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ABSTRACT
We employ a network theory mediated analysis to investigate the affordances of a precalculus curriculum, exemplified by the *Pathways* precalculus textbook and, in particular, its alignment with the research-based Algebra and Precalculus Concept Readiness Taxonomy (referred to as APCR). A network perspective of the precalculus curriculum has been helpful in identifying key features of the subject as they appear in several precalculus texts including the one being studied here by Carlson et al.. Of these, "hubs" have been particularly useful in mapping the alignment between the preset goals of the precalculus course, as identified in the previous literature, and its execution. Probabilistic tools, such as cumulative distribution functions, prove to be effective in analyzing the distribution and accumulation of precalculus content throughout the APCR and CCR. At the same time, this analysis has also been valuable in identifying the trajectory of the textbook curriculum in adequately preparing students for success in calculus.

KEYWORDS
Precalculus; calculus; networks; trajectory

1. INTRODUCTION

Education – teaching and learning – is a complex system, with multiple evolving parts in constant synergistic exchange. The goal of this exchange is the achievement of an equilibrium state which can persist, despite perturbations from the environment. This is to suggest that when different parts of the education system are in harmony then we see the emergence of new knowledge, which requires each player in the process of education to morph and adapt in response to feedback [19,37]. While it is difficult to point to definitive strategies about what makes for ideal teaching and learning, it is still possible to identify general characteristics of effective instruction that have a degree of universal applicability. One such characteristic, which has intuitive appeal, has reemerged under various guises in the form of "connections". In this paper, "connections" refer to the relationships between ideas, and the paths taken to meaningfully navigate between and amongst these sources of thought. This theme appears at the holistic level under the process philosophy [54], systems theory [4], cybernetic theory [30,55], or the more recent connected curriculum theory [22]. It also appears in recent calls by professional mathematics

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education organizations (MAA, AMS, etc.) that curricula and instruction emphasize a coherent, connected approach to organizing courses [20,28,43]. In this paper, we focus on a more specific case of connections, a local one, in the context of calculus preparation.

The low overall international standing of US students in mathematics courses has driven a significant amount of research on the efficacy of mathematics education [46,52,53]. With a considerable focus on high school mathematics forging a cumulative path toward higher-order mathematics in the form of precalculus and/or calculus, math educators are rightly concerned about the adequacy of student preparation for success in this course [9]. Reports on overall performance in calculus are rather bleak [6,13]. According to Carlson and Diefenderfer [7], "… only about 25% of students who start calculus and intend to take 3 semesters of calculus go on to complete the calculus sequence … ". Recent efforts have resulted in the development of assessment instruments such as the *Precalculus Concept Assessment* (PCA), *Algebra and Precalculus Concept Readiness* (APCR) and the *Calculus Concept Readiness* (CCR) assessments, based on a taxonomy which lists essential reasoning abilities and understandings in algebra, precalculus, and other prerequisite concepts. These instruments are based on analysis and synthesis of the previous decades' research on student learning in calculus [12,32,38,39]. The purpose of this paper is to investigate the effectiveness of the APCR and CCR recommended topics and their alignment in creating an effective trajectory for success in calculus. The approach adopted here is based on the network theory [47] and was introduced in a recent paper [41], which uses the connectedness of concepts in precalculus courses to mathematically assess the effectiveness of a curriculum as depicted by different texts on the subject. Specifically, the analysis here examines the alignment between the intended precalculus prerequisite and intended calculus curricula, mediated by the enacted precalculus curriculum (EPC) analyzed through the lens of network theory and probability theory. The underlying assumed mapping in this study is that the precalculus is represented by the APCR taxonomy, the calculus by the CCR taxonomy, and the EPC modeled by the *Pathways* precalculus textbook [10].

The choice of this text for the current study was based upon specific merits compared to several other textbooks, as seen through the lens of network analysis in a previous work [41]. This curriculum was also selected because one of its authors is a co-author of this paper and was also the lead author of the CCR Taxonomy, which helps us gain deep insights into both the precalculus and calculus segments of this work. Furthermore, the *Pathways* program follows a research-based curriculum which purports to be well designed to meet the needs of a calculus course, making it a particularly interesting book for further investigation.

In this study, we define "intended curriculum" to be the composition of taxonomy principles, choice of textbook and its contents, and any other external sources that influence the structure of the curriculum. The "enacted curriculum" acts as a mapping from the intended curriculum into the reality of the classroom based on interpretation and experiences of both the teacher and the students (see Figure 1). This study does not examine empirical data from classrooms for the purposes of

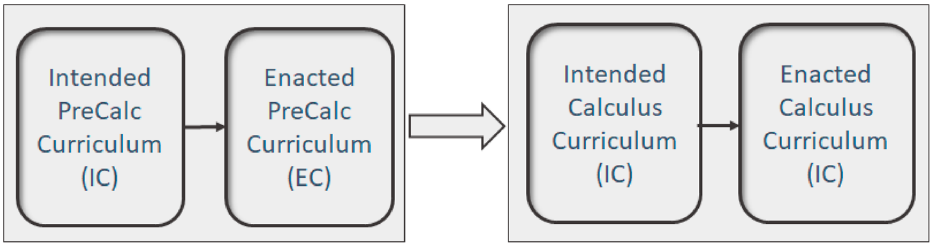

Figure 1. While intended curriculums ought to inform enacted curriculums in the form of classroom teachings and practices, one can observe the same transfer between precalculus and calculus preparation. The stakeholders in this process would likely be the professional bodies that govern local and national standards of education: the textbook authors, course coordinators, educators, administrative officials, etc. All of these influencers play a significant role in the development of the intended curriculum through genesis of taxonomic principles and choice of textbook. However, we acknowledge this forms an inherently complex system further subject to the experiences and interpretations of the teacher and classroom, making for diverse enacted curricula. This paper specifically focuses on the alignment between successful execution of precalculus and calculus curricula, holding all external factors equal otherwise for the sake of this particular study.

comparing intended and enacted curriculum. Therefore, we make the practical choice of assuming that the enacted curriculum is exactly the intended curriculum, i.e., to say the actual classroom pedagogical shifts are not considered and the assumption is that the text alone serves as the guide for students. This assumption allows us to compare the *potential* learning experiences for students using each set of curriculum materials under ideal circumstances with minimum outside supplementation. In making a distinction between intended and implemented curricula, our intention is to emphasize that, while good teachers do supplement their course materials, the intended curriculum serves as the basis for the implemented curriculum. It also often acts as the exact blueprint for teaching and learning for inexperienced educators – which is not so uncommon in mathematics classes. It has been shown that a teacher's supplemental teaching methods in conjunction with assigned readings have led to improved student experiences and outcomes [27,44].

While also not covered in the scope of this study, we assume that the intended calculus curriculum can be represented with a subsequent calculus course's assigned textbook, chosen in a rigorous and meaningful manner outlined in the aforementioned network-based study. Interestingly, the use of networks to examine how ideas of covariation in precalculus and calculus [51] are introduced and executed is both intuitively meaningful and analytically practical.

The primary contributions of this paper lie in (a) providing a new approach to aligning taxonomies to curriculum design through the use of network theory methods and (b) analyzing the trajectory of a curriculum and to what extent it aligns with subsequent courses. This is exemplified by examining the flow of ideas between precalculus and calculus courses. Such an analysis is particularly significant in mathematics (or STEM in general) which are prerequisite dependent. The techniques introduced here appear complex, but once understood, their application is both straightforward and worthwhile. The following sections of this paper provide both an overview of the mathematical methods and the details of the application to mathematics education. We believe that such an analysis will be valuable to a wide variety of scholars, instructors, curriculum designers, textbook companies, and even students. Due to the technicality of various concepts and topics that appear throughout this study, we make use of several acronyms to streamline the readability of this piece. A key to all acronyms that are used can be found in the Appendix for reference.

