Quantitative Reasoning and Symbolization Activity: Do Individuals Expect Calculations and Expressions to Have Quantitative Significance?

Alan E. O'Bryan
Arizona State University

This paper describes ways of thinking students employ when they choose to use calculations or produce algebraic expressions to respond to mathematical tasks and their expectations regarding the meanings of what they produce. My findings suggest that students' reasoning in symbolization activity is often guided by perceptual features of tasks, such as the numbers explicitly given in prompts and key words students identify. I describe the construct "emergent symbolization" as a potentially productive way of thinking in symbolization activity based on synthesizing prior work and the results of this study. I close by making connections between similar work in analyzing students' ways of thinking about graphs and discussing how my work contributes to the common instructional goal of promoting connections between representations.

Keywords: quantitative reasoning, student thinking, mathematical representations

Math standards commonly include the goal that students make connections between representations of mathematical relationships. For example, The American Association of Two-Year Colleges (2018) standards state that "Students should be provided opportunities to represent and communicate mathematical ideas using multiple representations such as numerical, graphical, symbolic, and verbal" (p. 25). However, intending that students make such connections does not guarantee they occur. Whether a student is successful in recognizing what a representation conveys about how two quantities are related and changing together relies on such things as the ways of thinking she engages in and the connections she is positioned to make.

Moore and Thompson (2015) studied students' ways of thinking about graphs and described advances derived from taking seriously how students interpret the meaning of graphs. Their work highlights complexities in students' reasoning about one kind of representation (graphs) and suggests potential benefits for students who envision graphs as emerging from coordinating pairs of covarying quantities they have conceptualized for a given context. In this paper I report results from studying students' reasoning about symbolization activity (the production of mathematical expressions and formulas that use operations and algebra to represent relationships). My results suggest that a key component of students' fluency and flexibility in representing relationships algebraically is also the degree to which they conceptualize quantities and quantitative relationships in a situation and foreground their activity based on these conceptualizations.

It seems likely that students inclined to engage in quantitative reasoning (Thompson, 1990, 1993, 1994, 2011), with a focus on coordinating relationships between quantities including using variables and variable expressions to represent the values of quantities, are well-positioned to make productive connections between mathematical representations. However, testing this hypothesis, and understanding the mechanisms by which students successfully make connections between representations, requires continued empirical studies designed to recognize and articulate students' ways of thinking about representations. This paper contributes to that work.

Theoretical Perspective

Quantitative reasoning (Thompson, 1990, 1993, 1994, 2011) describes a way of reasoning involving identifying objects' measurable attributes (*quantities*) and conceptualizing relationships between quantities. *Quantification* is the process of deciding on a measurement system for producing quantities' values. Quantities exist within the mind of an individual, and thus when talking about quantities, quantitative relationships, quantification, and so on, I take the position of describing them from an individual's perspective (either actual or hypothetical). As a lens for considering students' meanings and reasoning, I use Thompson and Harel's (Thompson, Carlson, Byerley, & Hatfield, 2014) descriptions that are extensions and elaboration of Piaget's (2001) genetic epistemology. An individual's *meaning* for an idea describes the set of implications inherent to the scheme(s) triggered by an in-the-moment *understanding* of a context or stimulus. Meanings may be temporary or they may be stable over time. A *way of thinking* describes a pattern in how an individual uses certain meanings. Descriptions of students' ways of thinking are useful for characterizing mental processes essential for specific types of mathematical reasoning (e.g., quantitative reasoning, proportional reasoning).

Emergent Symbolization

In O'Bryan and Carlson (2016) we described our work with a teacher (Tracy) involving a professional development intervention, classroom observations, and clinical interviews. Our analysis suggested a set of expectations she developed about the meanings for the calculations she performed and algebraic representations she generated that she independently established as mathematical learning goals for her students. These expectations included explaining what quantity she intended to calculate (represent) before performing a calculation (writing an expression) and how the order in which calculations are performed or the order of operations used to evaluate expressions reflects details about how individuals conceptualize a situation.

