You Can't Use What You Don't See: Quantitative Reasoning in Applied Contexts

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Abstract

At the heart of students' success with producing and utilizing mathematical models is the ability to conceptualize quantities in a situation and develop methods for communicating their ideas about how quantities in a situation are changing in tandem. We often underestimate how much effort it may require for students to develop productive conceptualizations of situations that allow them to make sense of solutions we present or empower them to produce and justify non-standard lines of reasoning. This article describes activities we use with students and teachers to highlight 1) the importance of providing ample time for individuals to explore and make sense of the contexts we want them to model, and 2) to realize that individuals cannot reason about quantities in a situation that they have not conceptualized. Both ideas have important implications for us as educators.

In our work to support student success in modeling applied problems in algebra and pre-calculus courses, we have repeatedly found that both teachers and students need more support in learning to visualize and conceptualize quantities (Moore & Carlson, 2012; O'Bryan & Carlson, 2016). These observations led us to design curriculum materials and professional development training programs aimed at providing such support.

Example 1:

One of the contexts that helped reveal sources of student difficulties is shown in Figure 1.

> Lisa and Sarah are 80 feet apart when they spot each other. They begin walking toward each other. As they do, whenever Lisa travels some distance Sarah travels the same distance.

Figure 1. A task that yields insights about student reasoning.

We often ask students to describe their understanding of the situation and generate models (e.g., drawings, applet constructions) relating the values of co-varying quantities. Students who succeed in the latter task tend to exhibit similar tendencies.

1. They drew diagrams and clearly identified quantities they imagined as varying and those they imagined as fixed. See Figure 2.

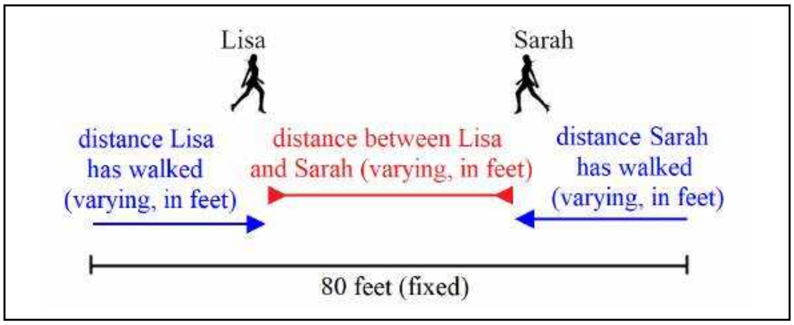

Figure 2. A diagram students may draw to reason about the context from Figure 1.

2. They included *the distance between Lisa and Sarah* as an important quantity to track and represent.

3. They described algebraic expressions as representing the values of quantities (not just as formulas for determining "answers"). For example, if x represents the varying distance (in feet) Lisa has walked and d represents the varying distance (in feet) between Lisa and Sarah, then a problem solver may write $d = 80 - 2x$ or $2x + d = 80$. In both instances it is productive for students to conceptualize the variables x and d as representing values of quantities that vary together. It is also productive for students to attach quantitative meaning in the expressions they write, such as describing $2x$ as the combined distance Lisa and Sarah have moved toward each other, and $2x + d$ as one way of representing the original 80-foot distance between Lisa and Sarah.

Alternatively, students who struggled to model relationships in this context did not talk about varying quantities or describe the constraints on their co-variation and often overlooked *distance between Lisa and Sarah* as a quantity they could track. This is an example of the necessity for students to first conceptualize the varying and fixed quantities in a problem context as a foundation for representing the relationship between two varying quantities using algebraic symbols.

Example 2:

A more complex example is shown in Figure 3. We have used this context both with secondary students and in professional development training sessions with in-service teachers.

> At 4:30 pm Bill drives from work in his car at a constant speed of 45 miles per hour. Following the same path as Bill, Maddox leaves work in his car twenty minutes later at a constant speed of 60 miles an hour. How long will it take for Maddox to catch Bill?

Figure 3. Task designed by Madison, Carlson, Oehrtman, and Tallman (2015).

"Catch-up" tasks like this are common in algebra courses (including assuming unrealistic aspects of the situation such as two cars maintaining constant speeds for long periods of time). However, Madison et al. (2015) and our own research show that only 8 to 19 percent of students can correctly solve the problem at the end of their Algebra II, College Algebra, or Pre-calculus course. Note: The most common incorrect answer is 15 minutes. The correct answer is either 1 hour since Maddox left work or 1 hour and 20 minutes since Bill left work.

