THE EMERGENCE OF NORMS FOR MATHEMATICAL ARGUMENTATION: CONTRIBUTING TO A FRAMEWORK FOR ACTION

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In this paper, we provide an account of the evolution of mathematical norms for argumentation that emerged during an ongoing teacher collaboration. The collaboration involves the mathematics and science teachers at Green Valley High School in the southwestern United States. As part of the collaboration, teachers were offered the opportunity to participate in college level courses offered at the school. In the fall of 2008, 21 of the 32 mathematics and science teachers chose to participate in a course that focused on functions and the covariation of the measures of two quantities. The activities that comprised the course were intended to serve as didactic objects around which productive conversations could emerge. Therefore, mathematical arguments developed around key significant mathematical issues. As a result, the norms for mathematical argumentation evolved during the semester.

Introduction

Balancing the tensions inherent in simultaneously attending to students' contributions and the mathematical agenda is a hallmark of deliberately facilitated discussions (cf. McClain, 2003). These discussions involve a plethora of decisions that must be made both prior to and while interacting with students. The image that results is that of the teacher constantly judging the nature and quality of the students' contributions against the mathematical agenda in order to ensure that the issues under discussion offer means of supporting the students' mathematical development. This view of mathematical discussions stands in stark contrast to open-ended sessions where all students are allowed to share their solutions without concern for potential mathematical contributions. In order to engage in the process of elevating discussions to the level of sophisticated mathematical argumentation, the teacher must have a deep understanding of the mathematics under discussion (cf. Ball, 1989; Ball, 1993, Ball, 1997; Bransford et al., 2000; Grossman, 1990; Grossman, Wilson, & Schulman, 1989; Ma, 1999; McClain, 2004; Morse, 2000; National Research Council, 2001; Shulman, 1986; Schifter, 1995; Sowder, et al., 1998; Stein, Baxter, & Leinhardt, 1990). This is critical in both being able to advance the mathematical agenda and in judging the quality and worth of student contributions. It requires decision-making in action concerning the pace, sequence and trajectory of discussions in order to ensure that the discussions are mathematically productive.

When focusing on students' offered explanations and justifications, the teacher is seen to actively guide the mathematical development of both the classroom community and individual students (Ball, 1993; Cobb, Wood, & Yackel, 1993). This guiding necessarily requires a sense of *knowing in action* on the part of the teacher as he or she attempts to capitalize on opportunities that emerge from students' activity and explanations. With this comes the responsibility of monitoring classroom discussions, engaging in productive mathematical discourse, and providing direction and guidance as judged appropriate. Similar pedagogical issues are addressed in Simon's (1995) account of the Mathematics Teaching Cycle that highlights the relationship between teachers' knowledge, their goals for students, and their interaction with students.

A focus on the importance of students' contributions also highlights the importance of norms that constitute the classroom participation structure. The importance attributed to classroom norms stems from the contention that students reorganize their specifically mathematical beliefs and values as they participate in and contribute to the establishment of these norms.

In the analysis in this paper, we focus on the discourse between and among the teachers and instructors in a setting in which the authors were the instructors. The analysis will make explicit the evolution of the discussions over the course of the interaction. In doing so we clarify the normative ways of speaking that evolved in the process. This work is significant in that it offers one framework for thinking about how to guide the evolution of productive mathematical argumentation.

Setting

The authors are involved in ongoing teacher collaboration with a group of high school mathematics and science teachers from Green Valley High School[1]. Green Valley High School serves a student population of approximately 2,800. It contains grades ten through twelve. As part of the collaboration, in the fall semester of 2008, 21 of the 32 mathematics and science teachers at Green Valley chose to participate in a course that was taught on Monday afternoons at the school. The authors served as instructors for the course that focused on covariational reasoning. The teachers were able to earn three hours of college credit for their participation. In addition, all of the mathematics and science teachers in the school attended weekly curriculum planning meetings. The teachers were assigned to groups according to the primary subject they taught. For instance, all of the Biology teachers met together as did the pre-calculus teachers. This meeting time was supported by the Principal as evidenced by his arranging for common planning periods for the teachers. However, for the purposes of the analysis in this paper, data is taken only from the discussions that occurred during the Monday night class.