I have since encapsulated aspects of my understanding of Tracy's expectations and ways of thinking into the construct *emergent symbolization (or emergent symbol meaning*) (O'Bryan, 2018). Emergent symbolization refers to actions motivated by one or more of the following expectations.

1. An expectation that performing calculations or generating expressions should reflect a quantification process for quantities that the individual conceptualizes.
2. An expectation that demonstrating calculations and producing expressions are attempts to communicate an individual's meanings. Thus, when given a set of calculations or expression/formula, we can hypothesize how the individual conceptualized a situation based on analyzing the products of their reasoning.
3. An expectation that the order of operations used to perform calculations, evaluate expressions, and solve equations "reflects the hierarchy of quantities within a conceptualized quantitative structure" (O'Bryan, 2018, p. 234).

I hypothesize that emergent symbolization is a potentially productive way of thinking for students and teachers and that targeting its development is a worthwhile instructional goal. Part of the important initial work in testing this hypothesis is describing nuances in how students reason about, and their expectations regarding, performing calculations or generating algebraic expressions in response to mathematical contexts and prompts.

Research Questions

Given that students develop stable ways of thinking over time, understanding nuances in these ways of thinking is important for educators and researchers. The work reported here focuses on results from considering two research questions.

1. When a student chooses to perform a series of calculations or generate an algebraic model to respond to a task, what does the student believe her work represents?

2. Do students' ways of thinking reflect an attempt to organize mathematical processes according to generalizations of quantitative relationships or is there a different set of expectations driving their mathematical activity?

Methods

During Spring 2018 seventy students enrolled in a precalculus course at a large public university in the United States using our research-based online materials (O'Bryan & Carlson, 2016). All students completed a pre-/post-test including multiple-choice items from the APCR and CCR exams (Madison, Carlson, Oehrtman, & Tallman, 2015) at the beginning and end of the course. In addition, I selected five students (Gina, Lisa, John, Marcus, and Shelby) for recorded clinical interviews (Clement, 2000) representing a wide range of majors and pre-test scores. I conducted interviews with each student at the beginning of the course and during the last month of the course using the same subset of questions from the pre-/post-test and the same interview protocols. During these interviews the multiple-choice options were hidden. I coded student responses to identify when students did or did not reference quantities, units, and quantitative relationships as a foundation for their calculations and symbolization activity.

Results

I share the following results from pre-/post tests and interviews, and to highlight common expectations and ways of thinking I observed. In this paper I am not focusing on shifts that may have resulted during the course, but rather on characterizing stable meanings and ways of thinking students exhibited.

Pre-Test and Pre-Interview Results

Students answered the Tomato Plant A task in Figure 1 on the test and during interviews. I chose this task to assess students' recognition of the need to update the reference quantity from one interval to the next (that is, update the answer to the question, "Percent of what?").

A tomato plant that is 4 inches tall when first planted in a garden grows by 50% each week during the first few weeks after it is planted. How tall is the tomato plant 2 weeks after it was planted?

a. 5 inches b. 6 inches c. 8 inches **d. 9 inches** e. 12 inches

Figure 1. The Tomato Plant A task (Madison et al., 2015).

On the pre-test, 54% of students selected the correct response. During interviews, three of the five students answered the question correctly, and all responses contained two similarities. First, all students focused on the amount to *add* each week and none of them described the relative size of the new height compared to the old height when explaining their reasoning. Second, all students described a percentage value as a number that results from moving a decimal place two positions rather than describing a measurement process. Of the two students who answered

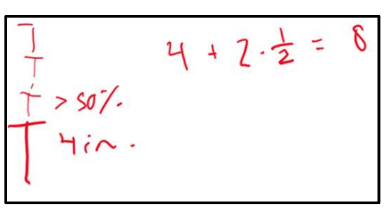

Figure 2. Marcus's work for the Tomato Plant A task.

incorrectly, Marcus' response proved interesting relative to later observations. See Figure 2.

