Understanding and advancing graduate teaching assistants' mathematical knowledge for teaching

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Graduate student teaching assistants (GTAs) usually teach introductory level courses at the undergraduate level. Since GTAs constitute the majority of future mathematics faculty, their image of effective teaching and preparedness to lead instructional improvements will impact future directions in undergraduate mathematics curriculum and instruction. Yet GTA training, when offered, tends to focus on the practice of teaching rather than the mathematics being taught. In this paper, we argue for the need to support GTAs' in improving their mathematical meanings of foundational ideas and their ability to support productive student thinking. By investigating GTAs' meanings for average rate of change, a key content area in precalculus and calculus, we found evidence that even mathematically sophisticated GTAs possess impoverished meanings of this key idea. We argue for the need, and highlight one approach, for supporting GTAs' to improve their understanding of foundational mathematical ideas and how these ideas are learned.

Key words: graduate student teaching assistant, mathematical meanings, average rate of change, precalculus

Many mathematicians believe that undergraduate curriculum and teaching is unproblematic. After all, it worked for them! However, evidence exists to suggest otherwise (Bressoud, et al., 2012; Seymour, 2006). As universities continue to explore how to improve the mathematical success of their students, we propose that consideration for the preparation of future faculty be a central component of shifting introductory university level instruction. Graduate student teaching assistants (GTAs), namely, those individuals who constitute future faculty, teach many of the courses that undergraduate students first encounter in college (i.e., courses up to and including calculus). As such, GTAs who offer high quality instructional experiences for their students can impact a large number of students early in their studies. Yet GTA instructional support and professional development, when offered, frequently focuses on general discussion of best practices in pedagogy and specific logistical and student issues for a given institution. As novice instructors, this information is important in supporting GTAs' development of facility with the day-to-day practices of teaching, but it does little to support them in engaging deeply with the mathematical content in a way that will translate to meaningful mathematical discussions in their classrooms.

As other researchers have noted, having completed many mathematics courses, as most GTAs have, does not necessarily improve a teacher's understandings and teaching practices (Speer, 2008; Speer, Gutmann, & Murphy, 2005). Speer and Wagner (2009) further argue that the work done by mathematics instructors in providing analytic scaffolding, which entails recognizing and figuring out the ideas expressed by students to build upon and push the mathematical discussion forward, involves *mathematical* work. Instructors with impoverished or strictly procedural meanings for the mathematical content of a lesson will struggle to engage their students in conceptually oriented discussions around this lesson. Thompson, Carlson & Silverman (2007) claim that:

> If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of

> mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students. We believe that it is mathematical understandings of the latter type that serve as a necessary condition for teachers to teach for students' high-quality understanding (pp. 416-417).

The possible attempts to support student learning are revealed in the nature of the instructor's questions, her questioning patterns and the quality of the discussion she leads. All of these activities are built from how the teacher makes sense of her students' thinking, which itself is done through the lens of the teacher's own mathematical meanings (Teuscher, Moore & Carlson, 2015).

In this paper, we argue the need for attention to the GTAs' mathematical meanings, fostering attentiveness to student thinking, and supporting GTAs to *speak with meaning* during their training (Clark, Moore, & Carlson, 2008). In the context of an intervention to support mathematics GTAs to act in productive pedagogical ways, we investigated GTAs' meanings for average rate of change (AROC), a key content area in precalculus and calculus. We found evidence that even mathematically sophisticated GTAs possess meanings that vary widely in productivity for teaching prior to our intervention, and that shifts in meanings required prolonged engagement in professional development focused on making meaning, building coherence and communicating with precision.

We provide an expanded description of our goals for high quality instruction, how the intervention supports GTAs in making progress towards those goals, and describe what research has reported to be a productive meaning for the idea of AROC (Thompson, 1994). We then report on results that reveal the varied fluency among participants in speaking with meaning about AROC when probed before, during and after the intervention. We conclude with a discussion about the impact of the intervention and implications for other GTA training and professional development for undergraduate mathematics instructors.

Theoretical Framework

We provide a brief explanation of our theoretical perspective on mathematical meanings and say what we mean by high quality instruction. We also include a detailed description of the intervention used to support GTAs in developing more robust meanings of the concepts of AROC.