2. MODEL OF PRECALCULUS CURRICULUM

O'Meara and Vaidya [41] employed a network model of precalculus to analyze several textbooks on the subject. The network structure of this book arising from the connections drawn between different topics discussed in the course formed the basis for the opportunities that the books provided for "creative meaning-making" by students, who could use the connections between topics to navigate complex mathematical arguments. Specific network metrics such as the average path length (APL), clustering coefficient (CC), degree distribution (DD), and hubs were found to be particularly useful and relevant in identifying characteristics of a "good" textbook conducive to an optimal learning environment.

2.1. Analytical Framework for Taxonomy-Textbook Trajectory

In this section, we lay out the theoretical framework for our study which is rooted in a Network and Probability theory approach. Figure 2 summarizes the model adopted here to understand the alignment between the ideal expectations of designing a precalculus course as a precursor to calculus. This goal is captured by the APCR and CCR taxonomies (where P_{exp} refers to the mapping between the two taxonomies); however, the enactment of this goal is mediated by a curriculum which extends the taxonomies into a rich and self-consistent story with longitudinal value. We propose a three-step approach to assess the effectiveness of the curricular alignment: (i) Convert the curriculum (or textbook) into a network using the approach outlined in Section 2.2; (ii) Map APCR to the curricular network (which we label P_1), and (iii) Map the curricular network to the CCR (labeled P_2). We consider the deviation, δ, between P_{exp}, the ideal path toward success in calculus, and $P_2\,P_1$, the empirical path that is stipulated and studied in this piece. While we are unable to observe P_{exp}, we consider the observable successes of $P_2\,P_1$ and where there might still be room for improvement. Additionally, we consider how these successes are of practical use to precalculus educators. In the following sections, we explain each of these modeling paradigms in further detail before exemplifying it via a specific precalculus curriculum.

2.2. Network Representation of the Text

We define a *network* representation of a course textbook as an undirected graph [47]. This graph's nodes represent topics/concepts introduced in the text, and edges

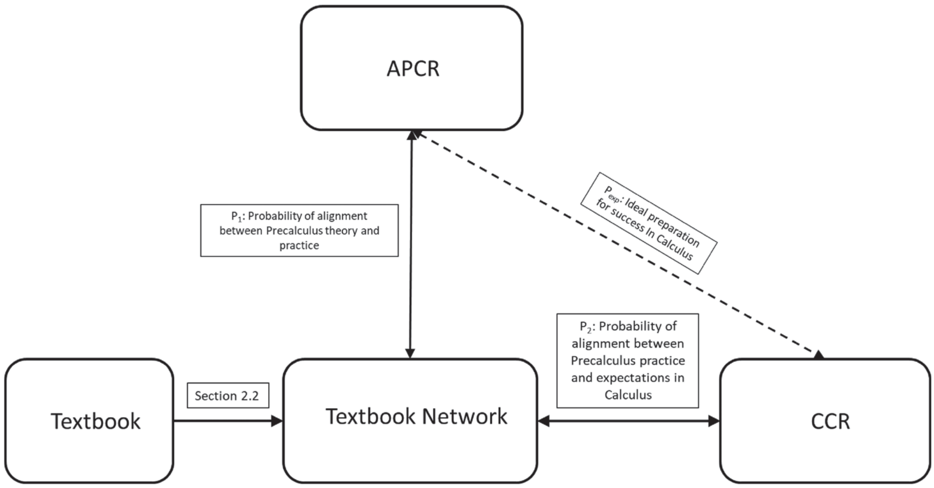

Figure 2. This schematic lays out the overall methodology of this work. The ideal goal of mapping APCR to CCR is actualized by means of the text and the Network-Probabilistic methods employed here which allow us to estimate the alignment between textbook design and theoretical expectations. Each set of arrows between boxes represents a mapping that correlates the content between these resources. We discuss the process of mapping the textbook into a network representation in Section 2.2, the mapping from the textbook network to the APCR in Section 3.1, the network to the CCR in Section 3.2, and our approximation of the mapping from APCR to CCR in Section 3.3.

represent explicit connections formed between them. Textbooks were encoded into network representations by two master's students at the time (one of whom is now an author of this manuscript) who would read the text section by section, construct network representations independently, and then compare. After discussing and confirming that the network's information was derived strictly from the text with no external nor implicit biases, students went on to construct networks for the entire textbook. In the following paragraph, we describe the qualitative approach taken by these two students to identify meaningful connections in several precalculus textbooks.

Two topics/concepts are considered "connected" if there is an identifiable and sufficiently strong intentional relationship between them in the text, i.e., the text explicitly refers to one when introducing and/or discussing the other. For example, consider the following line that might appear in a traditional precalculus textbook: "One clear distinction that arises between the functions we have studied thus far is that while a linear function exhibits a constant rate of change, an exponential function exhibits a constant percent change". The connections that can be observed in this line are: (a) linear functions and rate of change, (b) linear functions and exponential functions, (c) exponential functions and percent change, and (d) rate of change and percent change. These connections are provided in Table 1.

Codes are then assigned for each topic/concept, and their edges are organized in an array as in Table 1. This information is entered into a network visualization software for deeper analysis and computation of relevant metrics. For a given textbook, the "union graph" refers to the aggregation of all individual sections' networks, as

Table 1. Sample connections that might be formed amongst four common topics in a given precalculus textbook. Adapted from O'Meara & Vaidya [41].

Introduced topic	Directly connected topic
Linear Functions	Rate of Change
Linear Functions	Exponential Functions
Exponential Functions	Percent Change
Rate of Change	Percent Change

well as the networks of any other materials that are provided with the textbook, such as a student handbook or online problem sets. More recent NLP-based approaches allow for the creation of "knowledge graphs" from mathematics texts [29]. Such methods will eventually need to be utilized to scale up the creation of curricular networks for different mathematical topics.

One significant observation is that proximity is not enough for concepts to be considered connected. For example, "Linear Functions" are not meaningfully connected to "Percent Change". While elements of this procedure are somewhat subjective, establishing validity between two students experienced in teaching precalculus lent itself to a fairly closely aligned network representation of these textbooks. For a more detailed discussion on the construction of these networks and the calculation of metrics such as average path length and clustering coefficient, please refer to O'Meara and Vaidya [41]. That work provides an in-depth analysis of nine precalculus textbooks, of which three are shown below in Figure 3.

2.3. Network Metrics

With the textbook acting as a basis for the classroom's enacted curriculum (EC), these metrics are of particular interest to math educators, as their applications and analogues to the classroom environment are able to provide a measure of the potential effectiveness of prospective learning goals. In a graph-based mapping such that nodes represent introduced topics/concepts and unweighted, undirected edges represent both proximal and meaningful linguistic connections between topics, the average path length can be thought of as the mean number of "steps", or topics that must be traversed to move from one given concept to another. The Clustering coefficient (CC) and Average path length (APL) are given by

$$CC = \frac{\text{Number of all triplets}}{\text{Number of closed triplets}} \tag{1}$$

and

$$APL = \frac{1}{n(n-1)} \sum_{i \neq j} d(v_i, v_j) \tag{2}$$

where n refers to the number of nodes, v_i refers to vertex i, and $d(v_i, v_j)$ is the distance between any two vertices [3].