There are several subtle issues in reasoning about this context. For example, someone writing a system of equations to model each driver's distance from work must decide whether to measure time elapsed since Bill left work or since Maddox left work. That person also needs to deal with inconsistencies in units given (speeds are given in miles per hour, yet Bill's head start is given in minutes). Since students struggled with this context, we were curious how teachers would answer this question, the sources of challenges they encounter, and what their work can tell us about supporting productive reasoning about applied contexts.

Using the Task in a Professional Development Workshop

The first time we used this task in a professional development setting, we asked the 10 teachers to begin by drawing a diagram of the situation and labeling **all** of the quantities they could imagine measuring and tracking. Two examples of teachers' work appear in Figures 4 and 5. It is noteworthy that most secondary teachers required 20 to 45 minutes to complete the task and create solutions they could justify, highlighting its corresponding challenge level for secondary students.

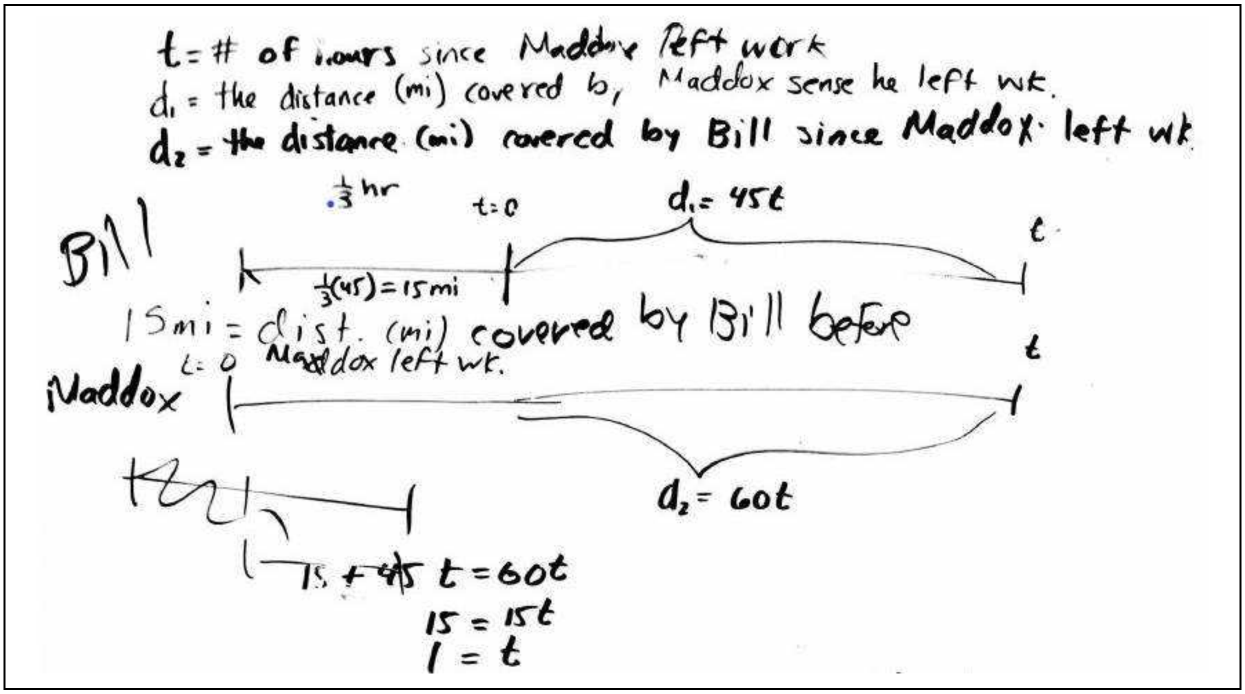

Figure 4. First group’s solution.

