EXPLORING SHIFTS IN A TEACHER'S KEY DEVELOPMENTAL UNDERSTANDINGS AND PEDAGOGICAL ACTIONS

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Research has identified mathematical knowledge for teaching (MKT) as specialized knowledge that contributes to a teacher's ability to teach for understanding. This report discusses one teacher's (Claudia) MKT as she implemented research-based precalculus materials. Claudia initially encountered difficulty implementing the materials in ways that supported her students' learning. As the semester progressed, Claudia's understanding of the central ideas of the course deepened and specific ideas emerged as key for understanding other ideas. Once she became aware of these connections she made more informed pedagogical choices that supported her students' thinking and learning.

Background

It is well established that although content knowledge is an important attribute of a successful teacher, content knowledge alone is not sufficient to support students' learning processes (Shulman, 1986; Silverman & Thompson, 2008). Explorations into relationships between a teacher's content knowledge and her/his practice have identified *mathematical knowledge for teaching* (MKT) as a critical link between content knowledge and supporting students' learning (Ball, 1990; Simon, 2006). Recent studies (Silverman & Thompson, 2008) have emphasized the importance of exploring teachers' ability to transform content knowledge from personal understandings to knowledge that has pedagogical power in the classroom.

Project Pathways is an ongoing initiative with a focus on improving secondary mathematics teachers' practice and secondary students' learning. The project's interventions (e.g., graduate courses, professional learning communities, and workshops) intend to improve teachers' content knowledge and their ability to achieve conceptually oriented classrooms that engage students in problem solving activity. During initial phases (years 1-4) of the project the research team observed gains in teachers' content knowledge, but shifts in the teachers' pedagogical actions lagged behind the content knowledge shifts. When investigating the obstacles preventing the teachers from shifting their practice, project members identified that the mathematical content in their curriculum was incompatible with the mathematical ideas developed during the project's interventions.

In an attempt to better support the teachers in improving the conceptual focus of their teaching, the project shifted the concentration of the summer work and professional learning communities (PLCs) to examining and modifying the teachers' curriculum, but the teachers found revising their curriculum to be a daunting and exhaustive task. The heavy burden of writing curriculum led the project leaders to offer a research-based, conceptually oriented precalculus curriculum (Carlson & Oehrtman, 2010) for use by one precalculus teacher (Claudia). We conjectured that the combination of the *Project Pathways* interventions and curriculum tasks would support Claudia in realizing the desired shifts in her practice and students' learning. Our report discusses the role of Claudia's MKT in her implementation of the curriculum. Claudia's difficulties and successes with the curriculum illustrate the relationship between Claudia's content knowledge, her transformation of this knowledge in ways that inform her practice, and her ability to support her students' learning processes.

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Theoretical Perspective

We leverage the construct of *key developmental understanding* (KDU) (Silverman & Thompson, 2008; Simon, 2006) to examine MKT in the context of secondary precalculus mathematics. Like Silverman and Thompson (2008), we believe that a teacher's mathematical conceptions and reasoning abilities influence her/his pedagogical choices and actions. In order to describe understandings that support MKT, Simon (2006) and Silverman and Thompson (2008) characterized KDUs as one's understandings and ways of reasoning that connect various ideas, provide foundations for learning other ideas, and support solving novel but related problems as a consequence of these understandings.

Although KDUs are critical features of a person's mathematical knowledge, Silverman and Thompson (2008) highlighted that a teacher's development of KDUs does not imply these KDUs will inform her/his practice. Silverman and Thompson suggested that MKT is developed when a teacher transforms a KDU in a way that shapes and informs their interactions with students. Specifically, Silverman and Thompson described:

> A teacher has developed knowledge that supports conceptual teaching of a particular mathematical topic when he or she (1) has developed a KDU within which that topic exists, (2) has constructed models of the variety of ways students may understand the content (decentering); (3) has an image of how someone else might come to think of the mathematical idea in a similar way; (4) has an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea; (5) has an image of how students who have come to think about the mathematical idea in the specified way are empowered to learn other, related mathematical ideas. (p. 508)

As the *Pathways* interventions transitioned to focus more on applying the teachers' content knowledge to their teaching practice through curriculum design, we conjectured that the interventions would better support the teachers in developing KDUs that informed their pedagogical practices. We expected that our providing Claudia with quality curriculum would better position her to construct the MKT necessary to achieve shifts in her practice. Specifically, we conjectured that the use of quality curriculum would lead to Claudia developing KDUs and transforming these understandings in ways that informed her pedagogical actions.