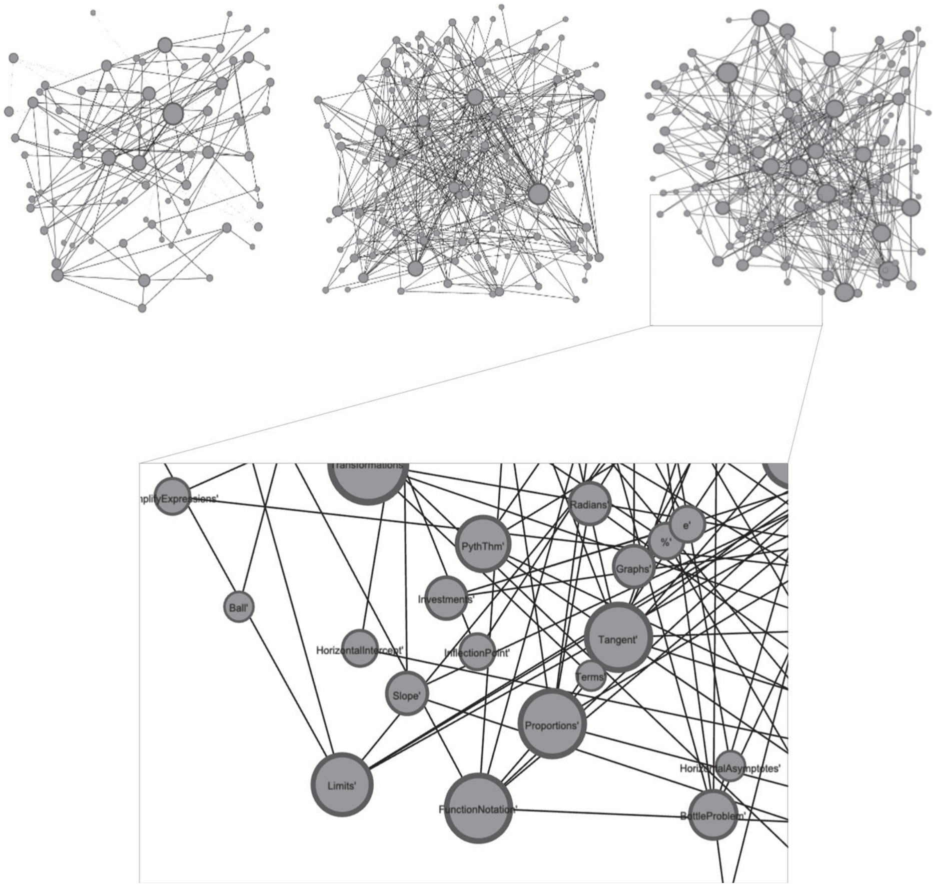

Figure 3. A network representation of three distinct precalculus textbooks, including Abramson et al. *Precalculus* [1] (top left), COMAP's *Precalculus: Modeling our World* [16] (top center), & Carlson et al. *Precalculus: Pathways to Calculus: A Problem Solving Approach* [10] textbook & student handbook (top right), where nodes represent topics introduced in the text and edges represent connections formed between topics. An enhancement of a portion of the network representation of the *Pathways* textbook is provided (bottom), showcasing the complexity of connections amongst topics (such as "Limits" and "Function Notation") in a text. This highlights the distinct ways in which meaning is formed across various materials, even when intended to represent the same content area (in this case, precalculus). It is reasonable to infer that the patterns exhibited in the materials for a course necessarily influence both teaching and learning within the course.

The minimization of the APL metric and maximization of the CC metric indicate a streamlining of content delivery. Because the clustering coefficient is a measure whose codomain is on [0,1] that indicates how close a graph on n nodes is to the complete graph K_n (where 1 indicates the input graph is a complete graph on all n vertices), this highlights the extent to which a given text attempts to make all possible connections between every single topic. However, recent research indicates that a computed metric approaching a value of 1 does not necessarily make for an increasingly effective text [41]. Finally, hubs are defined to be nodes which contain a degree of at least the minimum number of chapters covered in the collective sample of precalculus texts in the aforementioned study: in this particular study, the minimum requirement to be considered a hub happened to be a degree of 6. This indicates the set of topics that are not only the most frequently introduced but also the most highly connected. Therefore, a meaningful proportion of hubs is indicative of establishing and reinforcing the most apparently important concepts throughout the entire trajectory of the intended curriculum.

It is therefore our collective hypothesis that a text may be considered "effective" if it retains a relatively low average path length, a relatively high clustering coefficient, and whose hubs are not only provided in sufficient representation with respect to the aggregate collection of curriculum topics, but also whose hubs are representative of the overall learning goals of the current course. The relative magnitude of each of these metrics is provided upon the basis of a union graph, in which all connections between all topics introduced across every sampled text is made, establishing meaningful insight into how one ought to consider these metrics in a practical sense. This may then be extended to represent the core prerequisite knowledge for future courses scaffolded upon precalculus, of which preparation for calculus is among the focal points of this investigation.

Based on a composite ranking of the texts, elicited by a specific set of criteria selected by the authors, the *Pathways* curriculum appears to rank high among prominent precalculus texts (see Table 2). The criteria include comparing each individual text's network profile to the *union network*[1] with respect to the APL, CC, and proportion of hubs. A standardized error metric entitled the Composite Root Mean Square Deviation (or RMSD in Table 2) was computed to form the ranking system below. The metric chosen in this study is one of several that could be selected upon the preferences of the educator, among other factors. However, the assumption that an ideal model for enacted curriculum ought to closely resemble the properties of a curriculum in which all possible connections from each sampled book are formed serves as a highly meaningful basis for evaluating the efficacy of the provided material and the choices made by those involved in its authorship, publication, and execution.

2.4. Use of Probability Theory in Trajectory Analysis

When considering how content is distributed throughout a semester, a discussion of the accumulation of content arises naturally when considering the teaching and learning of mathematical content. Studying the distribution of accumulation is of particular interest throughout many industries and environments. Studying the accumulation and distribution of resources can be used to determine probabilities of random variables that follow normal distributions [17], analysis of hazard

[1] The Union Network refers to the union of all the network representations of the eight textbooks mentioned in Table 2. This collective representation can be treated as a case which contains the best features of all the individual texts and deviations of each text network from the union network has been used to rank texts. Naturally, there is nothing absolute about the choice of metrics used to assess the deviation which are dependent upon the preferences of the authors and curriculum designers.

Table 2. A composite list of all texts sampled in a network theory-based analysis [41] concerning identification of trends among precalculus texts acting as frameworks for enacted curricula. Here, the Composite RMSD is computed by squaring the differences in each text's metric and the union graph's corresponding values, averaging these squared differences, and then taking the square root of this average to normalize the deviations. This measure of error indicates to what extent each sampled text forms the same quality of connections as the union graph on a domain of zero towards increasing without bound.

