elapsed since the ball hit the water?

$r =$

c) Define a function f that determines the area of the ripple, A, in square centimeters, in terms of the number of seconds that have elapsed since the ball hit the water, t?

$f(t) =$

Figure 6. A function composition problem from Pathways WeBWork where students are asked to create a composition of functions.

the very last attempt was graded. These WeBWork problems allowed students to receive immediate feedback on their conceptual understandings. This was essential to support the implementation of the Pathways curriculum in the large lecture format.

7 Assessment

The main goal of this investigation was to determine if the Pathways curriculum could be successfully adapted to a large-lecture format. In particular, how well would PP prepare students for Calculus I? Would its performance be comparable to smaller section implementations of Pathways? How would its performance compare to CA and TAG?

In order to answer these questions, the team took the following steps.

(1 point) **local/Pathways/M3/I4/P02.pg**
Suppose function a determines the number of hours of sleep Susie gets in terms of the temperature, in degrees Fahrenheit, of her bedroom. Another function b determines the temperature of Susieâs room, in degrees Fahrenheit, in terms of the number of days since January 1, 2014.

a) What does 5 **represent** in the expression $a(b(5))$? Select the best answer.

- **A.** It is 5 am.
- **B.** Susie slept for 5 hours.
- **C.** 5 is the number you will plug into the rule for b.
- **D.** 5 days have elapsed since January 1, 2014.
- **E.** None of the above

b) What does $b(5)$ **represent** in the expression $a(b(5))$? Select the best answer.

- **A.** The temperature, in degrees Fahrenheit, of Susieâs room on Jan 6, 2014
- **B.** The value of 5 is going to be plugged in to the rule for b.
- **C.** The number of hours of sleep Susie gets
- **D.** The number of hours since 6:00am
- **E.** None of the above

Figure 7. A function composition problem from Pathways WeBWork where students are asked to think about the meaning of the function composition notation.

- The Precalculus Concept Assessment (PCA) was administered as a pre-test and post-test in PP, TAG, and CA. The change in the average score was used as a measure of how much better prepared students were for Calculus I than at the beginning of the semester. The average post-test score was used as a measure of how well they were prepared for Calculus I.
- The pass rates in Calculus I of students who took PP were compared with those who took TAG. [4]
- The pass rates in PP were compared with those in TAG and CA.

In the subsections below, we discuss the details of how these steps were carried out and the interpretation of the results. However, we first

[4]Students who took CA were required to take TAG before calculus.

review the development and validation of the PCA.

7.1 What is the PCA?

The PCA is a 25-item multiple-choice instrument that has been validated to assess student learning in precalculus and readiness for calculus. The PCA items assess student understanding of key ideas of precalculus that have been documented to be foundational for learning calculus ([7],[5],[2], [14],[20],[24], [11]). The PCA items are linked to a taxonomy of the reasoning abilities, understandings, and computational abilities that research has identified as essential for student learning and understanding of ideas of limit, derivative, accumulation, and related rate problems in calculus. These foundational ideas include ideas of function, function composition, function inverse, rate of change, and exponential growth, as well as the ability to define and reference function relationships using function notation and interpret the meaning conveyed by a function's graph. The multiple-choice options for each PCA item are linked to common student responses and ways of thinking that were revealed in interviews with students ([7], [5]). There are 25 questions on the PCA. Over 77% of the students who received a 13 on the PCA received a grade of C or better in beginning calculus, while over 60% of the students who received a score of 11 or below on the PCA withdrew or received a grade of D or lower in beginning calculus. PCA has been used to assess student learning at over 40 universities and colleges to assess various traditional and reform efforts, with the highest non-Pathways PCA mean score recorded of 10.2 when administered to 902 students taking precalculus using a conceptually focused precalculus textbook from a commercial publisher.

7.2 Results of pre-test and post-test administration

In Fall 2013, the mathematics department offered two large lecture sections of TAG and one large section of PP. One of the instructors taught both a TAG and the PP section in classes sizes of about 200 students. In each of these sections, the PCA was administered on the first day of classes as a pre-test, and again during the last week of classes as a post-test. In Spring 2014, the PCA was similarly administered as a pre-test and post-test in CA which consisted of two large-lecture sections of about 160 and 290 students each. The results of these exams are summarized in Table 1.