Research Question

How does a teacher's MKT impact her pedagogical choices and ability to support student learning? How does a teacher's MKT impact her ability to successfully implement research-based curriculum?

Methodology

During the semester in which Claudia used the Pathways curriculum, she attended a weekly 50-minute PLC composed of 4-6 trigonometry and precalculus teachers. She also sporadically attended a 3-hour trigonometry PLC after school with two other trigonometry teachers and a project member (the first author of the report) who served as the PLC facilitator. The PLC members primarily examined the knowledge involved in understanding and learning key ideas of their courses. The PLC members also spent time developing conceptually oriented tasks for classroom use.

Three *Project Pathways* members regularly observed Claudia's classroom and worked with Claudia to answer her questions about the curriculum. Claudia's classroom was videotaped two class periods a day and these videos were digitized for analysis. The project members took field notes during Claudia's PLC meetings, when observing Claudia's classroom, and during her meetings with project members. The project members directed the field notes towards Claudia's content knowledge as revealed by her interactions with her colleagues, her students, and the project members. As Claudia interacted with her students during class sessions, her responses and questioning offered insights into her mathematical understandings and thinking. Similarly, her interactions during the PLC meetings offered glimpses into her content knowledge, particularly when the PLC sessions focused on discussing how students learn various topics. Relative to Claudia's meetings with *Pathways* members, she often asked for clarification of the mathematical ideas driving the curriculum. Claudia also discussed questions she had about the solutions and reasoning that her students were providing. Collectively, the project members used these data sources to characterize Claudia's teaching practice by describing her mathematical content knowledge, pedagogical actions, and improvements in her students' learning.

As the semester progressed, the project members who observed Claudia's classroom noted shifts in her practice and her ability to scaffold her students' thinking. In order to characterize shifts in her teaching and her questioning, the research team analyzed her classroom videos to document relationships between her propensity to focus on student thinking and her questioning techniques (Teuscher, Moore, Marfai, Tallman, & Carlson, submitted). After analyzing her classroom sessions, the research team chose three particular sessions that were representative of her shifts over the course of the semester.

The present study extends the aforementioned study (Teuscher et al., submitted) by analyzing Claudia's thinking and MKT during the classroom sessions that the research team deemed to be representative of Claudia's practice (see Teuscher et al. for an explanation of the case selection). Compatible with Thompson's description of a conceptual analysis (Thompson, 2008), we analyzed Claudia's interactions with her students in an attempt to characterize the mental actions driving Claudia's pedagogical actions (e.g., the mathematical content knowledge that informed her questioning and utterances). We then compared the results of this analysis to the project members' field notes that were taken during PLC meetings in which the teachers discussed similar content topics. By comparing these data sources, we characterized information about Claudia's content knowledge and MKT and the influences of these knowledge bases on her pedagogical actions.

Results

In the following section we report the results of Claudia's interactions with her students when they were completing two in-class activities. We also discuss Claudia's meetings with project members and her PLC members in the context of her MKT and ability to support her students' learning.

During the first week of class Claudia's students completed The Diving Task (Figure 1) while working in groups of 3 or 4. The task asked the students to compare and contrast the meaning of average rate of change and average as an arithmetic mean. The previous lesson developed a meaning of average rate of change in the context of average speed (e.g., the constant rate of change needed to cover the same distance in the same amount of time). *Excerpt 1* presents Claudia's interactions with one group of students.

In a diving competition, a diver received the following scores from 4 judges after making a dive.