Rank	Title	Average path length	Average local clustering coefficient	% Hubs (in relation to total number of nodes)	Composite RMSD
1	Pathways (8th ed, Precalc: Pathways to Calc, 2020)	4.09	0.35	18.75	1.2989
2	COMAP (The Consortium for Mathematics and its Applications) – (prelim ed, Precalc: Modeling Our World, 2002 [16])	3.69	0.34	21.19	1.7228
3	CME (Common Core) – (Precalc, 2008 [15])	3.42	0.42	32.03	4.8429
4	Stewart (6th, Precalc: Math for Calc, 2012 [49])	3.36	0.04	15.00	5.1918
5	Larson (Precalc: Functions and Graphs – A Graphing Approach, 3rd ed, 2000 [31])	3.69	0.16	14.63	5.3651
6	Blitzer (5th ed, 2013, Precalc [5])	3.77	0.05	12.33	6.6795
7	Faires (Precalc, 5th ed, 2011 [21])	3.21	0.34	35.25	6.6892
8	Gary Rockswold (Precalc: with Modeling and Visualization – a Right Triangle Approach, 4th ed, 2010 [42])	3.41	0.28	11.76	7.0268
9	Abramson (OpenStax Precalc [1])	19.00	0.27	11.54	10.7050
	Union Graph	**5.11**	**0.32**	**23.82**	**0**

functions and portfolio management in finance [34], and disease progression in medicine [24]. In the field of education, a probabilistic perspective can be taken to compare the alignment between curricular plans and a teacher's enacted lesson plan [35] as well comparing empirical student performance to expected performance in a mathematics course [36]. In each of these settings, their primary value comes from assigning meaning to the accumulation of outcomes and their underlying distribution as they accumulate.

Because accumulation and distribution of precalculus content are of particular interest to us in this study, probabilistic methods are useful to compare the ways in which the APCR and CCR incorporate essential precalculus content over time. While many tools might be appropriate for analyzing accumulation of resources, we are primarily concerned with tools that indicate the density of relevant precalculus topics from the APCR into the precalculus textbook (P_1; see Figure 2) and then from the textbook to the CCR (P_2; see Figure 2). Analyzing such a transfer of content across these three resources enables us to consider to what extent students ideally prepared for success in calculus throughout this trajectory (P_{exp}; see Figure 2). Density and accumulation of content (or topics) lead us to considering the extent to which the APCR and CCR align in their accumulation of content deemed central for learning in precalculus. Such content can be meaningfully and naturally represented as a cumulative distribution function, where the degree of the hubs in the textbook is compared to the frequency with which these concepts are stipulated throughout the APCR and the CCR. Although the presented ordering of the topics is arbitrary, this technique provides insight into the distribution of topics across each resource and how well-represented these highly connected topics are in the precalculus curriculum. We elaborate more on the underlying assumptions and the process of defining our random variables and probability mass functions using this probabilistic method in Section 3.1. In particular, we highlight two essential concepts here: the Cumulative Distribution Function and Quantile Plots.

Cumulative distribution functions (CDFs) are defined to output the probability that a given random variable (e.g., X) is less than or equal to some value on the random variable's domain (e.g., x) [14]. When a random variable is discrete these might be given the name "cumulative mass functions", or CMFs, but the principle remains the same. Essentially, CDFs describe how a random variable "picks up" probabilities across its entire domain, yielding insight into the distribution of the probability function with respect to the given random variable. While random variables are often conflated with the traditional mathematical concept of randomness [17], we use "random variable" according to its definition in traditional probability theory. In this setting, a random variable refers to a function whose domain is the set of possible outcomes in a sample space and whose range is a measurable space [23]. While random processes underlie the behavior of a random variable, we can take this to mean that no single topic is expected to be weighted in any particular manner other than its empirical distribution.

To measure the relationship between CDFs, we make use of *quantile plots* [2] to measure the behavior of the CDFs at each quantile to determine how their alignments relate to one another over time. If two probability distributions have the same CDF, their quantile values will be identically distributed. If they vary with respect to one another in any way, one can examine the quantile plot to determine precisely where in each of their distributions these deviations might occur. It is with this method we are able to assess to what extent the APCR aligns with the textbook as a basis for precalculus curricula and to what extent the textbook then aligns with the CCR. If the quantile plots reveal nearly identical behavior, we conclude that the distributions are nearly identical and thus there is a high degree of cohesion and transfer of knowledge across these resources as a pathway toward student learning of precalculus and success in calculus.

3. APPLICATION TO A PRECALCULUS CURRICULUM

In this section, we employ the theory introduced in Section 2 to bear upon a specific precalculus curriculum. Each section takes up specific mappings – P_1 and P_2, and P_{exp}, introduced in Figure 2 and the key findings from our analysis are discussed in detail.

3.1. APCR Taxonomy and Precalculus Curricular Network (Mapping P_1)

While math placement tests have been used in the United States for several decades now, often skills assessed in such tests are somewhat disconnected from the essential needs of Calculus [32]. Recent research [12,38,39] has highlighted the importance of conceptual reasoning, alongside quantitative skills as being a prerequisite for success in upper level calculus courses. In recent times, many professional mathematics education organizations (MAA, AMS, etc.) are calling for a shift away from isolated skills and toward coherence, conceptual understanding, and deep engagement with mathematics [20,28,43].

The Algebra and Precalculus Concept Readiness Taxonomy (APCR) compiles a list of concepts which are essential for success in a precalculus course, heavily sourced from the Precalculus Concept Assessment (PCA) tool [11]. Such concepts are formed upon the basis of an analysis of the vast body of literature on learning calculus from the decades prior to the creation of these taxonomies. In particular, studies show that high performance on PCA-based assessment in students correlates highly with future success in calculus courses [11]. These include: Quantitative Reasoning (QR), Proportional Reasoning (PR), Covariational Reasoning (CR), Variable Reasoning (VR), Function Reasoning (FR), Computational Abilities (CA), Reasoning with Representations (RR), Notations, Conventions, and Definitions (NCD), Modeling (Mod), Measurement (Msm), Rate of Change (Rate), Function Concepts (FC), Solving Equations (SolEq), Inequalities (Ineq), and Properties of Reals (PptyReals).

Mathematics educators make a significant case for repetition and intra-topic connection as highly significant features at the forefront of educational efficacy [25,56]. With this assumption in mind, an effective text (such as *Precalculus: Pathways to Calculus* [10]) ought to remain well-aligned between its initial taxonomical goals and its most highly featured and repeated topics introduced throughout the curriculum, as indicated by the presence of hubs. Based on the description of the goals in the APCR, identified hubs in the *Pathways* text, for instance, can be aligned with the APCR based on either direct linguistic proximity or by indirect connotative proximity of related ideas, which has proven to be an effective method of determining student and curriculum success [45]. The relational mapping between the *Pathways* course materials and the APCR is represented visually by the Sankey diagram in Figure 4.

One meaningful question to consider in this research effort is, how ought we to consider the alignment between taxonomical principles and the prevalence of such

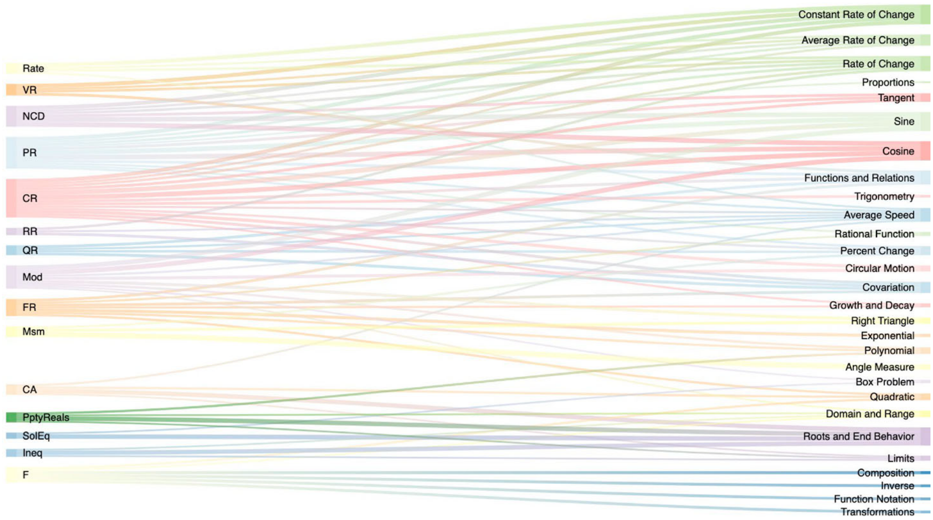

Figure 4. A Sankey Diagram displaying the representation of the APCR taxonomy (left) through the highly connected topics present in the *Pathways* textbook and accompanying student handbook (right). The text and handbook were synthesized via a union of their individual graphs, in which the union combines all connections made between the two sources to simulate the potential classroom experience for both teacher and student body.