Researchers have proposed *mathematical meanings* as the organization of an individual's experiences with an idea, also referred to as a scheme (Thompson, 1994). It is through repeated reasoning and reconstruction that an individual constructs schemes to organize experiences in an internally consistent way (Piaget & Garcia, 1991; Thompson, 2013; Thompson, Carlson, Byerley, & Hatfield, 2013). For example, an individual's meaning for the idea of *average rate of change* might consist of the calculation for the slope of a secant line, or simply $\Delta y/\Delta x$. An individual who has committed to memory that the average rate of change is the slope of a secant line does not possess the same meaning as someone who sees the slope of a secant line as the constant rate of change that yields the same change in the dependent quantity (as some original non-linear relationship) over the interval of the independent quantity that is of interest. These two individuals hold different meanings for the same idea, and the consequences of such differences can be profound. For instance, the former individual may or may not have an understanding of AROC as a *rate of change*, but instead conceptualizes AROC as the numerical output of a computation. This individual will struggle to generate problem solving strategies relying on AROC, for instance, in situations where linear approximations could help generate characterizations for how two quantities covary over a given interval.

In studying individuals' meanings for various mathematical constructs, researchers are at a disadvantage – we cannot *see* schemes. Rather, we only have access to what an individual says, writes and gestures when engaging in mathematical activity. In our research and in the intervention, we focus on an individual's *expressed meaning of an idea*, or the spontaneous utterances that an individual conveys about an idea. From these utterances we can make inferences about how an individual has organized her experiences with the idea. We find the expressed meanings of particular interest in studying instructors' mathematical meanings because it is these spontaneous verbalizations that emerge during class instruction when students ask questions or pose solutions that deviate from what the instructor had prepared. What the instructors say *in the moment*, in turn, affects the ways of thinking their students develop about the mathematical idea(s) central to the discussion. We further propose that these expressed meanings provide insight into an instructor's meanings, which influence nearly all instructional activities, including: how the instructor interprets and responds to students' utterances and written products, the nature of the questions the instructor poses to students, and the selection and implementation of curricular tasks during class meetings.

We further note that an individual's meanings can be more or less productive for teaching. By more productive for teaching, we intend to describe those meanings that would support the individual in constructing new mathematical ideas and connections to other ideas, and correspondingly create the possibility for the individual to foster those connections within her students. An individual's meaning for specific ideas can be further developed through reflection, which occurs when the individual is faced with perturbations to her current meanings for those ideas (Dewey, 1910). With this perspective in mind, the intervention for GTAs is designed, in part, to perturb the GTAs' thinking about mathematical content areas specific to the courses they teach, namely precalculus and beginning calculus.

High quality instruction

For the purposes of this paper and the intervention being discussed, we delineate some of the characteristics we take as essential components of high quality instruction. A teacher is engaging in high quality instructional practices when she

- Supports students in constructing deep understandings and rich connections among central ideas of a course,
- Supports students in developing flexible problem solving abilities that enable them to solve problems that are novel to them and require that they apply their understandings
- Interprets and acts on student thinking when teaching, and
- Reflects on student thinking and learning to improve teaching.

One of the behaviors that might indicate high quality instruction to an observer is that the instructor is speaking precisely and meaningfully about mathematical ideas (Clark, Moore, & Carlson, 2008). Another indicator is that the instructor asks questions that elicit student thinking and then interprets, analyzes, clarifies and, whenever possible uses students' contributions to push forward the mathematical activity in the classroom (Johnson, 2013; Steffe & Thompson, 2000). A teacher's ability to engage in these practices depends heavily on the teacher's mathematical meanings and her mathematical knowledge for teaching (Ball & Bass, 2003; Ball, Hill, & Bass, 2004; Silverman & Thompson, 2008).

The intervention

The graduate students involved in the intervention volunteered to participate in the program and were compensated for their participation. All but one had met their respective university requirements to teach and agreed to further engage in a yearlong professional development program, which we call *the intervention*. The GTAs participated in a 2-3 day workshop before the start of their first semester of teaching with research-based, *Pathways Precalculus* curriculum materials (Carlson, Oehrtman, & Moore, 2015). During this intensive workshop, the GTAs completed mathematical tasks that were designed and sequenced to support graduate students in constructing a productive meaning for the key ideas in the precalculus curriculum, including the ideas of constant rate of change, AROC, exponential growth, angle measure, and the sine and cosine functions[1]. The graduate students confronted problems and questions designed to perturb their meanings for these topics. Sometimes these questions were as simple as "Explain the meaning of _____." Other times, the workshop tasks were more advanced mathematical problems that could be solved with precalculus tools, provided the individual had a deep understanding of the mathematical ideas central to the lesson. Still other intervention tasks were situated in the act of teaching, requiring participants to respond to hypothetical student questions and responses. The intent of these questions and tasks was to prompt reflection and subsequent shifts in the GTAs' meanings for the mathematical ideas that were the focus of their instruction.