Marcus's answer (8 inches) assumed that the weekly 50% increase was always based on the plant's initial height. But what is most interesting is the expression he used to represent his answer. Apart from being an incorrect statement (which he recognized), all five students utilized the structure "initial value + (change in independent value)(1-unit percent change as a decimal or fraction)" to create a general model in the Tomato Plant B task. See Figure 3.

José plants a 7-inch tomato plant in his garden. The plant grows by about 13% per week for several months. Which formula represents the height *h* of the tomato plant (in inches) as a function of the time *t* in weeks since it was planted?

a. $h = 7(0.13)^t$ b. $h = 7 + 1.13t$ c. $h = 7(1.13t)$ **d. $h = 7(1.13)^t$** e. $h = 7 + 0.13t$

Figure 3. The Tomato Plant B task (Madison et al., 2015).

During the pre-test, 20% of students selected the correct answer, but none of the students interviewed provided the correct response. In their interview responses, all five students generated a formula equivalent to "$h = 7 + 0.13t$" where *h* is the tomato plant's height (in inches) and *t* is the elapsed time (in weeks) since the initial height measurement. The important observation here is not that students failed to provide (or select) the correct algebraic model. Rather, *the most important result is what their reasoning tells us about their expectations regarding the quantitative significance of the calculations they perform and the expressions they write.* Figure 4 shows Shelby's work in completing the Tomato Plant B task.

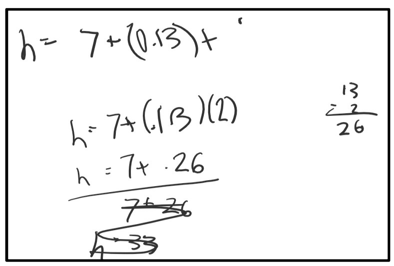

Figure 4. Shelby's work on the Tomato Plant B task.

Shelby: Okay so we're trying to find the height [*she underlines "height h" in the problem statement*] and then t is gonna be a variable [*she underlines "t in" in the problem statement*]. So height [*she writes "h="*] and then we start with seven [*she writes "7"*]. So we start at seven, so that's just gonna keep increasing [*she writes "+" after h = 7*] so and then put a plus. Um [*she writes "(0.13)t" after h=7+*]. Yeah, I'm gonna go with that.

At this point I asked Shelby to determine the plant's height after two weeks. She evaluated her formula but hesitated after writing "7+.26". She considered that maybe her answer was wrong and tried "7+26" based on moving the decimal two places as an alternative. She decided that "33" was too large, so her original answer must be correct and settled on "7.26" as her answer.

In Shelby's final answer we see a form similar to what appeared in Marcus's response to the Tomato Plant A task, and several aspects of her response appeared throughout interviews with all students. First, Shelby was not inclined to check the reasonableness of her solution without prompting. Second, she appeared to be using keywords to guide her work (such as "increase" means "add"). Third, her criteria for judging the accuracy of her solution was based on whether the values it produced seemed reasonable. None of the students explained how parts of their algebraic representations modeled relationships between quantities within the situation or explained the meaning of the term "0.13*t*"

Consistently in the Tomato Plant B task, as well as similar tasks such as modeling a bacteria colony doubling over set time intervals, students appeared to establish a goal and then create a mathematical representation as a literal translation of this goal from English. See Figure 5.

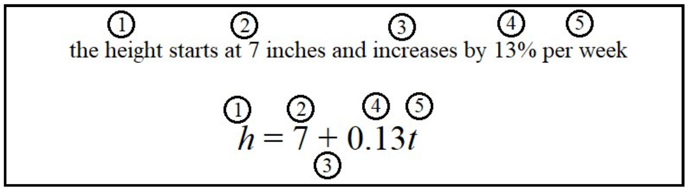

Figure 5. Students may produce the formula h = 7 + 0.13t as a literal translation of a goal statement written in English focusing on key words.

