CALCULUS READINESS: COMPARING STUDENT OUTCOMES FROM TRADITIONAL PRECALCULUS AND AP CALCULUS AB WITH A NOVEL PRECALCULUS PROGRAM

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This comparative study examines three groups of high school students' understanding of the foundational concepts needed for calculus. Students completed one of three courses: (1) traditional precalculus, (2) novel precalculus, or (3) Advanced Placement (AP) Calculus AB. Student scores on the Precalculus Concept Assessment (PCA) and two open-ended tasks which focused on functions and rate of change provided the data. Student work was analyzed for strategies employed and efficiencies with their strategies on the two tasks. The results revealed that students who completed the novel precalculus curriculum gained a deeper understanding of the fundamental concepts for calculus by performing higher than the traditional precalculus students and comparably to the AP Calculus AB students.

Introduction

Precalculus is a mathematics course offered in high schools across the United States with approximately 18.5% of all high school students enrolling during their third or fourth year of high school. Yet a growing student population is required to enroll in precalculus when they enter college because they either did not complete it in high school, but more often students do not score high enough on the individual college mathematics placement exams and therefore are required to enroll in a course they may have completed during high school. Parents, students and educators are asking: (1) What are students not learning in high school precalculus that is causing them to have to retake the course when they enter college and (2) What mathematics content is missing from the high school precalculus curriculum that could help students better prepare to enter college calculus?

Over the past 20 years researchers have focused on college students understanding of function and the different reasoning patterns they need to be successful in higher level mathematics, such as calculus (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson & Oehrtman, 2005; Engelke, Oehrtman, & Carlson, 2005, October; Ferrini-Mundy & Gaudard, 1992; Ferrini-Mundy & Graham, 1991, 1994; Markovits, Eylon, & Bruckheimer, 1988; David Tall, 1992; David Tall, 1997; Thompson, 1994a, 1994b). However, less research has been conducted at the high school level to understand what secondary students who are enrolled in precalculus understand or do not understand about function and if the reasoning patterns and frameworks used for undergraduate students can or should be applied to high school precalculus students. Therefore, in this study we seek to answer the following two research questions: (1) How do high school precalculus students who used either a traditional precalculus curriculum or a research based conceptually–oriented curriculum compare on the foundational concepts needed for calculus and their efficiency and use of mathematical strategies? (2) How do high school precalculus students who completed a conceptually-oriented curriculum compare with AP Calculus AB students on the foundational concepts needed for calculus and their efficiency and use of mathematical strategies?

Theoretical Framework

The concept of function is an essential knowledge students need to be successful in higher level mathematics. For students to understand the concept of function they must first obtain a process view of function (Breidenbach et al., 1992), meaning they understand that they are mapping a set of input values to a set of output values and it is a continuum. Second, students must develop covariational reasoning (Carlson, 1998), described as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354).

Typically precalculus teachers and the curriculum they use focus student attention on learning the algebraic manipulations and procedures rather than the conceptual understanding of function that provides foundational knowledge needed to build on for studying higher level mathematics. Breidenback et al. (1992) and Dubinsky and Harel (1992) described two views of functions students have: *action* or *process*. An action view of function is described as "the ability to plug numbers into an algebraic expression and calculate" (Dubinsky & Harel, 1992, p. 85). Students tend to think about the function in a procedural way as they input a value into a specific function to calculate the correct output doing "plug and chug" without any conceptual understanding of function which may eventually hinder their understanding of calculus concepts such as limit, derivative, and integrals.

Carlson and Oehrtman (2005) extended Breidenbach et al. (1992) and Dubinsky and Harel's (1992) research on the action and process views students have of function arguing that student's ability to answer function focused tasks was related to two types of reasoning.

> First, … students must develop an understanding of functions as general processes that accept input and produce output. Second, they must be able to attend to both the changing value of output and rate of its change as the independent variable is varied through an interval in the domain. (Oehrtman, Carlson, & Thompson, 2008, p. 5)

For example, students who hold an action view of function have multiple misconceptions about piecewise functions being several functions and do not reason over an entire interval. Students tend to focus on specific points instead of interpreting the entire function. They also tend to not relate the domain and range to the inputs and outputs of the function. Without an understanding of functions accepting inputs and producing outputs students will struggle to reverse (inverse functions) the process. Most students are able to find the inverse of a function algebraically (switch the x and y and solve for the y variable) or geometrically (reflect the function over the line y = x); however, their answer typically has no meaning (Carlson & Oehrtman, 2005). Students who hold an action view are able to work with functions procedurally, but have little conceptual understanding.