related goals that appear in the textbook? An initial assumption we make is that the relation between the two be as directly translated as possible so that the taxonomy is fully and significantly represented in the enactment of the course content (thus representing the ideal potential scenario). To meaningfully assess the extent to which a bijection between the taxonomy and the EC is achieved, we assume the underlying and ideal distribution of the APCR is indicated by the density of sub-topics introduced. That is, on the empirical list of the taxonomy goals, there are indicated sub-goals to be accomplished within the scope of the overall concept. For example, the QR concept goal includes ten listed sub-goals.

By this observation, we can then assign relative weights to each overarching goal of the taxonomy based on the number of subtopics plus the goal itself. Thus Quantitative Reasoning has a relative weight of $10 + 1 = 11$ (where 10 represents the number of subtopics as shown in Table 3 and 1 represents the topic itself), and a goal with no subtopics such as Proportional Reasoning has a relative weight of $1 + 0 = 1$. We can then assign an empirical discrete probability distribution to the APCR where, while ordering of goals is somewhat arbitrary, we can assume the order in which they appear in [11] is the official ordering of the data. From this schema, we can determine the corresponding probability mass function (PMF) and cumulative density function (CDF) of the APCR. We make the assumption that W is a discrete random variable measuring the relative weight or "importance" of a given taxonomy goal. The PMF is then indicated by the probability that W obtains any value between one and the maximum empirical weight observed, denoted $\mathrm{P}(W = w)$. Similarly, the CDF is the probability that an observed W is less than or equal to the weight of an indicated value, denoted $P(W \leq w)$. This data is indicated in Table 4.

Table 3. Description of sub-goals of the QR concept as outlined in the APCR [11]. In the APCR, taxonomic goals such as QR (Quantitative Reasoning) are provided as an overarching indicator of student preparation for precalculus, while each of the sub-goals (as listed in the table for QR) are specific measures of student actions in pursuit of meeting this developmental goal.

Taxonomy sub-goal	Description of sub-goal
QR-C	Conceptualizing a given quantity entails attending to both the object and measurable attribute of a quantity already made explicit in a situation
QR-I	Introducing and using a new quantity entails identifying a measurable attribute of an object in a situation that was not previously explicit
QR-R	Relating quantities entails identifying a relationship between the measures of two or more quantities in a situation as the values of the two quantities vary together
QR-RF	Representing a quantitative relationship with a formula
QR-RG	Representing a quantitative relationship with a graph
QR-O	Performing a quantitative operation entails combining two or more quantities to form a new quantity
QR-IF	Interpreting a quantitative relationship represented by a formula
QR-IG	Interpreting a quantitative relationship represented by a graph
QR-VF	Viewing a formula as a means of relating the values of two quantities
QR-VG	Viewing a graph as a means of relating the values of two quantities

Table 4. APCR goals with empirical weights, PMF values, and CDF values. Note that $\sum_{i=1}^{15} W_i = 43$ where W_i refers to the APCR weights. These weights are used to assess how many sub-goals are assigned to each overarching taxonomic goal. The role of probability in this analysis is to assess how these goals accumulate over time as students learn and prepare for precalculus, which is then correlated to the representation of hubs in the course textbook.

APCR goal name	Weights, W (subtopics plus one)	$P(W = w)$	$P(W \leq w)$
QR	11	11/43	11/43
PR	1	1/43	12/43
CR	1	1/43	13/43
VR	5	5/43	18/43
FR	3	3/43	21/43
CA	1	1/43	22/43
RR	6	6/43	28/43
NCD	1	1/43	29/43
Mod	3	3/43	32/43
Msm	1	1/43	33/43
Rate	3	3/43	36/43
F	4	4/43	40/43
SolEq	1	1/43	41/43
Ineq	1	1/43	42/43
PptyReals	1	1/43	43/43

We can now meaningfully assess the success of the translation of the APCR from taxonomy to textbook by performing a similar analysis on the frequency with which hubs from the text are categorized amongst each APCR goal. Rather than aim for exact alignment of weights, the goal of this comparison is to assess the magnitude of the Probability Mass Function and Cumulative Density Function of the relationship between taxonomy goals and related textbook hubs, as derived from direct encoding of course materials. Hubs from the textbook were assigned to an APCR goal if they were mentioned in the taxonomy's goal itself.

Table 5. APCR goals with empirical hub degree weights, PMF values, and CDF values. Note that $\sum_{i=1}^{15} H_i = 826$ where H_i refers to the hub degree weights. These weights are used in order to assess how many sub-goals are assigned to each overarching taxonomic goal. The role of probability in this analysis is to assess how these goals accumulate over time as students learn and prepare for precalculus, which is then correlated to the representation of hubs in the course textbook.

APCR goal name	Hub density weights, H	$P(H = h)$	$P(H \leq h)$
QR	35	35/826	35/826
PR	113	113/826	148/826
CR	138	138/826	286/826
VR	41	41/826	327/826
FR	61	61/826	388/826
CA	37	37/826	425/826
RR	26	26/826	451/826
NCD	75	75/826	526/826
Mod	83	83/826	609/826
Msm	38	38/826	647/826
Rate	38	38/826	685/826
F	55	55/826	740/826
SolEq	22	22/826	762/826
Ineq	28	28/826	790/826
PptyReals	36	36/826	826/826

By the same method as displayed in Table 3, the weights of the alignment between the APCR and the hubs from the precalculus text are determined by the sum of the degrees of the hubs that correspond to each listed APCR goal. These hub weights are given in Table 5.

Upon establishment of these empirical distributions, the taxonomy and hub alignment were analyzed via a quantile plot comparison to assess the distribution of the density of hubs with respect to outlined goals. Furthermore, the Cumulative Density Functions of both distributions were plotted against one another to examine how the two representations of knowledge trajectories accumulate over the course of the listed APCR. A Pearson correlation test was conducted between their Cumulative Density Functions, uncovering an R^2 value of 0.9612. While there appears to be a relatively moderate concentration in the lowest quantile of the function, this is to be expected, as the weight of the first listed APCR goal (Quantitative Reasoning) is the largest in the taxonomy and of significant magnitude in the sum of hub degree from the textbook. In conjunction with the relative linearity of the quantile plots, this indicates that the two follow similar distributions and thus maintain a highly significant level of alignment.