Supporting this idea is the consistent absence in students' responses of any numbers not explicitly stated in task prompts. For example, the correct model for the Tomato Plant B task requires using "1.13", and neither "1.13" nor "113%" appear in the prompt. Similarly, I observed students model a doubling-time situation with a function formula missing the number two when the prompt included the word "double" but did not include "2" in the prompt.

The students interviewed exhibited similar behaviors during symbolization activities throughout the course. Students rarely mentioned units when referencing numbers given in the problem text and rarely referenced the quantities being measured. In addition, students were not inclined to justify their answers unless prompted and, when I did prompt them, judged their models based on whether they produced numerical values the student deemed reasonable rather than referencing the structure of quantitative relationships the student conceptualized.

Post-Test and Post-Interview Results

All five students interviewed successfully completed the class, and all earned at least a "B" on the course final exam. Therefore, based on grades and exam performance, interviewees all met course requirements and objectives. However, data from the post-test, post-interviews, and analysis on students' interactions with course materials shows that the ways of thinking identified in the pre-test and pre-interviews was remarkably stable for many students.

Post-Results for Gina, Marcus, and Shelby. Gina, Marcus, and Shelby all continued to display many of the same ways of thinking during their post-interviews. All three had difficulty explaining how claims they made about features of one task or representation appeared in other tasks or representations and their reasoning often involved generating sets of potential answers then picking from among the results. For example, I asked Gina to determine the percent change from a price of $131 to $195. Gina first evaluated the difference "195 – 131 = 64". She then said, "I kinda remember how to do this" and used a calculator to compute "64/195 = 0.328", then "ans * 100 = 32.8", then "195/64 = 3.047", then "ans * 100 = 304.7", and finally "195/131 = 1.489". After reviewing these results she told me that the price increased by 32.8%. I asked her what that number was a measurement of (what quantities were being compared). Rather than answering my question, she computed "131/195 = 0.672" and said that her answer should have been "67%". "Sixty-seven percent of what?" I asked. She responded "131".

Analyzing these students' behavior during course lessons showed that all three tended to require high numbers of attempts to complete tasks in the lessons I examined. When their initial attempt was incorrect, they required seven or more attempts over one-third of the time, and nearly 20% of those instances they required 11 or more attempts before finally completing or abandoning tasks. For example, one lesson task asked students to determine a percent change from one value of a quantity to another value. The correct answer was 40%, but Shelby entered 140%. She then completed *19 additional attempts* in quick succession (240, 1.40, 139, 239, 299, 399, 499, 199, 240, 1.4, 39.9, 29.9, 49.9, 2, 200, 100, 140, 14, and 40) before finally entering the correct answer. In these instances the students were not coordinating their reasoning by first conceptualizing the quantities in a situation and the quantitative relationship they wanted to communicate and then considering how to represent or evaluate that relationship.

Finally, all three students tended to revert to the general form "$a + rt$" when completing tasks like Tomato Plant B. While Marcus and Shelby often caught themselves and modified their answers to exponential models, their continued challenges in justifying aspects of their models quantitatively suggest that this might be "pattern-matching" behavior rather than shifts in their expectations for the meaning and goals of generating algebraic representations.

The case of John and Lisa. Both John and Lisa exhibited shifts in describing the quantitative significance for the calculations they performed and the representations they generated. Since the focus of my paper is not on analyzing the results of a teaching intervention, I will not dwell on these changes beyond demonstrating some differences I observed in their reasoning and expectations during the post-test and post-interviews.

Figure 6 shows Lisa's work on the Tomato Plant B task during the post-interview.

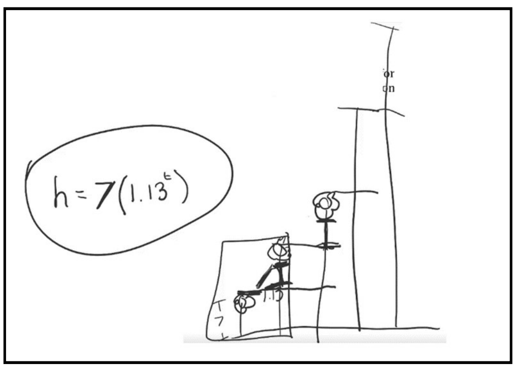

Figure 6. Lisa's work on the Tomato Plant B task (post-interview).