On the other hand, students who hold a process view of function can imagine an entire function and how the input values affected the output values. This is described as covariational reasoning, the ability to vary inputs and outputs at the same time and interpret or understand their influence on the rate of change. Students who demonstrate a process view of inverse functions use a reversal process that defines a mapping of output values to input values. A process view of function is essential for understanding calculus concepts such as the limit, derivative, and integral.

Carlson, Oehrtman, and Engelke (2010) developed a taxonomy of foundational knowledge for beginning calculus representing the key concepts of precalculus students should know and understand prior to entering calculus. The *PCA* taxonomy was used to develop the *Precalculus Concept Assessment (PCA)*, a 25 multiple–choice item exam that assesses rate of change, function, and covariational reasoning that students will build upon in their study of calculus. The *PCA* and PCA taxonomy have gone through multiple refinements as research on student thinking is conducted (Carlson et al., 2010). Although the details of the PCA taxonomy are not part of this paper it is imperative to note that the existence of such a taxonomy is one of the reasons why the *PCA* is a valid tool for assessing students understanding of the foundational concepts of precalculus prior to entering calculus.

The *PCA* was developed over a 15 year time period based on numerous college students' interviews of what they thought and understood as they answered open–ended questions about functions. The distracters on the *PCA* are common misconceptions that college students have about functions (Carlson et al., 2010). More recently, the *PCA* was used by high school precalculus teachers to determine their students' understandings and misconceptions after completing the year long course and entering AP calculus (Teuscher, 2008). These data were influential in helping teachers discover holes in their mathematics curriculum and making changes to help students develop the key precalculus concepts prior to entering calculus.

Methods

In this study we examined high school students' understanding of the foundational concepts of calculus as described by the *PCA* and two open–ended tasks focused on rate of change. We coded students' strategies in solving these tasks and compared their efficiency in using the strategies. In this section we describe the instruments and participants.

Precalculus Concept Assessment

The *PCA* was developed by faculty from the Department of Mathematics at Arizona State University and was designed to reflect core content and common misconceptions students have about functions (Carlson et al., 2010). Carlson et al. (2010) administered the *PCA* to 902 college precalculus students and found the "Cronbach's alpha of 0.73 indicating a high degree of overall coherence" (p. 137). Carlson and colleagues also conducted clinical interviews with more than 150 college students so as to validate the *PCA* items. Each question was tested and validated to guarantee that students who selected a specific distracter (i.e., answer choice) consistently provided similar justifications during interviews (Engelke et al., 2005, October).

Open–ended Tasks

The *Piecewise Functions* (*PF*) task was adapted from a released 2003 AP Calculus free response item by an AP Calculus teacher for precalculus students; it gives students the graph of a piecewise function and has them find rates of change and then write equations for the function. The *Filling the Tank* (*FT*) task was taken from examples developed by Peter Taylor (1992) and it requires students to compare average flow (rate of change) with instantaneous flow as water is flowing in and out of a tank simultaneously. Both tasks build on the foundational concepts of functions and rate of change that students need before calculus and require students to use reasoning and demonstrate their understanding of the meaning of functions; however, one task is based in a contextual setting and the other is not.

Students worked on the two tasks individually during a scheduled class period during the first week of class and the student work was scored for correctness using scoring guides and coded for solution strategies on specific questions for both open–ended tasks. The items coded for solution strategies were *PF* questions 1, 2 and 4 and *FT* questions B and E. The five categories of strategies were: Formula, Graph, Table, Multiple, and None or can't tell. The scoring guides for the open–ended tasks required 22 scoring decisions on each of the *PF* and *FT* tasks and the average reliability of scoring was 95% and 96% for the *PF* and *FT* tasks respectively.

Participants

High school students from Rover High School located in the southwestern region of the United States, who completed either Precalculus or AP Calculus AB during the 2009-2010 school year and enrolled in AP Calculus BC for the 2010–2011 school year were asked to participate in the study. All mathematics teachers at Rover High School are part of a National Science Foundation Math and Science Partnership grant (No. EHR–0412537). The students in this study (N = 36) were taught by one of four different teachers in their previous mathematics course (Precalculus or AP Calculus AB). Four mathematics teachers taught students in precalculus during the 2009–2010 school year. Three of the four teachers used *Precalculus: Mathematics for Calculus* (Stewart, Redlin, & Watson, 2005) and the fourth teacher used a research based conceptually–oriented curriculum *Precalculus: A Pathway to Precalculus* (Carlson & Oehrtman, 2010). All precalculus students took the *PCA* at the end of the 2009–2010 school year. Students who enrolled in AP Calculus BC for the 2010–2011 school year completed the two open–ended tasks, *PF* and *FT* during the first week of class.