Perhaps the most meaningful result of this analysis is that the moments in which misalignment occurs between the two exhibits a greater representation of importance in the taxonomy instead of in the execution of the density of hubs with respect to the APCR. This is of great significance because this informs us that these course goals are apparently underrepresented in the curriculum. If these goals are to be considered of greater importance due to their higher level of detail/attention as indicated by a greater number of sub goals, an underrepresentation implies that these goals may not have been communicated to their greatest potential through

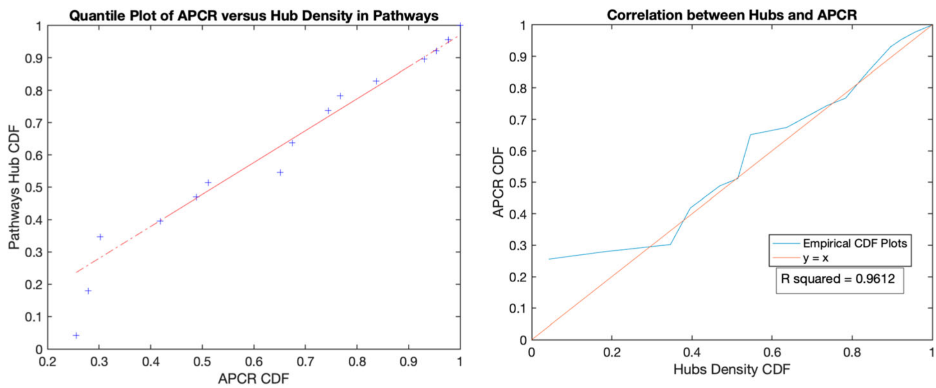

Figure 5. Quantile plots and assessment of correlation between APCR taxonomy components and hubs present in course materials. These quantile plots allow us to analyze to what extent the hubs of the textbook and the APCR align with one another as the materials accumulate over time. Deviations from the line $y = x$ indicate periods in which misalignment occurs (left), which can then be analyzed using coefficients of determination (right) to explore to what extent alignment is achieved and illuminate opportunities for improvement.

the course materials, possibly causing a greater gap between a student's desired understanding of precalculus concepts and/or their potential to inform and comprehend calculus concepts. The specific taxonomic components in which this is observed are: Quantitative Reasoning, Rational Reasoning, and Inequalities, with a slight over-representation in the textbook of Covariational Reasoning. However, we may assume that over-representation is not necessarily an issue in such an analysis, as extra attention might be paid to certain concepts in the classroom to facilitate individualized learning goals, as long as it does not detract from other such goals. These results are visually represented in Figure 5.

In light of these moments of misalignment between APCR and representation in the course materials, the creators of the taxonomy note the following with respect to its creation and implementation [7]:

(1) The APCR was created by a committee without the same level of years-long validation that PCA experienced. This can skew the distribution of topics to not necessarily reflect their relative importance to succeeding in calculus.
(2) Some topics (such as covariational reasoning) can be difficult to assess, and as a result are sometimes incidentally underrepresented in assessments (in the case of CR, the static nature of the assessment contributes to this). This can cause a disconnect between prevalence in the curriculum and the assessment.
(3) Since students have received largely incoherent, ineffective instruction up to this point in their educational careers, some topics might need additional weight within curriculum materials to scaffold student success with the remaining topics in the course, causing some mismatch with comparing representations.

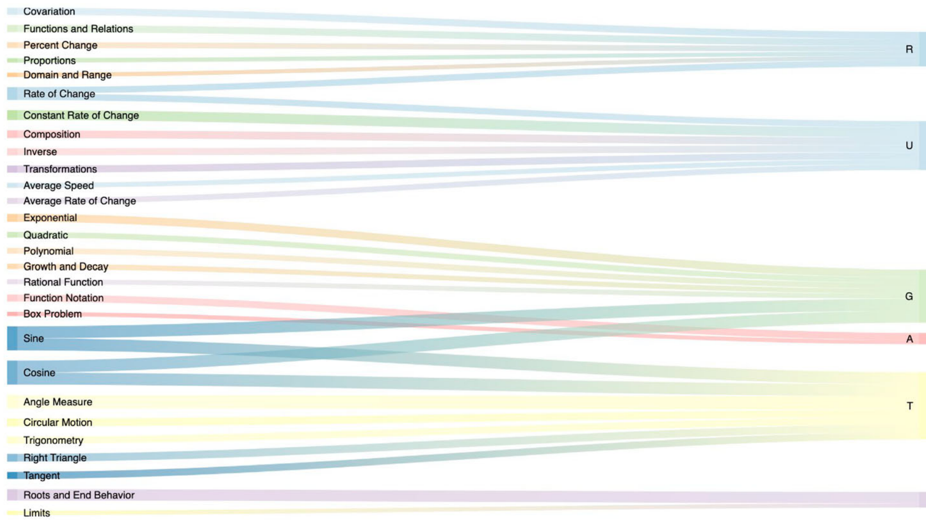

Figure 6. Sankey diagram outlining the representation of CCR taxonomy goals (right) within the hubs present in *Pathways* precalculus textbook and student handbook (left).

These considerations are excellent reminders that the classroom experience is highly qualitative and subjective, with many traits that are difficult to capture in a mathematical analysis. Observing intended and enacted curriculum alignment through a quantitative lens may yield highly valuable insights that motivate refinement and reform in the classroom, although complex systems exhibit many facets of unpredictable behavior that require thorough consideration on all analytical levels.

3.2. Mapping PCN to CCR (Mapping P_2)

The Calculus Concept Readiness (CCR) Taxonomy provides a description of reasoning abilities and understandings associated with precalculus ideas that students' success in calculus. Among these are markedly broad learning goals that are reinforced by significant documentation that current math education research indicates as essential for the learning of key calculus concepts [8]. The particular abilities outlined in the CCR Taxonomy include Reasoning Abilities (R), Understanding of concepts of quantity and function (U), Understanding of representations and interpretations of function growth patterns (G), Understanding of central ideas from trigonometry (T), and miscellaneous other abilities that involve using algebraic skills to determine values of one quantity in a function relationship, when values of the other are known (A).

Based on the descriptions of each taxonomy goal in the CCR, the hubs derived from the *Pathways* text were mapped according to the schema outlined in alignment between the APCR and the text. One ought to note that the mappings need not be one to one in either the APCR or the CCR, as topics may be applicable to several different learning goals simultaneously. A Sankey plot of the alignment between the text and the CCR is displayed in Figure 6.

Table 6. CCR goals with empirical weights, PMF values, and CDF values. Note that $\sum_{i=1}^{5} V_i = 29$ where V_i refers to the CCR weights. These weights are used in order to assess how many sub-goals are assigned to each overarching goal in the CCR. The role of probability in this analysis is to assess how these goals accumulate over time as students learn precalculus in the curriculum, which is then correlated to the representation of these hubs in the CCR as they prepare for their studies in calculus.

CCR goal name	Weights, V (subtopics plus one)	$P(V = v)$	$P(V \leq v)$
R	4	4/29	4/29
U	8	8/29	12/29
G	6	6/29	18/29
T	5	5/29	23/29
A	6	6/29	29/29

Table 7. CCR goals with empirical hub degree weights, PMF values, and CDF values. Note that $\sum_{i=1}^{5} J_i = 304$ where J_i refers to the hub degree weights. These weights are used in order to assess how many sub-goals are assigned to each overarching goal in the CCR. The role of probability in this analysis is to assess how these goals accumulate over time as students learn precalculus in the curriculum, which is then correlated to the representation of these hubs in the CCR as they prepare for their studies in calculus.

CCR goal name	Hub density weights, J	$P(J = j)$	$P(J \leq j)$
R	49	49/304	49/304
U	69	69/304	118/304
G	75	75/304	193/304
T	95	95/304	288/304
A	16	16/304	304/304

By the same methodology that was applied in Section 3.1, we can determine the distributions independent of each other and seek whether or not there exists a correlation between the densities of the two. The empirical distribution of goals in the CCR can be found in Table 6, and the density of hubs in each learning goal are given in Table 7:

As can be observed in the figures below, there is a high level of alignment achieved between the representation of identified necessary calculus concepts in the course materials, and the relative importance of such concepts as listed in the CCR. *The only taxonomy goal in which CCR representation exceeds textbook density is in the understanding of specific concepts involving the idea of quantity and function-related concepts* (U), which is highly oriented around the understanding of representations of quantities and analyzing function relations between specified variables and/or constants.