The intervention leaders conducted workshops in a manner to encourage the participants to speak with meaning about the ideas. This meant GTAs were encouraged to avoid using vague language (e.g., use the quantity descriptions instead of pronouns, as in "The distance increases" instead of "It goes up.") and to explain basic vocabulary (e.g., *proportional*) instead of taking terms as understood. As this type of communication is new to most of the participants, this focus on speaking with meaning about the mathematics continued during the intervention. During the fall and spring semesters, the GTAs attended weekly 90-minute seminars concurrent with teaching a course using *Pathways Precalculus* materials, instructional resources designed using research on student thinking (e.g., Carlson, 1998; Carlson et al., 2002; Moore, 2012; Strom, 2008; Smith, 2008; Engelke, 2007) and scaffolded to support students' construction of key ideas of precalculus that are foundational for calculus.

The primary goals of the weekly seminars were to support the graduate students in developing more productive meanings of the key ideas to be taught during the upcoming week, and to support them in clearly explaining their meanings for those ideas to others. As part of the intervention to support them in achieving these goals, each of the graduate students reviewed the course materials prior to the weekly seminar and came prepared to discuss what is involved in understanding and learning the ideas central to the lesson for the subsequent week. They also came prepared to give presentations to their peers about how they intended to support their students' learning of these ideas. During the seminars, participants frequently worked in small groups to develop mini-presentations focused on implementing the most conceptually challenging tasks to be used in the lessons for the upcoming week. At some universities, GTAs were further asked to videotape their instruction and write reflections on their instruction with respect to the ways of thinking they supported during that class, the types of questions they posed to students, and how they might have responded differently taking student thinking into account.

The intervention had a primary focus on the mathematical content and what was entailed in understanding and learning key ideas of each lesson. Student thinking was discussed and described using constructs from relevant research literature and the GTAs' classroom experiences. Student thinking was analyzed for the purpose of describing student reasoning and understanding, and identifying productive ways for advancing students' learning. To

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[1] The idea of *average rate of change* is the culminating idea of the first instructional unit of the

achieve the goal of improving the GTA's ability to make sense of and advance student thinking, the intervention leaders and accompanying instructor materials had a primary focus on supporting the GTAs in developing rich and well-connected meanings for the mathematical ideas that were the focus of each lesson.

The content for the course began with a focus on developing students' ability to conceptualize quantities in a problem context and consider how pairs of varying quantities change together—these reasoning abilities have been identified to be necessary for constructing meaningful formulas and graphs (Moore & Carlson, 2012) to define functional relationships in applied contexts. As students explore the patterns of change for non-linear functions, such as exponential, quadratic, polynomial, rational, and trigonometric functions, they use the idea of *average rate of change* to characterize and compare function behavior over intervals of a function's domain. Since the idea of *average rate of change* is an important cross cutting idea that is challenging to teach with a conceptual focus, we selected this idea for studying the GTAs' mathematical meanings.

A productive meaning for the idea of average rate of change

Constructing a rich meaning of *average rate of change* entails conceptualizing a hypothetical relationship between two varying quantities in a dynamic situation. Given a relationship between the independent quantity A and the dependent quantity B, and a fixed interval of measure of quantity A, the *average rate of change* of quantity B with respect to quantity A is the *constant rate of change* that yields the same change in quantity B as the original relationship over the given interval. In order to understand this complex idea meaningfully, an individual must first conceptualize the idea of *quantity* as a measurable attribute of an object (e.g., the distance a car travels from home, number of minutes elapsed since noon). Next, provided a situation in which two quantities vary in tandem, an individual must develop an understanding for what it means to describe the *rate of change* of one quantity relative to the other. Namely, the individual must conceptualize the multiplicative comparison of changes in the two quantities (the change in the output quantity is always some amount times as large as the change in the input quantity). In the special case that the relative size of changes in one quantity relative to the other remains constant, we say the quantities vary with a constant rate of change (CROC) (see Figure 1 for the mental actions involved in constructing the idea of CROC). Individuals with a robust meaning will draw connections between AROC and CROC and view those two connected ideas as a means for approximating values of varying quantities in dynamic scenarios.