Description of the Course

The course was designed to focus on a significant mathematical concept, that of covariational reasoning. Oehrtman and colleagues (Oehrtman, Carlson, & Thompson, 2008) have argued that covariational reasoning is foundational to high school mathematics and should serve as the organizing concept for all courses. We agree with this stance and therefore took covariational reasoning as our mathematical endpoint. In order to achieve the envisioned endpoint of covariational reasoning playing a significant role in instruction, we initially engaged the teachers in activities that were used in an Algebra I class where covariational reasoning guided the development of the mathematics. (These activities were part of a project conducted by Pat Thompson and his research team.) The teachers in the class worked through the series of activities *as students* and then reflected on their prior activity from their position *as teachers*. Following their mathematical investigations, they explored the classroom in which the instructional unit was implemented. This was made possible by video-based case development efforts from Thompson's ongoing grant. Two of the authors, McClain and Coe, had participated in the case development and were, therefore, well equipped to provide instruction based on the case.

Following the case investigation, class sessions turned to materials developed from Project Pathways that was funded by the National Science Foundation under Carlson's direction. She therefore took the lead on the instruction for this portion of the course. The materials begin with an investigation of proportional reasoning as it relates to functions, and they form the basis for an exploration of linear and exponential growth. Throughout the investigations the teachers were encouraged to consider how the measures of the two quantities co-varied in each situation. The grounding of the problems in contextual situations allowed the teachers to work in small intervals to investigate the phenomena (e.g. small increments of time). It also helped support a shift away from what Thompson (personal communication, 2008) calls "shape thinking." Shape thinking involves imagining the "path" of the phenomena and then seeing the graph as a "static" representation of the completed trace of the path. In other words, the graph is static and has already occurred. An example can be seen when students trace the path of a car given the time and distance it has traveled instead of trying to coordinate the measures of the quantities of time and distance.

It is important to note that throughout the semester, the teacher participants were constantly encouraged to *speak with meaning.* Elsewhere we have described speaking with meaning as ensuring that explanations carry meaning for all participants. This requires conceptually based conversation about quantity. The negotiation of the norms for argumentation that resulted lead to improved understanding between the teachers and deeper knowledge of the content. It is these discussions that provide the basis of our analysis.

Methodology

The general methodology falls under the heading of design research (Brown, 1992; Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003). Following from Brown's characterization of design research, the teacher collaboration involved *engineering* the process of supporting teacher change. Like Brown, we attempted to "engineer innovative educational environments and simultaneously conduct experimental studies of those innovations" (p. 141). This involved iterative cycles of design and research where conjectures about the learning route of the teachers and the means of supporting it were continually tested and revised in the course of ongoing interactions. This is a highly interventionist activity in which decisions about how to proceed were constantly being analyzed against the current activity of the teachers.

The particular lens that guided our analysis of the data was a focus on the normative ways of arguing about solutions, or what Cobb and Yackel (1996) define as the classroom mathematical norms. Classroom mathematical norms focus on the collective mathematical learning of the teacher cohort[2] (cf. Cobb, Stephan, McClain, & Gravemeijer, 2001). This theoretical lens therefore enabled us to document the collective mathematical development of the teacher cohort over a period of time. In order to conduct an analysis of the communal learning, it is important to focus on the diverse ways in which the teachers participate in communal practices. For this reason, the participation of the teachers in discussions where their mathematical activity is the focus then becomes the data for analysis. The diversity in reasoning also serves as a primary means of support of the collective mathematical learning of the teacher cohort. An analysis focused on the emergence of classroom mathematical norms is therefore a conceptual tool that reflects particular interests and concerns (Cobb, et al., 2001).

Analysis

Research on effective teaching often characterizes the teacher's classroom decision-making process as informed by the mathematical agenda, but constantly being revised and modified in action based on students' contributions. These characterizations take account of the students' contributions while attending to the mathematics. Attempting to balance the tension inherent in simultaneously attending to students' offered solutions and the mathematical agenda is the hallmark of deliberately facilitated discussions. A critical resource for the teacher in this process is therefore the *means of support* available to help him achieve his mathematical agenda. This support manifests itself in the form of the instructional tasks and the tools available for solving the tasks. For this reason, tools, notation systems, and student generated inscriptions all serve an important role in the mathematics classroom. However, it is not the tool (or the notation or the inscription) in isolation that offers support for the teacher. It is instead the students' use of the tools and the meanings that they come to have as a result of this activity (Kaput, 1994; Meira, 1998; van Oers, 1996). In this way, the tool is not seen as standing apart from the activity of the student. When designed, these objects must be thought of as didactic objects that will form the basis for reflection and discussion (cf. Thompson, 2002). For this reason, the teacher generated artifacts from the course served an important role in supporting the evolution of mathematical discourse.

As an example, one of the first tasks posed to the teachers in the Monday afternoon class is called the *Sprinter Task*[3]. In this task, the teachers first watched a video of Florence Griffith Joyner's Gold Medal 100 meter race. After viewing the race a couple of times, the teachers were asked to qualitatively track the distance from the start against time since the start. In this introductory task, the teachers had to begin to coordinate two quantities[4], distance from start and time from start. As part of the coordination, they were asked to create a graph and then explain the graph in terms of the two co-varying quantities.