Judge 1	Judge 2	Judge 3	Judge 4
8.7	9.3	8.0	8.2

How does the meaning of the word *average* when computing a diver's average score compare with the meaning of the word *average* when computing a diver's average speed?

Figure 1 – The Diving Task

Excerpt 1.

S1:	Do you know the meaning of average in this problem?
C:	Tell me what you have so far.
S2:	If the average is 8.55…
C:	Okay, so…
S1:	I said that 8.55 would mean average would be like the constant or the, I don't want to say constant.
S2:	You said approximate.
S1:	The approximate score.
C:	Keep going.
S3:	Can we use mean to describe average?
C:	(*laughs*) No.
S1:	The approximate score to get your score.
S4:	No, cause…
C:	Your…
S4:	…the approximate score given by the judges for that dive…
C:	Keep going, the approximate score by the judges, what…
S4:	…for the dive.
C:	Okay, for what dive?
S2:	For the diver's dive.
C:	For just one dive?
S:	Yeah (*students in unison*).
S2:	They only did one dive.
S1:	If each, if one judge per dive or isn't it…
C:	Four judges for the one dive, okay, alright. Okay so tell me again what you said.
S4:	The approximate score given by the judges for the one dive.
C:	Okay, by which judges?
S1:	By the four judges.
C:	By what?
S2:	By the four judges. (*Claudia confirming the student's answer and moving on)*

During this interaction, Claudia's questioning did not result in her students making explicit connections between average as an arithmetic mean and average rate of change. For instance, she failed to leverage a student's reference to "constant" (lines 5-6) and she did not ask the students to expand on their meaning of "approximate" (lines 7-8, 12, & 15), nor did she ask them to compare and contrast their thinking with the outcomes of previous tasks. Additionally, Claudia's questions were mathematically shallow and she allowed the interaction to end with the students failing to develop the understandings as intended by the curriculum.

Claudia approached a project member after this class session and expressed an uncertainty of the ideas driving the diver task. She also expressed confusion about the tasks relation to previous tasks and claimed that her confusion about the intent of the questions prevented her from determining how to guide and support her students' thinking. Claudia's interactions with the project members, in combination with her interactions with her students in *Excerpt 1*, suggest that Claudia did not understand the content in ways compatible with the curriculum's intent. In lieu of her lacking the understandings necessary to make appropriate sense of the activity, she did not have the foundational knowledge necessary to identify ways to support her students in learning the key ideas of the lesson. Furthermore, Claudia's actions indicate that her understanding of arithmetic average consisted of a vague connection to an approximate value and that she had not cognitively distinguished notions of average rate of change and an arithmetic average in a way that she could support her students in making similar comparisons.

As the semester progressed, the project members observed an increase in the mathematical depth of Claudia's questioning in both her classroom and during her meetings with project members. During a lesson on linearity, during which the students created a graph, Claudia questioned her students in an attempt to draw their attention to determining changes of output for equal changes of input. The shifts the project members observed near the middle of the semester indicated that Claudia was developing understandings that enabled her to interpret the curriculum tasks in ways consistent with the curriculum's design. However, our analysis of Claudia's classroom also revealed that she was having difficulty engendering these understandings in her students (Teuscher et al., submitted). Such a finding suggests that Claudia had not transformed her understandings into mathematical knowledge that was useful in her teaching (e.g., MKT).

As the semester progressed the project members observed Claudia continuing to improve her content knowledge. The project members also observed a transition in Claudia's ability to support her students' learning. The concluding module of the precalculus curriculum introduces trigonometric functions, beginning with a lesson on angle measure. Following the lesson on angle measure, a lesson introduces the sine and cosine functions. Specifically, the curriculum introduces these two trigonometric functions using a circular motion context (e.g., a bug on the tip of a fan blade) and asks the students to determine how the bug's vertical distance above the center of the fan varies as the fan rotates. When beginning this problem, Claudia posed the following question: Does the angle measure change if the radius of the fan is changed, but the distance the bug travels (measured in radians) is not changed? *Excerpt 2* presents Claudia and two students discussing their response to this question.