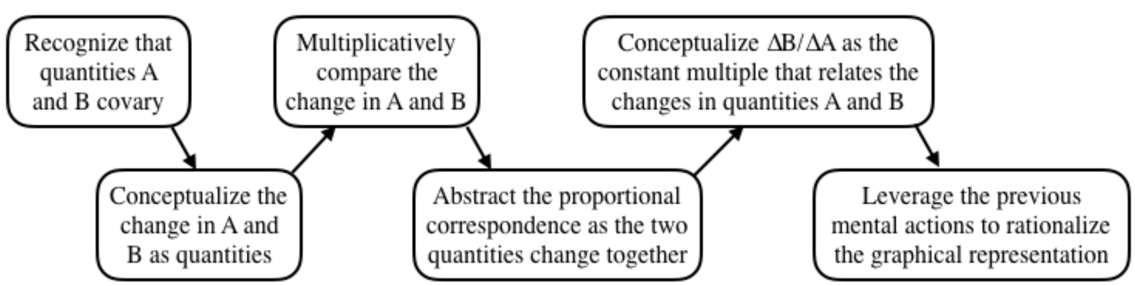

Figure 1. Mental actions involved in understanding constant rate of change

Methods

We collected data from mathematics graduate students and instructors at three large, public, PhD-granting universities in the United States. Participants' teaching experience varied between zero and 11 years, at both the K-12 and tertiary level. We conducted semi-structured clinical interviews with 19 graduate teaching assistants, all of whom had at least one semester experience teaching a small section or acting as the recitation leader for a Precalculus course that used the *Pathways Precalculus* materials (Clement, 2000). The lead author conducted interviews to probe shifts in their beliefs about the roles of students and teachers in teaching and learning mathematics, and gain insights about their understandings of mathematical ideas, teaching practices and goals for student learning. Interviews were recorded using both a video camera and Livescribe technology to capture audio-matched written responses to sample teaching scenarios provided during the interviews. Interviews lasted 1-2 hours, and were transcribed and coded by three members of the research team. Members of our team analyzed videos in pairs at first, identifying themes of interest relative to our conception of a productive expressed meaning for AROC before working individually to continue coding and reconvening as a group to discuss our findings (Strauss & Corbin, 1990).

Results

We first share data that reveals the expressed meanings that the graduate students conveyed for the idea of AROC when entering the program. We then illustrate their varied fluency in describing their meaning for AROC by sharing excerpts from clinical interviews with experienced participants. This is followed by our contrasting this data with their written descriptions of participants' meaning for AROC provided the week after they had completed teaching the investigations on AROC in a precalculus course using the Pathways materials for the first time. Collectively, our data reveals that the meanings for AROC conveyed by the novice and experienced GTAs differ in terms of the level to which they are able to spontaneously provide a meaningful description of what is involved in understanding AROC; however, it is noteworthy that even after completing the intervention, some GTAs in our study did not shift to speak fluently about the idea of AROC.

Pre-Intervention Meanings for AROC

As a warm-up activity for the start of a Summer 2015 teaching assistant workshop, we asked seven math graduate students to describe the meaning of "average rate of change." Each participant's response is recorded in Figure 2, in the order in which they verbalized their meaning to the group. Their responses align with the authors' prior experiences with both students and teachers at the secondary and tertiary levels; most of the participants provided computational or geometric interpretations based on imagining a secant line between two points on the graph of a function. In particular, we see that Alan[2] described AROC both computationally (i.e., $\Delta y/\Delta x$) and geometrically as a line, instead of highlighting the key attribute of the line—its slope. Another student, Frank, provided two equivalent descriptions of how to *compute* the AROC over a given interval, but did not convey what the result of those computations represent. When prompted by the workshop leader to explain the meaning of the result of the described computations, Frank struggled to communicate his thinking beyond referencing speed, saying, "[The result] represents how fast you're moving in effect. I always think of it as distance and time. It's difficult when you don't know what the quantities are." This issues of describing a meaning for AROC devoid of context arises repeatedly, even among the most experienced GTAs.

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[2] Psuedonyms are used throughout the reporting to protect the identity of participants.

Responses to the question: What does "average rate of change" mean to you?	
Alan:	Delta *y* over delta *x*. A straight line between two points on a graph
Edgar:	Rate of change over an interval and talk about interval as a whole. Describe the rate of change.
Frank:	The amount the dependent variable changes divided by the amount the independent variable changes. Delta *y* divided by delta *x*.
Cassie:	Steepness of a graph, like how steep or how flat it is.
Diane:	Steepness of a graph. […] Uh, I don't have actual words.[…] Slope or derivative.
Brian:	As one variable changes for every one unit, how much is the other variable changing. Slope. Speed.
Greg:	I lost all the words…It's the predictive effect of changing one variable and the amount and how it's going to affect the other variable. One quantity affecting change in another quantity.