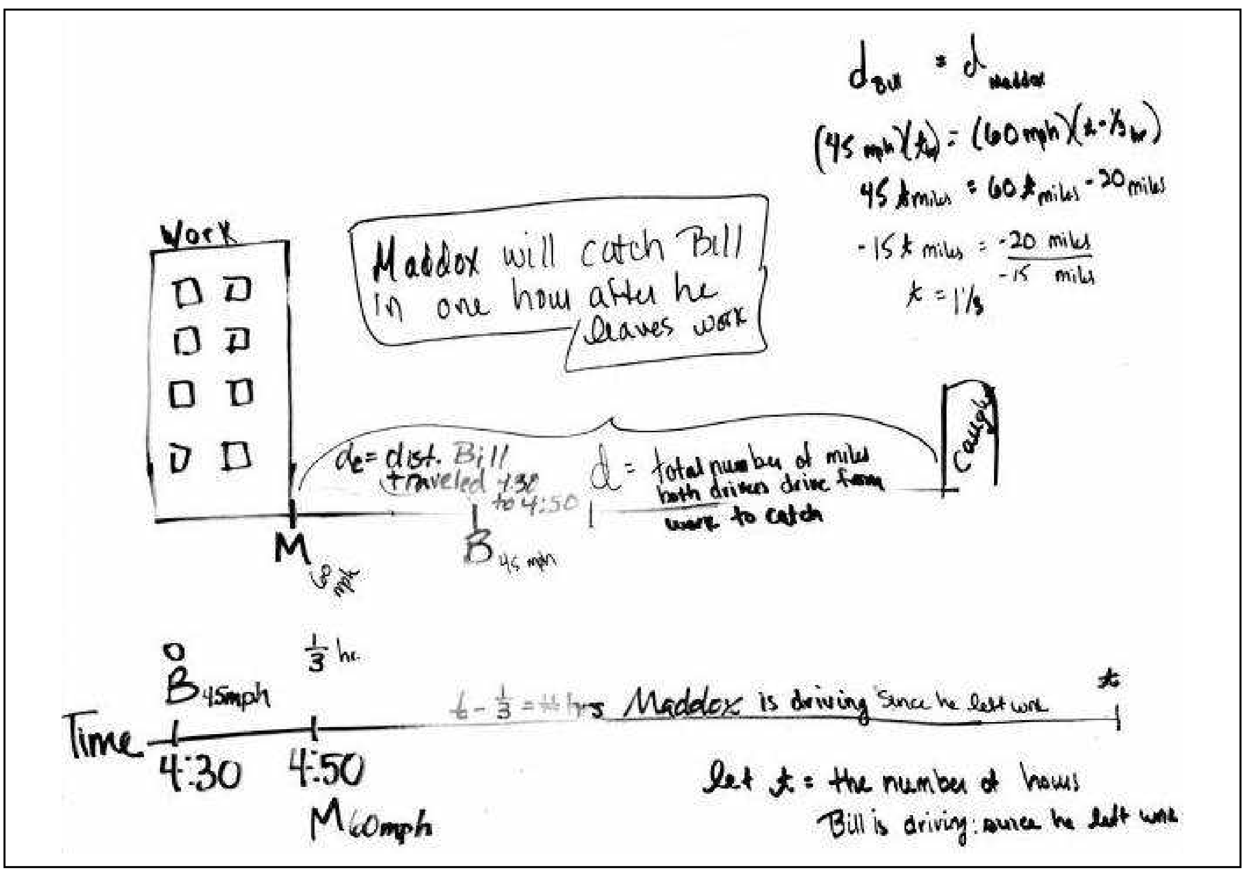

Figure 5. Second group’s solution.

These examples show teachers opting to track time elapsed differently (number of hours since Maddox left work in Figure 4 and number of hours since Bill left work in Figure 5). Of interest is that the *distance between Maddox and Bill* was not a quantity identified by teachers as something they could track, nor was it a quantity that any teacher mentioned in oral explanations. I suspected that this was due partly to the static nature of their work with the context, so I thought about how I might modify this activity in a future workshop.

Using the Task in a Second Professional Development Workshop

I used the context and activity described above in a second professional development setting with 18 teachers. Teachers' solution methods were very similar, and again none of the teachers identified *distance between Maddox and Bill* as a quantity to track either in their diagrams or in their explanations. Thus, after their presentations I led a group activity where we collectively programmed a GeoGebra (2020) applet as a dynamic model of the situation. (See Figure 6.) As a user moves the slider value, Maddox and Bill move in tandem.

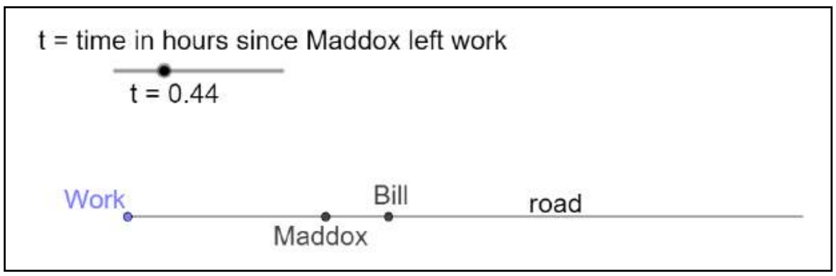

Figure 6. A screenshot from GeoGebra applet allowing.

Once the applet was complete, a conversation similar to the following occurred.
Instructor: What do you notice as I move the slider value to simulate time elapsing?
Teachers: Maddox is catching up to Bill.
Instructor: What do you mean that Maddox is catching up to Bill?
Teachers: The distance between them is getting smaller.

This is noteworthy because in two sessions involving 28 teachers, including all groups presenting their solutions, the idea that the *distance between Maddox and Bill* was decreasing as time elapsed was never mentioned. The conversation continued.