Excerpt 2

S1:	How does the angle change if the radius changes? The angle stays the same.
S2:	Yeah, but if the distance traveled around stays the same, but the radius changes then it will be a different amount of radians.
S1:	Oh yeah.
S2:	It will be inversely proportional.
C:	It will be a different amount of radians?
S2:	The angle measure.
S1:	Well the distance the bug travels.
S2:	Because as the radius gets bigger, the arc length will be a lesser portion of the circle meaning a lesser angle.
C:	Ok, so draw, ok so let's take any two circles, if you were to cut out a 90 degree angle, actually, draw a circle inside of a circle for me.
S2:	The arc length is…
C:	Now draw another circle - yeah.
C:	Ok, now do you have a Wikki Stix, does anyone, you do?
C:	Ok, first for the inside circle I want you to mark off one radian of the

17		circumference.
18	S2:	Can we just take a Wikki measure, do I have to use radians?
19	C:	She is trying to find the center of her circle though, is what she is doing. Here let
20		me get you a smaller Wikki Stix. Here is a skinner one it might be easier to use.
21	S1:	It is two and half so it would be like in the middle
22	C:	Yeah so for the small circle mark off one radian of the circumference. Ok do the
23		same thing for the bigger circle.
24	S2:	One radian?
25	C:	Yeah one radian
26	S2:	But the radian of an angle measure is not arc measure, the radian and arc measure
27		are the same. No, the angle measure is the same…
28	S1:	One radian is the same…
29	C:	What is the same about one radian?
30	S1:	The same angle.
31	C:	Yes, what is the not the same about one radian?
32	S1:	The arc length.
33	C:	Yes, do you see that S2?
34	S1:	But then what is distance traveled? Is that the arc length?
35	S2:	Yes.

Claudia first asked the students to clarify their claim that the radian measure changes when the radius changes (line 6). Student 2 made an incorrect claim (lines 9-10) and in order to support the student in confronting his misconception, Claudia focused the students' attention on the idea of measuring arcs in radii (lines 11-29). Claudia's line of questioning led the students to realize that an arc of one radian, regardless of the size of the circle, corresponds to an angle with the same amount of openness (lines 28-34). As their interactions continued, Claudia probed her students to elaborate on the implications of measuring arcs relative to the radius, with Student 1 concluding, "The angle measure doesn't change if the radius is changed."

Later in the class session, the same group of students attempted to graph the bug's vertical distance above the center of the fan in relation to the bug's distance traveled around the fan. The students were experiencing difficulty relating the two quantities when Claudia joined their discussion (*Excerpt 3*).

Excerpt 3.

1	C:	Let's back up a step…Your bug's going to start moving. Now what's going to
2		happen to the total distance?
3	S1:	It's going to increase.
4	C:	Ok, alright. And what's happening to the vertical distance, in general?
5	S1:	It's increasing at a decreasing rate.
6	C:	Ok, it's going to increase, and then, why did you say it's going to increase and then
7		decrease (*referring to a student's statement given previous to this interaction*)?
8	S1:	'Cause once it gets to 12 o'clock, then it's going to, the vertical distance is going to
9		decrease back to zero.
10	C:	Ok, what is the vertical distance at 12 o'clock.
11	S1:	About 2.6 feet.
12	C:	And how do you know that?
13	S1:	Because that's the radius.
14	C:	Ok, so, you know here (*pointing to the top of the fan*) that your vertical distance is

15		2.6 feet. And then what does it do?
16	S1:	It decreases.
17	C:	So, at a total of distance of zero, your vertical distance is zero, right?
18	S1:	(*nodding yes*)
19	C:	Now, what is your…
20	S1:	Then your total distance is 2.6 and your vertical distance is 2.6 as well.
21	C:	So when you get to the top of the circle, when you get to here (*pointing to the top*
22		*of the circle*), have you traveled one radian?
23	S1:	Ya.
24	C:	How many radians are there in a half circle?
25	S1:	Oh, um, pi.
26	C:	So how many radians have you traveled to here?
27	S1:	Pi over two.
28	C:	Ok, so what is that?
29	S1:	That is 1.57 (*identifying values on graph*).