Course	**pre-test Average**	**Post-test Average**	**Δ**
CA	6.23	8.35	2.12
TAG	7.17	8.94	1.77
PP	8.08	15.11	7.03

Table 1. PCA pre-test and post-test results (out of 25)

The average gain in PCA score by students enrolled in PP was about 5 points higher than in CA or TAG. More significantly, the average post-test score in TAG was significantly less than the threshold of 13 that indicates preparedness for calculus whereas the average post-test score in PP exceeded this threshold. In addition, the gains in PP were significantly higher than the gains in CA although CA and PP covered several of the same topics (function, composition, function inverse, average rate of change)

7.3 Pass rates in Calculus I

Effectiveness was also tested by examining student success in first-semester calculus. The pass rate in first-semester calculus of the students who took PP was compared with that of the students who took TAG. The results are summarized in Table 2. The overall pass rate in this class

Fall 2013 Course	Pass Rate in Calculus I (C- or better)
TAG	53%
PP	61 %
All students	59%

Table 2. Comparison of pass rates in first-semester Calculus

was fairly low. At the same time, the pass rate of the students who took PP in the Fall was 8 percentage points higher than those who took TAG in the Fall.

7.4 DFW rates

The authors compared the DFW rates for TAG, CA, and PP in Fall 2013. The results are summarized in Table 3.

Fall 2013 Course	DFW rate
TAG	33.1%
CA	35.1 %
PP	26.8 %

Table 3. Comparison of DFW rates

7.5 Interpretation of results

After examination of these data, the development team concluded that PP prepared students for Calculus I; in particular it better prepared students for Calculus I than either CA or TAG. In particular, although there was considerable overlap in the topics covered by CA and PP (functions, function composition, inverse functions), the learning gains in CA were very small compared with PP, as measured by the PCA.

The significantly lower learning gains in CA and TAG could be explained in part by the somewhat weaker mathematical abilities of the students therein as evidenced by the lower pre-test scores. However, as the pre-test scores in TAG were not significantly lower than the pre-test scores in PP, it seems unlikely that these weak gains could be explained entirely - or even largely - by the weaker abilities of the students at the start of the semester. Since there was considerable overlap in the topics covered by CA and PP, our data supports that the larger gains in student learning could be attributed to the PP curriculum that supports student engagement and student learning of fundamental precalculus concepts. We note that whereas the students in PP were almost entirely STEM majors, the populations of TAG and CA were more heterogeneous. It is possible that non-STEM majors were less motivated in these classes than the STEM majors. However, we are skeptical that this could explain a significant part of the differences in performance.

Since Fall 2013, improvements have been made to the PP group projects, clicker questions, homework assignments, and exams. During this same time period DFW rates in PP continued to decline (in one semester only 19% of students received a D,F, or W) and mean learning gains of PP students on the PCA have remained in the 6-7 point range. The Iowa State gains on PCA and pass rate are comparable to those of Arizona State University small sections of Pathways Precalculus, and better than other universities using Pathways in a precalculus course for non-STEM majors.

8 Conclusion

We have described how we adapted the small-class intervention of Pathways materials to a large class environment. Our adaptations include recitations in which students complete and discuss key ideas from Pathways investigation questions prior to reviewing them in the 200 student lecture. The large lecture clicker questions gives students opportunities to discuss their thinking with a peer, while holding them accountable for making sense of the ideas and discussions led by the course instructor. The online homework sets are assigned after each class meeting and focus on student understanding and applying the main ideas from the recitation and lecture. We observed that each of these implementation features supported student engagement in productive reasoning and led to students acquiring improved understandings of ideas that have been repeatedly documented to be essential for student learning and success. Our collective evidence suggests the learning gains and student success is a result of the research based scaffolding that is intentional about targeting specific reasoning abilities that are critical for understanding the course's central ideas. Specifically, the structural support we provide for meaningful engagement using clicker questions is effective in supporting all students to engage with the ideas being addressed during the large lecture. The conceptually focused online homework that is linked to the recitations and lecture supports students in developing essential fluency in reasoning productively and using their understandings to solve problems. The tasks in the investigations, clicker questions, and homework are research based and engineered to support student learning and fluency with ideas and skills.