Figure 2. Pre-intervention participant descriptions of AROC

Cassie and Diane spoke explicitly about a graph's steepness, a visual aspect of a graph that is simultaneously restricted to the Cartesian coordinate system and, in that setting, is potentially misleading when the coordinate axes do not have the same scale (see Figure 3). Another GTA, Brian, also mentioned slope, though he did so while conveying the idea that slope is an amount of change in the dependent quantity for each unit change in the independent quantity, a restrictive meaning for slope as it fails to support reasoning about variation when changes in the independent quantity have magnitude other than 1. His mention of speed suggests he might have been imagining more than he communicated, but his inability, or perceived lack of need, to coherently communicate with precision about his thinking is exactly one of the characteristic behaviors of the novice GTAs the intervention aims to transform.

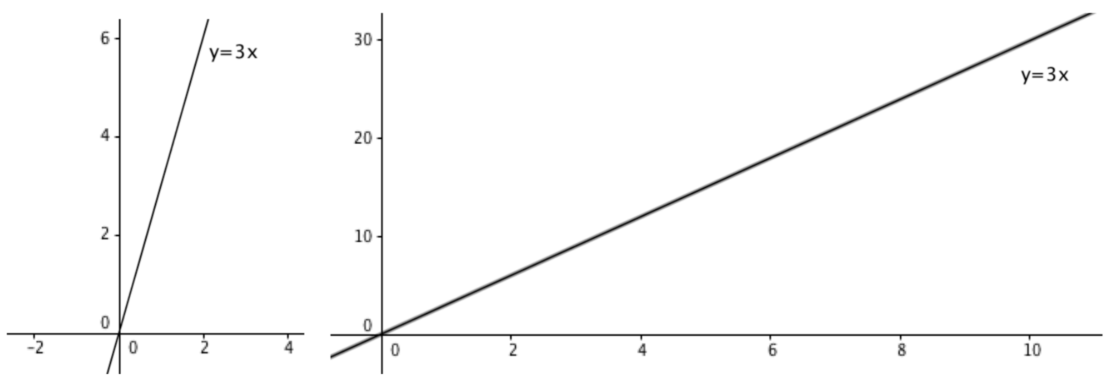

Figure 3. Same relationship but different visual "steepness"

Edgar's description of average rate of change as the rate of change over a whole interval lacks specificity and fails to communicate new information to the label of "average rate of change" beyond involving an interval. Taken literally, saying the AROC describes the rate of change loses all nuances about the possibility of a situation involving a varying rate of change and how AROC allows one to capture information about a dynamic situation even in the absence of information about the actual rate of change. Greg commented on the "predictive" quality of AROC, making him the only participant to explicitly highlight the idea that AROC provides an alternate means for characterizing how two quantities change together. This thinking, however, is missing many elements of what we have previously characterized as a productive meaning for AROC.

The pre-intervention participants' expressed meanings were predominantly geometric, computational, or restricted to particular representation formats (i.e., a graphical representation of a function in the Cartesian coordinate system); moreover, only one of the seven participants spontaneously hinted at the idea that the AROC serves as a tool for characterizing a function's change over some interval of its domain.

Post-Intervention Meanings for AROC

We analyzed 19 clinical interviews with participants who experienced at least one summer workshop and one semester of our intervention. In contrast to the predominantly geometric and computational descriptions of AROC from our pre-intervention participants, 15 of the 19 participants attempted to describe a meaning for AROC that conveyed some significance of the concept beyond a computation to perform (i.e., $\Delta y/\Delta x$) or an image to consider of a particular representation (i.e., secant line connecting two points). These descriptions can be classified as: the productive, general meaning described in our theoretical framework; a special case of that meaning for average speed; or, in one instance, a distinct interpretation the participant called "linearization." The other four participants offered explanations that fall strictly into the last four categories described in Table 1.