As the teachers worked, one group of mathematics teachers was very focused on the accuracy of their graph. They were unable to think about the measures qualitatively and struggled to get exact measures for exact times by continually starting and stopping the video. They were reluctant to share their solution until they were sure that their graph was correct. Our goal was more global. We wanted the graphs to show a qualitative relationship between the measures of the two quantities that could be described generally with wording such as, "As time passes, her distance from start increases." We were also interested to know if the teachers viewed Griffith-Joyner traveling at a constant rate, an increasing rate or other, and we were interested in their understanding of the meaning of these. Our ability to support the teachers' ability to focus on the coordination of the measures of the two quantities depended upon their ability to reason about the situation, not read a graph in the canonical sense.

Unfortunately, we did not achieve our goal on this first task. This is not surprising either now nor was it at the time. The mathematics teachers in particular argued that scaling the axes was an important part of creating a graph. We eventually had to tell them to leave them unmarked. Even so, they were hesitant to present their results.

Although all of the groups of teachers were able to generate a graph that gave a qualitative sense of the how the measures of the two quantities co-varied, conversations at this point were characterized by telling. In this early phase of the class, the teachers were focused on the correctness of the answer and their contributions were a recitation of that correct answer and the procedure used to arrive at it. In addition, they did not question each other, but sat quietly as each group shared. This portion of the lesson took on characteristics of a "show and tell" instead of an intellectual conversation about ideas. We were, in fact, unable to prompt significant conversations at this point. We therefore describe the first mathematical norm that emerged as that of *argumentation as telling.*

A week later, we posed a question that involved the teachers watching another video and then creating a graph of the situation. In this video, a skateboarder skates back and forth on a half pipe. The *Skateboarder task* required the teachers to coordinate the boarder's horizontal distance from start with the time from start. In this task, the teachers had to attend to the quantities being tracked rather than the position of the boarder. This is in sharp contrast to the *Sprinter task* where Griffith-Joyner's position gave information on the distance from start. This task, therefore, focuses on the issue of shape thinking in that if teachers tried to rely on the *position* of the skateboarder, the graph would be incorrect as shown below in Figure 1.

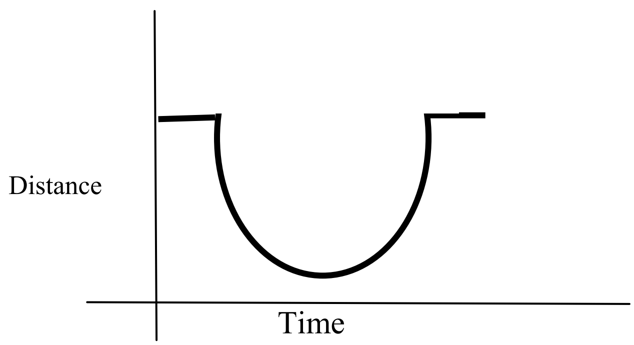

Figure 1. Path of the skateboarder on a half pipe.

The other significant issue that emerged was that of explicitly labeling the axes. For instance, using the descriptor "distance" to label the vertical axis does not clarify what quantity is varying. It could be the total distance traveled. When graphed correctly, the vertical axis should be labeled *horizontal distance from start* and the horizontal axis labeled *time from start.* For this reason, the graph is not the trace of the half pipe as shown above, but the coordination of the measures of the two quantities. Therefore, in creating their graphs, the teachers had to wrestle with first understanding what two quantities were being measured and then coordinating the variation in those quantities. In the process of creating their graphs, the different groups came up with differences in their interpretations as shown below (Figure 2).

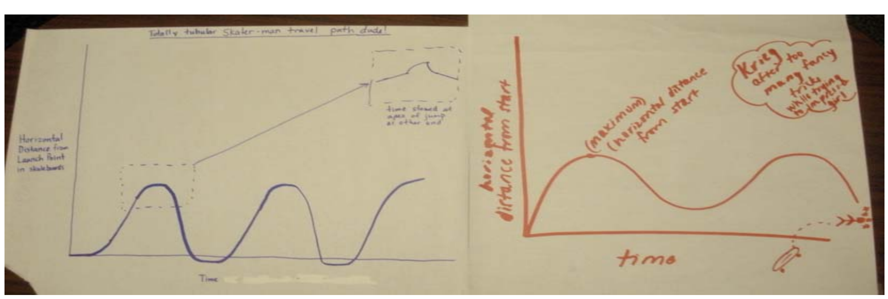

Figure 2. Graphs of the skateboarder's distance from start as a function of time.