As noted in the introduction, although the large-lecture may be far from an ideal learning environment, it is here to stay and will be the format in which many large public universities must offer precalculus.

As shown by Sonnert and Sadler [35], university precalculus courses do not improve students ability to pass Calculus I. It makes sense then to explore ways in which students can learn precalculus mathematics better in the large lecture section. We believe our approach offers a path by which universities could do so.

9 Acknowledgements

We thank the referees for many helpful comments. Timothy McNicholl's work on this project was funded in part by Simons Foundation Grant #317870. Marilyn Carlson's work on this project was funded in part by NSF grant # 1323753.

REFERENCES

[1] D. M. Bressoud, *Should students be allowed to vote?*, Launchings (2009), Available at http://www.maa.org/columns/launchings/launchings_03_09.html.

[2] M. Carlson, S. Jacobs, E. Coe, S. Larsen, and E. Hsu, *Applying covariational reasoning while modeling dynamic events: A framework and study*, Journal for Research in Mathematics Education **33** (2002), 352 – 378.

[3] M. Carlson, S. Larsen, and R. Lesh, *Integrating models and modeling perspective with existing research and practice*, Beyond constructivism: A models and modeling perspective (R. Lesh and H. Doerr, eds.), Lawrence Erlbaum Associates, Mahwah, New Jersey USA, 2003, pp. 465–478.

[4] M. Carlson and M. Oehrtman, *Precalculus: pathways to calculus-student workbook*, Rational Reasoning, Arizona, USA, 2013.

[5] M. Carlson, M. Oehrtman, and N. Engelke, *The precalculus concept assessment: a tool for assessing students' reasoning abilities and understandings*, Cognition and Instruction **28** (2010), no. 2, 113 – 145.

[6] Marilyn Carlson, *Obstacles for college algebra students in understanding functions: What do high performing students really know?*, AMATYC Review **19** (1997), no. 1, 48–59.

[7] ______, *A cross-sectional investigation of the development of the function concept*, Research in collegiate mathematics education III (E. Dubinsky, A.H. Schoenfeld, and J. Kaput, eds.), American Mathematical Society, Providence, R.I., 1998, pp. 114 – 162.

[8] Marilyn P. Carlson, Sean Larsen, and Sally Jacobs, *An investigation of covariational reasoning and its role in learning the concepts of limit and accumulation*, Proceedings of the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, 2001, pp. 517–523.

[9] Kelly S. Cline and Holly Zullo, *Teaching mathematics with classroom voting: With and without clickers*, Mathematical Association of America, 2011.

[10] E. Connally, D. Hughes-Hallet, A.M. Gleason, F. Avenoso, P. Cheifetz, and A. Pasquale, *Functions modeling change: A preparation for calculus*, 1 ed., John Wiley & Sons, Inc., New York, NY, 2000.

[11] I. Cottrill, E. Dubinsky, D. Nichols, K. Schwingendorf, K. Thomas, and D. Vidakovic, *Understanding the limit concept: Beginning with a coordinated process schema*, Journal of Mathematical Behavior **15** (1996), no. 2, 167–192.

[12] C.H. Crouch and E. Mazur, *Peer instruction: Ten years of experience and results*, American Journal of Physics **69** (2001), no. 9, 970–977.

[13] D.Tall and S. Vinner, *Concept image and concept definition in mathematics with particular reference to limits and continuity*, Educational Studies in Mathematics **12** (1981), no. 2, 151–169.

[14] E. Dubinsky and I. Cottrill, *The nature of the process conception of function*, The concept of function: Aspects of epistemology and pedagogy (G. Harel and E. Dubinsky, eds.), MAA Notes, vol. 25, Mathematical Association of America, Washington, D.C., 1992, pp. 85 – 106.

[15] N. Engelke, *Developing the solution process for related rates problems using computer simulations*, Proceedings of the 11th Annual Research in Undergraduate Mathematics Education Conference (San Diego, CA), 2008.

[16] N. Engelke, M. Carlson, and M. Oehrtman, *Composition of functions: Precalculus students' understandings*, Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Eugene, OR) (G.M. Lloyd, M.R. Wilson, J.L.M. Wilkins, and S.L. Behm, eds.), All Academic, 2005.