Table 1. Experienced participant descriptions of AROC

Expressed Meaning Category	*Sample Excerpts from Clinical Interviews*	*Number of Instances**
Productive – General	[Students] have to understand constant rate of change because the average rate of change is the constant rate of change someone else would have to go, and I'm talking about average speed now, to achieve the same change in output for a given change in input. So, if you don't have meaning for constant rate of change, well, then average rate of change is just this number.	9
Average Speed	[AROC] is a constant rate of change for that specific time and distance, or uh, you know how I mean…	8
Conceptual Other	I would like to say linearization. Right, this idea of approximating something that isn't linear in a linear fashion.	1
Computational	… this final minus initial over the outputs and this final minus initial over the inputs and that's a rate.	4
Geometric	Average rate of change is the constant rate of change to go between two points.	2
Incorrect	I want my students to understand that constant rate of change is a special case, I guess of average rate of change. It's this special case that exists when the corresponding changes in our two quantities are proportional.	3
None		1

* Total exceeds 19 because some interviewees conveyed more than one expressed meaning.

The excerpts in Table 1 highlight the fact that the impact of the intervention on participants is far from uniform. One participant failed to provide a clear statement of a meaning for AROC as he talked around the issue for 14 minutes during his interview. We found this surprising in light of the fact that this participant had four semesters of experience teaching the idea of AROC using curriculum materials designed to support the productive meaning described above. Of the 9 people who conveyed the productive meaning for AROC, four started with the more specific version of average speed and then generalized to a description in terms of general quantities. The other five produced explanations with various degrees of fluency, taking one participant almost 4 minutes to construct his explanation. Four of the eight participants who conveyed a meaning tied to average speed did not convey a meaning for AROC beyond the context of comparing distance and time. In fact, two of those four GTAs only discussed average speed when prompted from the interviewer following their initial response of describing a computation.

The sample excerpt for an incorrect meaning suggests that the participant developed a meaning for AROC linked to CROC in a non-standard way; conventional treatment of the two ideas typically describes AROC as a CROC approximation instead of viewing CROC as a special case of AROC. Another GTA stated that an average rate of change can be found by adding up and dividing, conveying a meaning for AROC that is conventionally identified as an arithmetic mean. Yet another participant proclaimed, "I will forever think of average rate of change as the slope of the secant line." The fact that some of these GTAs did not immediately produce the meaning for AROC supported both by the intervention and the curriculum materials points to the complexity of the idea of AROC and the difficulty that even graduate students had in modifying their strongly held geometric and computational images of the idea of AROC to a more robust scheme with connections that are rooted in a quantitative meaning that can be expressed in multiple representational contexts.

Nonetheless, many participants' expressed meanings did align with our productive meaning for AROC as a way to describe a characteristic of a relationship between varying quantities, even if only in the special case of average speed. Recall that during the intervention, leaders encouraged participants to *speak with meaning* as a tool to support their students in reasoning about quantities. Participants were asked to use appropriate language, describe the underlying meanings of specialized vocabulary (e.g., reference quantities instead of using pronouns like "it", explain "proportional" instead of just using that word), and offer multiple ways of explaining a concept. We see evidence of this practice in the "Productive Meaning" excerpt from Table 1 that was conveyed by a participant with 3 years of experience with the intervention, first as a participant and more recently as a leader. Not only did she express a productive meaning for AROC, using appropriate descriptions that highlighted *changes* in quantities as opposed to *values* of quantities, she further made explicit the connection between CROC and AROC and described the mental imagery she hopes her students develop. She later elaborated the importance of students imagining a second object or scenario that displays a CROC relationship that would yield the same change in output over the given interval of the input quantity.

Similarly, Hannah stumbled slightly, but ultimately described AROC in terms of changes in quantities, as seen in the following interview excerpt:

> *I think one needs to understand that average rate of change means that [...] two quantities are varying but not necessarily at a constant rate of change—like the output quantity can, umm, not have a constant factor with respect to the input quantity. But the average rate of change of that relationship would be like if the…if there was a constant rate of change, the same output would be covered for a given amount of input. I think the easiest one for students to*

understand with that is the example of like distance and speed. So if you're driving your car at a constant speed and I am stopping and going and slowing down and speeding up, we will cover the same amount of distance in the same amount of time. And your—the constant rate that you go—is the same with my average rate. But I find that with that example it's really […] hard for students to talk about things not in terms of time. I also find that using the word "average" is confusing to students.

She continued to reflect on a driving context as a familiar example to support students' reasoning about AROC, but demonstrated an awareness of student thinking by highlighting that particular example as potentially problematic for students to generalize beyond contexts dependent on time. She also expressed an awareness of student difficulties with the multiple meanings of the word "average" appearing in the phrase *average rate of change*.

Interestingly, though Hannah demonstrated a relatively high level of fluency in speaking with meaning about AROC, she pointed out that this particular idea is usually difficult for her to discuss with her students, saying:

I was struggling with it, and […] it's just hard to word it in terms of input and output and varying quantities without having a concrete example. And so, to me I'm not even sure that [students are] not getting it so much as that they're not able to articulate it.