When these were juxtaposed on the board, a lively discussion ensued. The teachers initially fell into their original mode of discussion by engaging in a show and tell. However, the differences in their graphs initiated a shift in the conversations such that the teachers began to take the position of defending their own graph. There was no effort to make a comparison across the different graphs—only to justify why their graph was correct. As a result, the teachers did not attend to each other's argument, but focused only on their artifact. There was no attempt to revise and modify the current graphs to move toward a more accurate representation. That only occurred at our initiation. The teachers just kept pointing to their own solution. As a result, the second mathematical norm that emerged was that of *argumentation as disagreement*.

As the course continued and the teachers continued to investigate situations where they had to coordinate the measures of two quantities, the instructors began to push the teachers to speak meaningfully about their graphs. In particular, as the teachers explained their graphs to the class, they were pressed to talk about what was happening to one variable as the other changed or varied. As an example, in the skateboarder task, it was insufficient to say, "First he went down and then across and then back up." A more meaningful explanation was, "as he dropped down the left side of the pipe, his horizontal distance from start did not change. However, as he moved along the base of the pipe, his horizontal distance from start began to increase. As he went up the right side of the pipe, his horizontal distance from start was the same." In other words, the explanation had to be in terms of the two quantities and how they are covarying. In subsequent problems the teachers and instructors began to renegotiate what constituted an adequate explanation.

In this third phase there was, therefore, a significant shift in that the teachers began to engage in conceptual explanations. However, their goal was not to engage other members of the class in a discussion. The teachers spoke *to* the other members of the class, not *with* them (elsewhere Lima and colleagues have made a similar distinction, (personal communication, June, 2008)). Although the teachers were able to reconceptualize their own thinking about a particular solution, they were still unable to understand what it might mean to speak so that others could comprehend their thinking. As a result, the third mathematical norm for argumentation that emerged was that of *speaking conceptually* ***to*** *another*.

The last mathematical norm for argumentation that emerged was that of *speaking conceptually* ***with*** *another*. In this fourth and final phase, the teachers continued to speak meaningfully or engage in conceptual explanations with their colleagues. However, the teachers were not only able to reconceptualize their own thinking about a particular solution but also do this while thinking about what it might mean to speak so that others might comprehend their thinking. This shift in argumentation was apparent in discussions of the *Bungee Jumper* task. In this task, the teachers were shown a video of a man bungee jumping off of a bridge. The task was to create a graph that coordinated the time since the jumper leapt from the bridge with his distance from the ground. By making the distance quantity the *distance from the ground*, teachers were unable to create a graph that was merely the trace of the jumper's path. Instead, they had to coordinate the variation of the time since he leapt with his distance from the ground. As the teachers shared their solutions, they used their explanations and questions to clarify for both themselves and their colleagues how the distance varied as time changed. In this process there was a concerted effort on the part of the teachers and the instructors to communicate clearly. It was during these conversations that the teachers began to speak conceptually **with** one another.

Conclusion

In the analysis presented in this paper, we have provided an account of the evolution of the mathematical norms for argumentation that occurred in the course taught to the mathematics and science teachers at Green Valley High School. This work is significant in that it provides a framework for thinking about mathematical argumentation in the context of teacher development. This evolution is similar to data we have analyzed from other teacher development collaborations (see McClain, 2002). For this reason, we argue that we are providing the starting points for what diSessa and Cobb (2004) call a *framework for action*. diSessa and Cobb characterize a framework for action as a first step toward the development of a guiding theory. Therefore, we are not claiming that we have discovered a new theory. Our claims are much more modest. What we are offering is a first step in that direction. As a result, this pattern of evolution will be useful in our future work. The potential of its power as a theory is only determined by its use by others in similar and different situations. The messiness and complexity of teacher development in the context of design research "highlights the pressing need for theory while simultaneously making the development of useful theories more difficult" (p. 79). It is for that reason that we offer this process as a "way of looking" at this significant aspect of teacher collaborations.

Endnotes

1. Green Valley is a pseudonym.
2. In this analysis, we purposely refer to the group of teachers as a *cohort* instead of a *community*. Documentation of the evolution of the cohort into a community is beyond the scope of this paper. We therefore take the "easier road" by not making assumptions about the nature of the relationships within the cohort at the time of this analysis. We choose to do so because of the importance we place on both establishing *communities* of teachers and verifying their existence with established criteria (cf. Wenger, 1998).
3. The initial tasks were designed by Scott Adamson and Ted Coe as part of their grant work with Pat Thompson.
4. By quantity we mean an attribute that can be measured or that one can imagine measuring.

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