Instructor: How quickly is the distance between them changing?
Teacher 1: 15 miles per hour.
Instructor: Why?
Teacher 1: That's how much faster Maddox is driving.
Instructor: What was Bill's lead when Maddox left work?
Teacher 2: 15 miles.
Instructor: How do you know that?
Teacher 2: Bill drove 20 minutes, or a third of an hour, at 45 miles per hour. That's 15 miles.
Instructor: So how long did it take Maddox to catch Bill?
Teachers: 1 hour.

Some of the group's whiteboard solutions revealed that *the reasoning in the previous conversation was incorporated within the flow of the most common algebraic solution method presented*. See Figure 7.

t = hours since Maddox left work

$$\begin{array}{rcl} 60t & = & 45t+15 \\ -45t & & -45t \\ & 15t=15 & \\ & t=1 & \end{array}$$

Figure 7. One group's solution.

"$15t = 15$" is a representation of the reasoning in the previous conversation. "15" in "$15t$" is the difference in Bill's and Maddox's speeds ("$15t$" represents how much the distance between them is reduced t hours since Maddox left work), and "15" on the right side of the equation represents Bill's lead when Maddox leaves work (15 miles). Solving this equation is answering the question, "If Maddox is driving 15 mph faster than Bill, how many hours will it take for Maddox to make up Bill's 15-mile head start?" However, this became clear to the teachers only after working with the applet. In group presentations of their solutions, teachers only described the contextual meaning of the first and last steps in their algebraic work. The middle steps were "context-free" algebraic processes.

I next presented the tasks in Figure 8, and asked if the teachers could solve the tasks without recording anything, but rather, relying on the reasoning that emerged from working with the GeoGebra app.

1. At a hot dog stand hot dogs are sold for \$5 each. The vendor pays \$1.50 for each hot dog, and rents the space for the cart for \$210/day. How many hot dogs must he sell each day to break even?

2. Solve the system of equations: $y = 6x + 10$ and $y = 2x + 30$.

Figure 8. Two new tasks.

This group of teachers had no difficulty reasoning as follows:

1. Each hot dog sold makes up \$3.50 of the initial daily rental cost, so the vendor needs to sell enough hot dogs to pay off the \$210 rental fee. That is 210/3.50 hot dogs.
2. When $x = 0$, the second function has a value 20 units greater than the first function. As x changes, the first function's value increases by 4 units more per 1-unit change in x as compared to the second function. If x changes to $x = 5$, the first function will "overcome" the initial 20-unit difference in value, and the functions values will match.

Teachers claimed that this was a new way of looking at this class of problems and that they had never promoted this type of reasoning although *the standard algebraic solutions essentially incorporate this reasoning within the solution steps*. (See Figure 9.)

$5n = 1.50n + 210$	$6x + 10 = 2x + 30$
$3.50n = 210$	$4x = 20$
$n = \frac{210}{3.50}$	$x = 5$
60 hot dogs sold (at a cost/revenue of \$300)	$(5, 40)$

Figure 9. Common algebraic solution methods for the tasks from Figure 8. (Note that for the first solution, n represents the number of hot dogs sold on a given day.)

Discussion

In presenting these vignettes I am not suggesting that teachers change how they teach topics like solving systems of equations. But it is rare for teachers (or their students) to recognize differences in function values and differences in rates of change as quantities they can track, and if they do not identify these quantities, they cannot include them in their reasoning or justification processes. This suggests important lessons for us as educators.

- It can take students a long time to productively conceptualize the contexts we present (far longer, say, than a teacher might spend in reviewing homework problems).
- When students have time to explore relationships within contexts and interact with various representations of those relationships, they are better equipped to reason productively about standard and non-standard solution approaches.
- When we explain solution methods in class, we often present a clean, concise, "final draft" solution. Students who require more time to understand the context will struggle to follow our reasoning and may internalize unproductive beliefs such as, "If I don't immediately know how to solve a problem, I must be bad at math."

Some Implications for Teaching

In addition to using the contexts shared in this article to facilitate discussions with your students, there are several general takeaways that all teachers can apply in their instruction.

- Allow more time for students to explore contexts and explain their understanding of those contexts before you ask them to provide solutions to specific questions and explain their reasoning. A good way to facilitate this is to initially remove the "question" from the end of a word problem prompt. Just present the scenario and ask your students to explore and describe their understanding and any relationships they can identify. You can then ask your students to generate a list of questions they could ask and answer using data and relations presented and what reasoning would support those answers.
- Since it takes considerable time for students to productively conceptualize a problem context, consider developing a small set of "iconic word problems" for your course that are rich and deep enough to be reused when introducing multiple topics throughout your course. Any time you invest in helping students make sense of the contexts early in the course will pay off in future lessons without sacrificing the quality and depth of mathematical discourse.

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