The important distinction to be made is that this observation, as is also true in the APCR analysis, simply indicates that these topics in the textbook do not have as high a level of connectivity to other concepts in the coursework as is indicated by the relative representation of the taxonomy. The overarching recommendation to be made on this observation is that, if the CCR is assumed to be the ideal "checklist"

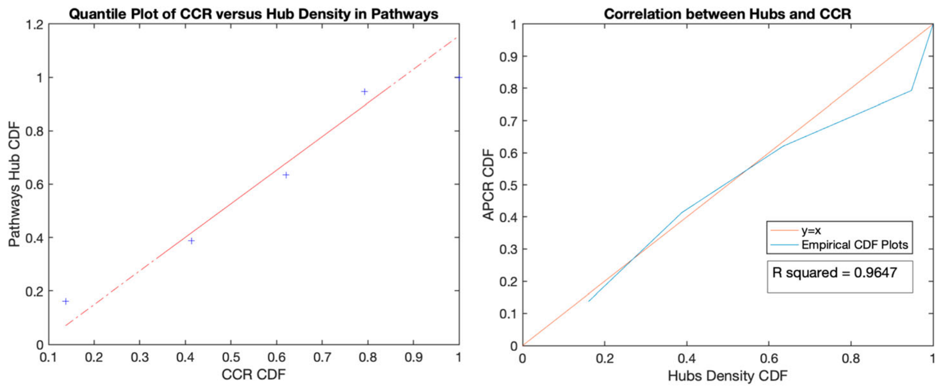

Figure 7. Quantile plots and assessment of correlation between CCR taxonomy goals and hubs present in course materials. These quantile plots allow us to analyze to what extent the hubs of the textbook and the CCR align with one another as the materials accumulate over time. Deviations from the line $y = x$ indicate periods in which misalignment occurs (left), which can then be analyzed using coefficients of determination (right) to explore to what extent alignment is achieved and illuminate opportunities for improvement.

for information and knowledge requirements to bring to calculus from precalculus, then an increased emphasis in the course materials ought to be placed on such concepts throughout the entire trajectory of the course. How this can be achieved is often complimented and decided by individual educators in the context of their teachings in pursuit of a greater degree of adaptive and consistent, desirable teaching practices well-aligned with the philosophy of the *Pathways* materials. It is this philosophy in which covariational reasoning and connected curriculum practices are placed at the forefront of instructional methods to meet essential research-based taxonomies, yielding successful transitional phases between courses for students.

The quantile plot and correlation between the CCR and hub density are given in Figure 7.

One additional noteworthy observation is that, according to the descriptions provided in the CCR, there are two hubs present in the *Pathways* text that do not appear to align with any of the given goals. These hubs are "Roots and End Behavior" (R&EB) and "Limits" (LIM). R&EB describes the process of identifying the x-intercepts (often synonymously labeled "roots", "solutions", "zeros", etc.) in polynomial functions, rational functions, and many other classes of functions studied in a typical precalculus course. Roots & End Behavior also includes the notion of studying how a function's output responds to allowing the input to tend towards extreme positive ($+\infty$) or negative ($-\infty$) values. This idea can be restated as, As our input x increases without bound, what does our function's output $f(x)$ tend towards? Symbolically, $(x \to +\infty) \Rightarrow (f(x) \to L)$, where "$L$" represents the value that $f(x)$ approaches. LIM is a direct extension of this idea; the "limit" of a function can be related to the end behavior of a function in the following definition: The limit as the input x of a function f(x) is equal to L if and only if, as x approaches some value a (where a is assumed to be a real number or positive/negative infinity), $f(x)$

approaches L. In the context of symbolic mathematical notation, we can express this definition as

$$\lim_{x \to a} f(x) = L \Leftrightarrow [(x \to a) \Rightarrow (f(x) \to L)].$$

These two concepts are inherently foundational to the understanding and execution of calculus at any level. While similar studies corroborate and validate the inclusion of these two concepts [48], the presence of these topics can be found in the first chapter of most popular calculus textbooks, dating well back to the eighteenth century and throughout the twentieth century and beyond [26,33,50].

3.3. The Aggregate Map from APCR to CCR (Mapping and Interpreting $P_2 P_1$)

While there are certainly many possible approaches to handling this observation, one possible solution might be to include additional taxonomy goals that highlight the necessity of highlighting these concepts in the precalculus classroom. This will ensure that students are well-prepared for the use of these foundational ideas in their future courses, whether that be restricted to calculus or beyond. Presenting taxonomies to educators of courses can ensure that courses will be taught in such a way that the transition between courses be as streamlined as possible. This also eases the consistent strain that exists on educators to refresh the student body on prerequisite knowledge.

While P_{exp} is an idealized optimal path by which the transfer of knowledge flows from the expectations before (APCR) and after (CCR) successful completion of precalculus, we gain insight into one potential representation of its structure, with some tolerance of error δ, via our composed mapping P_2P_1 in Figure 8.

This visual representation of the transfer of knowledge through a precalculus course yields several meaningful insights. First, we see that "Roots and End Behavior" and "Limits" are both densely represented in the APCR, but are not stipulated amongst any of the five CCR categories. In fact, many topics' hub density weights drop off dramatically upon transfer from APCR to CCR. There are notable exceptions to this trend (e.g., "Proportions", "Angle Measure", etc.); however, it is interesting to note such significant representation of frontloading of precalculus content relative to how it might be valued when considering one's readiness for calculus. Second, this allows one to trace the lineage of any highly connected topic in this particular precalculus textbook. This is valuable to both educators and curriculum designers with respect to how support in courses before or after precalculus emphasize particular concepts to ensure alignment within any sequence of math courses. Third, showcasing hub density weights by thickness allows one to immediately note which topics ought to be recurring throughout the course to ensure strong alignment with the goals of given curricular materials. For example, if one were to teach by the values of this particular set of curricular resources, "Sine" and "Cosine" stand out as topics with particularly high density as they transfer into the CCR. For a precalculus instructor, this could serve as a self-check to ensure strong development of these concepts by consistently connecting them to various other

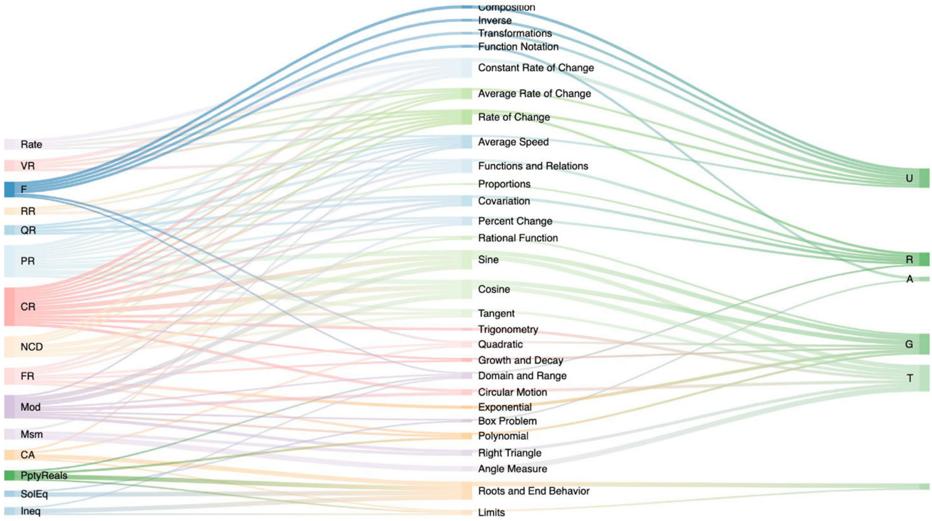

Figure 8. A visual representation of the "flow" or "transfer" of intention to enactment across curriculum goals, from APCR (left) to textbook representation of concepts (middle) to CCR (right), and thus onboarding into calculus. Widths of each flow line indicate the magnitude of hub density weights for the APCR and CCR.

concepts in the curriculum. These outcomes, among others that might be observed through this representation, provide a tangible asset to the precalculus teaching and learning community.