Please note that the three groups of students were not selected deliberately; they naturally emerged as a result of the mathematics classes they were enrolled and the choice of curriculum followed by their teachers. As a matter of fact, only one teacher chose to use *Precalculus: A Pathway to Precalculus* and this is how the PP students, who are of major interest in this paper, were selected.

Results

The goal of this study was to assess student understanding of the foundational concepts for calculus. Therefore, we compared precalculus students' performance based on the curriculum they used. We also compared the precalculus students (N = 25) who used the Pathways curriculum to the AP Calculus AB students (N = 11) who enrolled in AP Calculus BC the following year. To summarize and clarify, the three groups of students of interest in this study were: (1) students who were taught using P*recalculus: Mathematics for Calculus* (Stewart et al., 2005), which we refer to as Traditional Precalculus (TP; N = 11); (2) students who were taught using *Precalculus: A Pathway to Precalculus* (Carlson & Oehrtman, 2010), which we refer to as Pathways Precalculus (PP; N = 14); and (3) students who completed AP Calculus AB which we will refer to as AP Calculus AB (AB).

The analyses in this study are presented in four parts. First, we provide descriptive and comparative analyses to compare the PP students with both the TP students and the AB students. Second, we present a breakdown and distribution of student scores to the items on each task and compare the PP students with the other two groups (TP and AB). Third, we describe the solution strategies used by students based on the three groups. Finally, we compare the effectiveness of students' solution strategies for the PP students with the other two groups (TP and AB).

Statistical analyses and descriptives

The first analysis compares TP and PP students; in order to identify whether these groups were statistically different prior to entering AP Calculus and completing the open–ended tasks the *PCA* scores were compared. A t–test was conducted and PP students had a statistically significant higher mean on the *PCA* than TP students (t = –3.37, p = .003); therefore, further analysis to compare students' performance from these two groups on the open-ended tasks used *PCA* scores as a covariate to account for difference in learning prior to entering AP Calculus BC.

An analysis of covariance (ANCOVA) was conducted to determine the statistical difference between the mean total score on the *PF* task for the two groups (PP and TP) of students adjusted for *PCA* scores. No statistically significant difference was found between the two groups (PP and TP) for the mean total score on the *PF* task (F = 0.98, p = 0.333). A similar analysis was completed to determine the statistical difference between the mean total score on the *FT* task for the two groups (PP and TP) of students adjusted for PCA scores. Results indicated no statistically significant difference between the two groups (PP and TP) for the mean total score on the *FT* task (F = 0.20, p = 0.658).

An analysis of variance (ANOVA) was conducted to determine the statistical difference between the mean total score on the *PF* task for the two groups (PP and AB) of students. Results indicated no statistically significant difference between the two groups (PP and AB) for the mean total score on the *PF* task (F = 0.10, p = 0.753). A similar analysis was completed to determine the statistical difference between the mean total score on the *FT* task for the same two groups of students. No statistically significant difference was found between the two groups for the mean total scores on the *FT* task (F = 2.10, p = 0.161).

PCA scores for the AB students were not available because data collection on precalculus students for the MSP project began the year these students were in AP Calculus. However, the results of the two ANOVA's confirmed that there was no statistically significant difference between the PP and AB students for the mean total scores on both open-ended tasks meaning these students' prior knowledge was similar.