Other GTAs expressed similar difficulties in discussing the idea, both during the interview and while teaching. For those GTAs who were not actively involved in the intervention and were also not the lead course instructor (they led the break-out recitation sections while a professor or lecturer provided the lecture and assigned student grades), their discomfort in expressing their meaning for AROC appeared to be more severe; not only were they unable to verbalize a coherent meaning for AROC, but they appeared more uncomfortable in being asked to do so. Examination of the video data revealed that they were more likely to squirm in their chairs, cross their arms, or move away from the interview desk. After the interviewer asked one GTA to describe his meaning for CROC, the GTA first explained how many months it had been since he taught that idea. As the interview progressed and he was asked to explain his meaning for AROC, he stumbled through saying:

The average rate of change is…um…rather than having a, a, um, ok. So the average rate of change is the, um, is again, the relation of two quantities…[explanation omitted]…ahhh…blah. You understand what I'm getting at, I hope. I'm just putting it into words poorly.

Another GTA paused for several seconds as she debated how specific to be in her response and if she would "mess up writing it or verbally saying it." Both of these participants eventually conveyed the productive meaning for AROC, but not without discomfort, lengthy periods of reflection and scratch paper.

Mid-Intervention Meanings for AROC

Because of the extreme difficulty some of the GTAs had in describing a meaning for AROC, we gathered written data from six GTAs at one university while they were actively engaged in the intervention. Participants wrote responses to several tasks asking about their meanings for CROC and AROC during the week after they taught the idea in their respective classes, either as lead instructor or as recitation leader. We attribute some of the variation in productivity of responses to whether or not the GTA was lead instructor or recitation leader (see Table 2). All but one of the GTAs wrote a description of AROC as a CROC satisfying some condition. The first GTA produced a written description of AROC that conveys the productive meaning for AROC *and* uses the same level of precision in language supported in the intervention and curriculum materials. This GTA had extensive experience studying student thinking around the idea of AROC outside of her participation in the intervention and had been selected to help lead the weekly meetings of the intervention during the academic year.

In sharp contrast to the pre-intervention computational and geometric expressed meanings, GTAs 2-6 all refer to the CROC needed to achieve the "same" or "given" change in one quantity relative to the "same" change in the other quantity. At this stage of the intervention, the sharpest criticism of these responses is that they neglect to clarify what these changes are the same *as* (i.e., a mention of an existing relationship between two quantities and the corresponding changes under inspection). In the same set of tasks, the GTAs were further prompted to construct new problem scenarios that would require a student to use the idea of AROC to understand. All five of these GTAs produced coherent problems that did, in fact, rely on the productive meaning of AROC in the context of distance and time. Interestingly, GTA 1 was the only GTA to construct a problem involving two *non*-time quantities, with her example asking a student to determine a vehicle's average rate of change of miles traveled with respect to gallons used based on an odometer reading and information about number of gallons purchased at a gas pump.

Table 2. Written responses to the question "What is *your* meaning for 'average rate of change'?"

GTA #	Type of GTA	Written response	Meaning Category
1	Lead instructor	I imagine 2 quantities x and y changing together, not necessarily at a constant rate of change. As x changes from x_1 to x_2, I imagine that y changes from y_1 to y_2. x_1 and y_1 stand for an arbitrary ordered pair that is fixed. The average rate of change of y with respect to x over the interval (x_1, x_2) is the constant rate of change needed to have the same change in x (that is $x_2 - x_1$) and the same change in y (that is $y_2 - y_1$) as the original function.	Productive - General
2	Lead instructor	The constant rate of change needed to cover the change in the output given the change in input.	
3	Recitation Leader	The constant rate of change needed to produce the same change in output given the same change in input.	
4	Recitation leader	AROC is the constant rate of change required to cover the same change in the dependent variable over the same change in the independent variable.	
5	Recitation leader	It is the constant rate of change that you have to apply to your variable to get a fixed value of your answer. For example, if it is speed: it is the value of speed that you should pick to get a given distance in a given amount of time.	
6	Recitation Leader	It is the constant rate of change to have traveled the same amount distance in the same amount of time.	Average Speed
7	Not teaching	Sum all the rate of change, the divide by the total number.	Incorrect

Surprisingly, GTA 7, who participated in the intervention by attending the summer workshop and the weekly seminar, in hopes of getting a teaching assignment during the subsequent semester, again gave a response highlighting the stability of a meaning for AROC as the arithmetic mean by describing AROC as the arithmetic mean of rates of change. Even more than the interview setting, getting this type of response from a GTA in writing points to the difficulties of getting individuals, even those with sophisticated mathematical backgrounds, to shift their thinking and communication about the idea of AROC.