4. DISCUSSION

Effective mathematics learning involves making meaningful connections between mathematical topics and ties between mathematics and other disciplines, which includes connections with the individual's everyday life [18]. Network theory and analysis provides us with a rigorous approach to examine these qualities in a textbook; the novelty of this paper lies in the identification and demonstration of this mathematical tool in an educational context. Probability theory has been employed as an additional tool to examine the degree of alignment of the structured transfer of knowledge from taxonomical goals to the curricular materials. The language of quantile plots and cumulative distribution functions have allowed us to examine pointwise how taxonomical principles are represented in the hubs of the curriculum, as well as consider to what extent these materials are correlated to one another in terms of their given weights. A strict network analysis alone would not have sufficed here because the APCR and CCR do not lend themselves to the same lens of network analysis as the curriculum. Our confirmation of the alignment between the APCR, curriculum, and CCR stems from our use of quantile plots and an examination of the correlation between the development of these topics in precalculus.

While we found the transfer of information across these sources to be effective overall, we do see a need for further developments in the areas of Limits and Roots & End Behavior.

An immediate value of the network approach outlined here to educators can be seen from the "hubs" or density of specific topics in Figure 8. These hubs underscore the repeated significance of topics such as functions (F), covariation (CR), modeling (Mod), proportions (PR), etc., which are all essential to understanding various concepts in precalculus. We recommend the readers to an earlier paper [41] which elaborates on the outcomes of applying a network-based approach to various texts on precalculus, which are also highlighted in Table 2, including a list of hubs in different texts. While there are some topics that appear in the set of hubs that are either under-represented in the APCR and/or CCR, or do not retain any direct alignment in the CCR, this affords the opportunity for the educator to develop these concepts further in the pursuit of preparing students for studies in calculus and beyond.

In light of this analysis, we suggest adjustment of connectivity and repetition throughout the precalculus course to ensure that the completion of precalculus aligns optimally with the initiation of calculus. A major consequence of this study is that the prevalence of these amended qualities should provide the student and curricular environment with a sense of continuity throughout their studies. This in turn can reinforce individual meaning-making while laying the blueprint for a methodical process by which educators can refine their classroom practices to either supplement or better align with the given curricular materials and underlying learning goals.

For the author of the textbook that guides a course, this heavy lifting is important because you are laying out the basis for all teaching and learning that stems from this resource. While underlying taxonomic hierarchies might not be a common approach taken by most textbook authors, it is one that is worth considering to ensure the efficiency by which students master content and prepare for consequent coursework.

In practice, to alleviate the burden of heavy computation, practitioners can implement this approach by considering what students should know coming in to precalculus, what they should know entering calculus, and have students construct concept maps and analyze degree distribution to consider what concepts are not being effectively transferred for them, allowing opportunity for intervention. The suggested tools to analyze curricular alignment can also be more informally utilized on smaller scales. Students could be asked to solve specific problems while being asked to reflect on (and demonstrate) connections between specific fundamental principles and concepts of the courses highlighted in the APCR and/or CCR.

5. CONCLUSION

It is worth noting that while the current paper specifically focuses on precalculus and its trajectory, the network-based approach employed here is broad in scope and can be put to the test to examine the structure and evolution of any program in math, science or beyond. The outcomes of this study should have wide applicability and value to research scholars, research-based practitioners, curriculum designers, and all those who teach precalculus and/or within the calculus sequence. However, the extensions of this work can have an even wider appeal, across disciplines and while the approach is technical, the effort is well worth the time spent learning the details of the computation whose results can have value to teachers and students alike. While this work is not directly written for students, it has inspired other student-learning focused work which is the subject of an upcoming paper. [40].

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APPENDIX

Provided below is a quick reference to acronyms used in the paper.

PCA – Precalculus Concept Assessment
APCR – Algebra and Precalculus Concept Readiness
CCR – Calculus Concept Readiness
IPC – Intended Precalculus prerequisite Curricula
ICC – Intended Calculus prerequisite Curricula
EPC – Enacted Precalculus Curriculum
APL – Average Path Length
CC – Clustering Coefficient
DD – Degree Distribution
RMSD – Root Mean Square Deviation
PCN – Precalculus Curricular Network
CCR – Calculus Concept Readiness
PCA – Precalculus Concept Assessment
QR – Quantitative Reasoning
PR – Proportional Reasoning
CR – Covariational Reasoning
VR – Variable Reasoning
FR – Function Reasoning
CA – Computational Abilities
RR – Reasoning with Representations
NCD – Notations, Conventions, and Definitions
Mod – Modeling

Msm – Measurement
Rate – Rate of Change
FC – Function Concepts
SolEq – Solving Equations
Ineq – Inequalities
PptyReals – Properties of Reals

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BIOGRAPHICAL SKETCHES

John O'Meara received his M.S. in mathematics with a concentration in mathematics education this past spring from Montclair State University, where he will continue his studies this fall as a student in the Mathematics Education Ph.D. program. His current research focuses include complexity science, curriculum development, and network theory as a lens for deepening the understanding of interactions between students, teachers, and their respective environments, both individual and communal. Such interests are further motivated by his role as an instructor at Montclair State, where he teaches MATH111 (Applied Precalculus). In his teaching, he strives to actively apply research-based andragogical methods that align student perception, identity, and experience with adaptive curriculum development as a means of promoting educational equity for all.

Marilyn P. Carlson is a professor of mathematics education in the School of Mathematical and Statistical Sciences at Arizona State University. She researches the teaching and learning of precalculus and introductory calculus. In collaboration with her PhD students, she has generated knowledge of the reasoning abilities and understandings that lead to understanding and using key concepts in precalculus and beginning calculus. Her current research team has developed and is studying the impact of a professional development and curriculum intervention on precalculus instructors' instructional effectiveness and students' learning. She is studying the mechanisms for advancing precalculus instructors' MKT and responsiveness to students' thinking, and their impact on teachers' instructional choices and effectiveness in advancing students' thinking. She continues to refine this instructional innovation and is studying variables the lead to scaling it to new instructional settings.

Alan O'Bryan is a senior research analyst at the Arizona State University (ASU). He earned his Ph.D. from ASU in 2018 in mathematics education. He now has over 15 years of experience working in secondary and post-secondary mathematics education, focusing on professional development training, teacher preparation, and curriculum development.

Ashwin Vaidya is a professor in the Department of Mathematics at Montclair State University. His research interests lie in the general area of complex systems, more specifically, on problems of self-organization and emergence in physical systems. Besides application of complex systems theory to physical and biological systems, he also studies education as a complex system. His work in education in collaboration with colleagues at MSU focuses on issues surrounding creativity in math and science education, which he views as an emergent phenomenon resulting from the creation of the right environment where students can thrive. His education research aims to make "creativity" a central theme in teaching and learning.