Breakdown and distribution of the scores

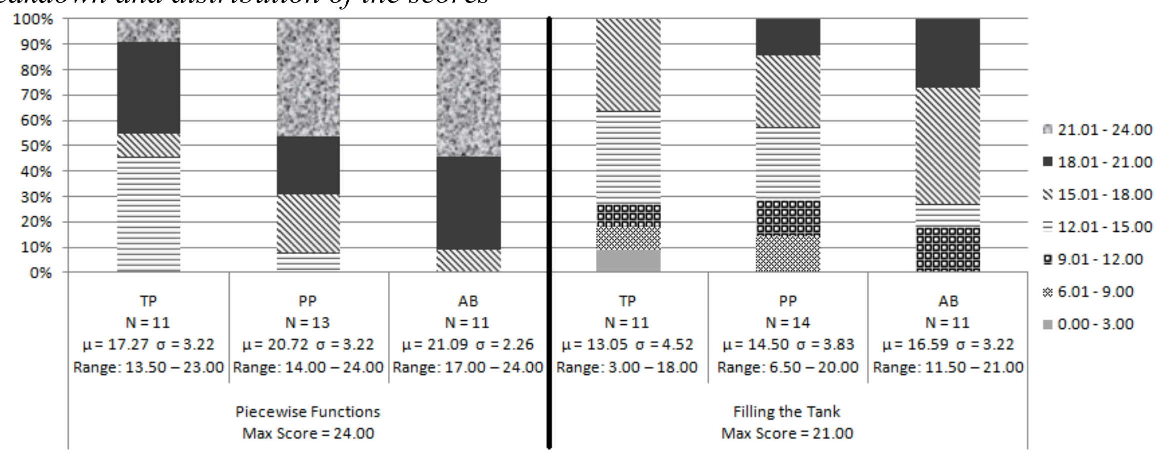

Figure 1. Score distributions on the total scores of the PF and FT tasks for the three groups of students

The distributions of the total scores for the three groups of students on the PF and FT tasks are displayed in Figure 1. For the PF task, respectively, 45% and 55% of the PP and AB students scored higher than 21 out of the total score of 24 while only 9% of the TP students scored in this range. Approximately 37% of the TP and AB students scored between 18 and 21 inclusively, out of 24. The TP group had the greatest percentage (37%) of students receive a score in the 15.01 – 18.00 range with 36% of AB students and 22% of PP students scoring in the same range. Finally, the standard deviation (3.22) was the same for the PP and TP students whereas it was less for the AP Calculus AB students (2.26).

For the *FT* task, respectively, 37%, 28%, 10% of the TP, PP and AB students scored in the 12.01 – 15 range. Meanwhile, 46% of the AB students, 37% of the TP students and 28% of the PP students scored in the 15.01 – 18 range. Although none of the TP students scored in the highest range of 18.01 – 21.00, 14% of PP students and 27% of AB students scored in this range.

Finally, the standard deviation was the greatest for TP students (4.52); whereas, the AB students had the smallest (3.22).

Distribution of the solution strategies

The distributions of the strategies for all three groups on items 1, 2, and 4 of the *PF* task and items b and e of the *FT* task are shown in Figure 2. For item 1 of the *PF* task, 77% of the PP students, 55% of the TP students and 37% of the AB students used graphing as their solution strategy. While none of the PP students used a formula, approximately 45% of AB students and 18% of TP students employed this strategy. Finally 18% of the PP students used multiple strategies.

For item 2 of the *PF* task, graphing again was the dominant strategy for a considerable number of students. PP students used graphing the most (77%); while the AB students used this method the least (37%). None of the PP students used the formula strategy; however, approximately 18% of AB students and 10% of TP students employed this strategy. Only PP students (7%) used multiple strategies when solving this question. The AB group had the highest percentage (46%) of students whose work for this item revealed no strategy and the TP group had a slightly less percentage (36%); however, the PP group only had 16% of students whose work did not reveal a strategy.

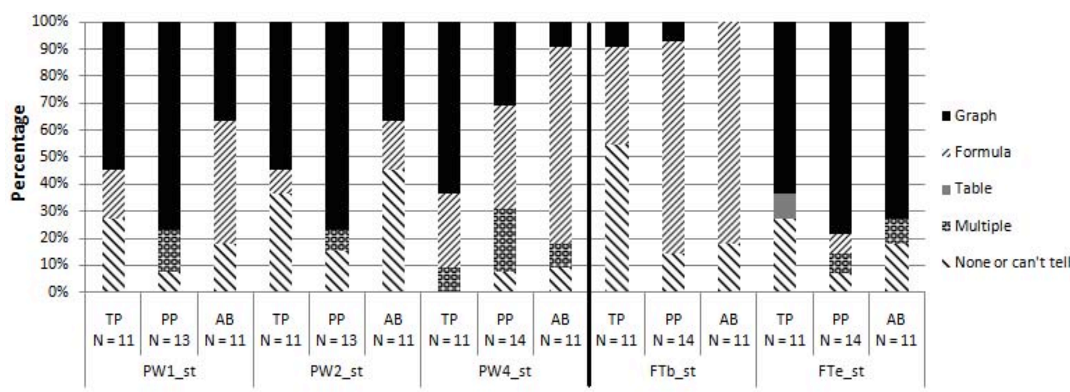

Figure 2. Distributions of strategies for the three groups of students on both of the open ended tasks.

For item 4 of the *PF* task, the TP group had the highest percentage (64%) of students who used graphing to solve this item. The PP group had the highest percentage (23%) of students who used multiple strategies, typically graphing and formula strategies, to solve this item. The AB group had the highest percentage (74%) of students who used a formula to solve *PF* item 4. Only 8% of PP and 9% of AB students work was not possible to code for a strategy.