Discussion

The vast majority of participants held impoverished meanings for the idea of AROC at the beginning of the study, primarily focused on computation or a geometric interpretation restricted to graphical representations of function relationships in the Cartesian coordinate system. Once the GTAs participated in a summer workshop and taught using Pathways research-based curriculum designed to support the productive meaning for AROC, the meanings GTAs conveyed in written responses more closely matched the meaning supported by the intervention. Our interviews with participants who were no longer attending the weekly seminars revealed that these GTAs had retained aspects of the productive meanings for the idea of AROC that we desired. In particular, the majority of post-intervention interviewees attempted to give a meaning for AROC that went beyond a computational or geometric meaning for the idea. What remained variable, however, was the fluency with which they conveyed these somewhat more productive, albeit unstable, meanings.

The initial impoverished meanings expressed by graduate students were widespread across all three institutions, suggesting that that the relatively impoverished meanings expressed by mathematics graduate students' prior to the intervention is likely a widespread phenomena and is in need of further investigation. Moreover, it bears noting that some of these graduate students had prior teaching experience, typically as recitation leader for a range of calculus classes. Yet their experiences in teaching these classes did not support them in building strong, connected meanings for ideas of slope, rate of change, constant rate of change and average rate of change. These findings challenge the assumptions that graduate students in mathematics have strong meanings of fundamental ideas of mathematics, and further, that having taught a course guarantees such meanings will be constructed. In fact, when asked what it means to talk about "rate of change," one GTA responded by saying, "Actually, I've taught calculus before, like Calc 1. Usually in that class, I don't talk about rate of change to them. So I haven't thought about that before." Such a statement suggests that interventions such as what we have described are necessary in fostering reflective teaching practices among future faculty that might lead to more productive and coherent mathematics instruction. Failure to take action to debunk these faulty assumptions by supporting GTAs' development of conceptually coherent and meaningful conceptions of key mathematical ideas may have severe consequences for improving the predominantly procedural focus that exists in many introductory undergraduate courses in colleges and universities across the United States of America (e.g., Tallman & Carlson, 2012).

Graduate mathematics students who hold a meaning for AROC that is strictly geometric (i.e., slope of secant line) will be unable to support their students in developing a quantitative meaning for AROC. It is the quantitative meaning for AROC that an individual leverages to build connections between accumulation functions and rate of change functions, an activity foundational to applying the tools of calculus in contextualized problem scenarios. In particular, those bound by a geometric meaning for AROC are similarly limited to thinking about derivatives in a calculus class as the result of some limiting process for slopes of secant lines. Individuals with this meaning will be hard pressed to make connections between this slope-of-tangent-line meaning and, say, using the derivative to predict population values given a model of population relative to time.

Though not the focus of this research, there was mention during some of the interviews of how the Pathways materials exposed the participants to new ways of thinking about the mathematical ideas. As one GTA said, her participation in the intervention gave her the experience of "living conceptual learning…all of a sudden [she] was being asked questions about linear functions [she] couldn't answer and that really transformed [her] image of what precalculus was and the math that [she] was teaching." Other GTAs emphasized that their new ways of thinking further had positive effects on their own performance as mathematics PhD students. Interestingly, however, these new ways of thinking did not necessarily translate to what the participants had as goals for their students' learning. There was evidence of tension between what the GTAs had experienced as undergraduate students and what they were being asked to do as instructors with regards to discussing mathematics meaningfully.

On a similar note, experiences in working with the graduate students during the interventions produced encouraging anecdotal evidence that the opportunity to reconceptualize fundamental ideas may have a lasting impact on their image of what effective mathematics teaching entails. GTAs with prolonged exposure to the intervention (more than one year, and sometimes in conjunction with a leadership role within the intervention) more frequently commented on both wanting students to develop a more coherent view of the mathematics and the role of understanding their students' thinking in helping them become more effective instructors. One GTA even suggested that, depending on the level of influence his future position will offer, he could envision leading a similar type of program for instructors of a common course to provide a space for discussing the mathematics, student thinking and ways of supporting student learning. This leaves us optimistic that ours and other similar efforts might motivate mathematicians to engage in work to make undergraduate mathematics instruction more meaningful for students.

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