In order to support the students in determining the graph, Claudia prompted the students to reason about variations in the total distance (lines 1-2). As the discussion progressed, Student 1 identified that the vertical distance increased and then decreased as the total distance increased (lines 5-16). Claudia then asked the student to describe specific pairs of values and Student 1 incorrectly identified that the bug travels an arc length of 2.6 feet from the starting position to the top of the circle (lines 17-20). In response, Claudia returned the student to reasoning about measuring the arc length in terms of a number of radii (lines 21-29), which supported the student in correcting her graph and creating the sine graph by the completion of this activity.

Claudia's continued focus (*Excerpts 2-3*) on measuring arcs in radii (which was covered in a previous lesson) suggests that she found this to be an important idea for understanding the sine and cosine functions. Claudia probed her students in ways that supported their measuring arcs in radii when determining the graph of a trigonometric function. These actions suggest that Claudia had developed an understanding of angle measure that she considered developmentally key for learning trigonometric functions (e.g., a KDU), and that she had transformed this understanding in a way that influenced her pedagogical actions and ability to support her students' learning (e.g., she developed MKT).Claudia's interactions with her trigonometry PLC further illustrated that her understanding of angle measure informed her pedagogical actions when teaching the sine and cosine functions. During the year prior to Claudia's use of the curriculum, it did not occur to her or her PLC members that reasoning about measuring arcs in radii is foundational for constructing understandings of the sine and cosine functions. After Claudia's use of the curriculum, her PLC returned to the topic of trigonometry and the PLC members chose to design a lesson exploring applications of trigonometric functions. Claudia became troubled about the topic choice and explained that she did not believe that they could teach applications of trigonometric functions conceptually without teaching angle measure in a way that supported students' understanding of trigonometric functions. Claudia's justification for her decision is consistent with our claim that Claudia's understanding of angle measure as measuring arc lengths in units of radii was key for the way she had come to conceptualize trigonometric functions, and that this understanding provided a basis for her pedagogical decisions.

Discussion and Conclusions

Claudia exhibited content knowledge shifts during the early phases of implementing the curriculum, but she had difficulty supporting her students' learning. At these early stages, her improved understandings did not include connections that supported her re-conceptualizing her teaching in terms of how her students may form similar understandings. However, Claudia's sustained focus on her students' thinking, along with the support she received from the project members, resulted in her building connections that she leveraged to support her students' learning. Specifically, as she developed and transformed KDUs to MKT, the research team observed Claudia scaffolding her students' learning by drawing connections between ideas. Future research should examine the supports (e.g., conceptual curriculum) necessary to help teachers transform their understandings to knowledge that has pedagogical power.

The findings from this study reveal that the nature of a teacher's understanding of an idea is key to her/his conceptualization of teaching that idea and her/his interactions with students when teaching that idea. Claudia's transitions over the course of the semester suggest that a teacher transforming her/his content knowledge in ways that inform her/his practice (e.g., developing MKT) is a complex process. As Stigler and Hiebert (1999) have noted, a majority of pedagogical shifts are superficial. However, Claudia's knowledge and pedagogical transitions were substantive. These shifts that we observed are consistent with Silverman and Thompson's (2008) claim that significant changes in teaching practices result from "pedagogical conceptualizations of the mathematics: both the sense they have made about the mathematics and their awareness of its conceptual development" (p. 507). Claudia was unable to support her students' learning in profound ways until she had acquired key developmental understandings (e.g., measuring arcs in radii to develop trigonometric functions) and used these understandings to scaffold her students' learning (e.g., she transformed these KDUs to MKT). The findings presented illustrate the necessity of future studies that explore the role of KDUs in teaching and the process by which teachers' transform KDUs to understandings (e.g., MKT) that improve their ability to scaffold and build upon their students' learning.

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