For item b of the *FT* task, the dominant strategy was formulas for all three groups. Of the TP students, 36% used formulas, 9% used graphing and the remaining students it was not possible to code for a strategy. Of the PP students, 79% used formulas, 7% used graphing and the remaining 14% of the students it was not possible to code for a strategy. The majority (82%) of the AB students used formulas and the remaining 18% of students it was not possible to code for a strategy.

For item e of the FT task, the dominant strategy (graphing) was the same for all three groups; respectively 64%, 79% and 74% of the TP, PP and AB students used this strategy. Only TP students (9%) used tables and this was the only instance where tables were employed as a strategy. The PP group had 7% of students use formulas; whereas, 7% of PP and 8% of AB students used multiple strategies. The TP group had the largest percentage of students (27%)

whose work was not able to be coded for a strategy, while the PP group had the lowest percentage (7%). The percentage of students for whom it was impossible to tell which strategy was used was the smallest for the PP group on items 1 and 2 of the *PF* tasks and items b and e of the *FT* task.

Simultaneous Comparison of the Solution Strategies and Scores for Evaluating the Efficiency among the Groups

While it is interesting to indicate strategies students used in solving the open-ended task, it is also vital to know their respective efficiencies (i.e. what strategies do students use when they receive higher scores?). For this purpose we examined simultaneously the scores and strategies to assess for each group the strategies employed within each score range. Particular interest was given to the students who could score 19 or above out of 24 on the *PF* task and 14 or above out of 21 on the *FT* task; these students constituted the majority (approximately 70%) of the PP and AB students and a minority (35%) of the TP students. For this highest scoring group, the dominant strategies were graphs and formulas, although AB students preferred formulas to graphs whenever they could whereas this was the opposite for the PP students. The TP students in this group primarily used graphs as well.

Discussion

This study assessed students' understanding of foundational concepts of calculus. Two open-ended tasks were used to compare students' understanding after completing a precalculus or AP Calculus course, yet prior to entering AP Calculus BC. Thus, there was a gap of three months between the administration of the open-ended tasks and the last time students were in school learning mathematics.

The statistical tests that compared the TP and PP students showed no statistical significant difference between these two groups based on their performance on the tasks; however, the statistical tests accounted for prior achievement by using the *PCA* scores as a covariate. The *PCA* is a powerful assessment tool used for evaluating students' understanding of the key concepts in precalculus (Carlson et al., 2010). The fact that the PP students mean *PCA* scores are statistically higher than the TP students mean *PCA* scores indicates that the PP students have a significantly better understanding of the key concepts in precalculus when compared to the TP students. Moreover a second conclusion is that the mean total score for the PP students on the open-ended tasks were consistently higher than those of the TP students. The PCA scores for the AB students were not available.

The statistical tests that compared PP and AB students showed no statistically significant difference between their mean scores on the two open-ended tasks (*PF* and *FT*); specifically, on individual items and total scores. The existence of no statistically significant differences between these groups (PP and AB) is particularly remarkable since theoretically the AB students should have performed better than precalculus student on the foundational concepts of calculus as they were exposed to a whole year of calculus. Yet, the AB students did not perform better than the PP students, which could be interpreted as AB students did not learn the foundational concepts of calculus deeply enough to retain them for three months over the summer break; however, the PP students did. This result also suggests that the precalculus curriculum, *Precalculus: A Pathway to Calculus* (Carlson & Oehrtman, 2010) is different than the traditional precalculus program.

In fact, *Precalculus: A Pathway to Calculus* was specifically designed to deepen the understanding of mathematical concepts by improving the learners' exploratory skills through analysis and synthesis; the learners in this curriculum continually study mathematical and scientific contexts and use mathematics as a means to perform scientific investigations. Accordingly, the classroom practice is shifted from the delivery of curricular material to inquiry and project-based approaches centered on content. This study proves that such a shift will help students develop a deeper and longer lasting understanding of the foundational concepts of calculus.

The sample sizes were 11 for the TP and AB students while it was 13 or 14 for the PP students. This is a considerable limitation which could have created power issues, however, this study was indeed a pilot one, carried out to assess the effectiveness of the *Precalculus: A Pathway to Calculus* program. Nevertheless, the relatively smaller sample sizes did not hinder the ability of the authors to come to statistically significant conclusions about the superior aspects of the program.

Based on the results of this study, the *Precalculus: A Pathway to Calculus* is likely to create a positive impact on the precalculus knowledge of students prior to entering calculus.

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