[17] K. Frank, *Examining the development of students' covariational reasoning in the context of graphing*, Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University, 2017.

[18] S. Freeman, S.L. Eddy, M. McDonough, M.K. Smith, N. Okoroafor, and H. Jordt, *Active learning increases student performance in sci-*

ence, engineering, and mathematics, Proceedings of the National Academy of Sciences **111** (2014), no. 23, 8410–8415.

[19] S. Jacobs, *Advanced placement bc calculus students' ways of thinking about variable*, Ph.D. dissertation, Arizona State University, Tempe, Arizona, 2002.

[20] I. Kaput, *Patterns in students' formalization of quantitative patterns*, The concept of function: Aspects of epistemology and pedagogy (G. Harel and E. Dubinsky, eds.), MAA Notes, vol. 25, Mathematical Association of America, Washington, D.C., 1992, pp. 290 – 318.

[21] E. Kuper, *Sparky the saguaro: Teaching experiments examining students' development of the idea of logarithm*, Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University, 2019.

[22] Eric Mazur, *Farewell, lecture?*, Science **323** (2009), no. 50, 50–51.

[23] L.K. Michaelson, A.B. Knight, and L. D. Fink, *Team-based learning: A transformational use of small groups in college teaching*, Stylus Publishing, LLC, Sterling, VA, 2004.

[24] S. Monk, *Students' understanding of a function given by a physical model*, The concept of function: Aspects of epistemology and pedagogy (G. Harel and E. Dubinsky, eds.), MAA Notes, vol. 25, Mathematical Association of America, Washington, D.C., 1992, pp. 175–193.

[25] K. Moore, *The role of quantitative reasoning in precalculus students learning central concepts of trigonometry*, Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University., 2010.

[26] K.C. Moore and M.P. Carlson, *Students' images of problem contexts when solving applied problems*, Journal of Mathematical Behavior **31** (2012), no. 1, 48 – 59.

[27] A. O'Bryan, *Exponential growth and online learning environments: Designing for and studying the development of student meanings in online courses*, Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University, 2018.

[28] President's Council of Advisors on Science and Technology, *Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics*, Available at http://www.whitehouse.gov/administration/eop/ostp/pcast/docsreports, 2012.

[29] National Academy of Sciences, *The Mathematical Sciences in 2025*, The National Academies Press, 2013.

[30] N.W. Reay, P. Li, and L. Bao, *Testing a new voting machine question methodology*, American Journal of Physics **76** (2008), no. 2, 171 – 178.

[31] O. Schak, I. Metzger, J. Bass, C. McCann, and J. English, *Developmental education challenges and strategies for reform*, Tech. report, U.S. Department of Education, 2017.

[32] A. Sierpinska, *On understanding the notion of function*, The concept of function: Aspects of epistemology and pedagogy (G. Harel and E. Dubinsky, eds.), MAA Notes, vol. 25, Mathematical Association of America, Washington, D.C., 1992, pp. 25 – 58.

[33] M.K. Smith, W.B. Wood, W.K. Adams, C. Wieman, J.K. Knight, N. Guild, and T.T. Su, *Why peer discussion improves student*

performance on in-class concept questions, Science **323** (2009), no. 5910, 122 – 124.

[34] N. Smith, *Students' emergent conceptions of the fundamental theorem of calculus*, Ph.D. dissertation, Arizona State University, Tempe, Arizona, 2008.

[35] G. Sonnert and P. Sadler, *The impact of taking a college precalculus course on students' college calculus performance*, International Journal of Mathematical Education in Science and Technology **45** (2014), no. 8, 1188–1207.

[36] A. Strom, *A case study of a secondary mathematics teacher's understanding of exponential function: An emerging theoretical framework*, Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University, 2018.

[37] S. Vinner and T. Dreyfus, *Images and definitions for the concept of a function*, Journal for Research in Mathematics Education **20** (1989), 356 – 366.

[38] P. White and M. Mitchelmore, *Conceptual knowledge in introductory calculus*, Journal for Research in Mathematics Education **27** (1996), 79–95.

[39] S.R. Williams, *Models of limit held by college calculus students*, Research in Mathematics Education **22** (1991), no. 3, 219 – 236.