In justifying her work, Lisa provided a clear explanation for the meaning of each value in her model and supported her explanations with drawings where she elaborated on how she imagined the relationship between *h* and *t*, as modeled by the formula, also being modeled using diagrams.

An example from John's post-interview is also worth sharing. In the Tomato Plant A task John was the only student who spontaneously represented the plant height as an exponential expression rather than calculating the amount to add each week and performing addition. He wrote "y = 4 x $(1.5)^2$" [*"x" representing multiplication*] as his answer and then determined that value with a calculator. In this process he first evaluated $(1.5)^2$ to get 2.25 and multiplied this by four. I asked him if 2.25 was an important number to understand or if it just represented an intermediate step in the calculation process. He explained that 2.25 represented the growth factor for a two-week change in time elapsed. This was an idea never mentioned by the other students I interviewed but seems important in terms of John's fluency with writing and explaining exponential models at the end of the course.

These examples hint at a connection between students' fluency with mathematical representations and the degree to which they expected these representations to reflect their conceptualization of relationships between quantities, at least in familiar contexts.

Discussion

Moore (2016) proposed the distinction between *operative thought* and *figurative thought* (Piaget, 2001; Steffe, 1991; Thompson, 1985) as a useful lens for understanding differences between students' ways of thinking about graphs. Figurative thought describes thought that relies on perceptions directly accessible during the reasoning process. One implication of a person limited to figurative reasoning within a context is the inability to extend beyond that context to organize his thinking relative to general relationships and connections to other contexts and ideas. Operative thought suggests a degree of control and coordination in the reasoning process that extends beyond perceptual features of a given context. An implication of operative thought is that the individual makes conscious decisions throughout his reasoning process and is aware of how work within one context connects to work in other contexts.

I see parallels in Moore's analysis with students' symbolization activity in the answers to my research questions. Students' symbolization activity was primarily driven by their assimilating interpretations of the tasks at hand to schemes where the resulting actions (performing specific calculations, generating algebraic representations, etc.) were not motivated by reflections on quantities, general relationships, or connections to other tasks. Their activity seemed subordinate to perceptual features of the tasks (such as the specific numbers given in a problem statement, their identification of key words, and perhaps even the structure of students' native language). The students whose ways of thinking did not shift during the course struggled to describe quantitative meanings for their calculations, often produced sets of potential responses and picked from among these choices rather than foregrounding their activities in conceptualizing the quantities' values and relationships they wanted to represent, and typically did not or could not make connections between tasks or between different representations.

Emergent symbolization is an example of operative thought "involv[ing] mental representations of actions and consideration of the consequences of those actions that allows students to make propitious decisions about next steps in their reasoning process and how those steps connect to conclusions already made" (O'Bryan, 2018, p. 315). My results do not prove that John and Lisa developed the full set of expectations that drive emergent symbolization reasoning (that would require a different study). They do show that students who were able to explain quantitative meanings for steps in their calculation processes and components of their mathematical expressions were also able to explain how aspects of their calculations or algebraic reasoning related to reasoning in other tasks and might also be represented in other forms.

Moore and Thompson (2015) argued that studying students' shape thinking is critical for providing perspective on how to foster students' connections between the variety of possible representations for mathematical relationships. It is important for "researchers to be clearer about what a graph represents to a student, and thus what students understand multiple representations to be representations of" (p. 784). This paper is an attempt to begin answering this call relative to students' symbolization activity with the long-term goal of better understanding how students see connections between mathematical representations. More work is needed to continue to flesh out the ways of thinking students employ in symbolization activity. I am particularly interested in examining students' symbolization activity in novel contexts to understand the potential implications of students' expectations regarding the models they generate and the meanings they identify in those models in less familiar contexts.

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