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Foundational ways of thinking for understanding the idea of logarithm

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ABSTRACT

This study explored two undergraduate precalculus students' understandings of the idea of logarithm as they completed conceptually oriented exploratory lessons on exponential and logarithmic functions. The students participated in consecutive, individual teaching experiments that focused on Sparky – an animated mystical saguaro that doubled in height every week. The exponential lesson focused on developing students' conceptions of growth factors and tupling (e.g., doubling) periods as a foundational understanding for conceptualizing logarithms and logarithmic properties meaningfully. This paper characterizes conceptions that are productive for students to acquire in introductory lessons on exponential and logarithmic growth, and discusses two understandings that were revealed to be foundational for students' development of productive meanings for exponents, logarithms and logarithmic properties.

1. Introduction

Instruction focused on teaching the idea of logarithm has not been broadly effective in supporting students to understand this idea (Kenney, 2005; Weber, 2002). Efforts to support student learning of the idea of logarithm include incorporating the history of logarithms into daily lessons (Panagiotou, 2011), coordinating arithmetic and geometric sequences (Ferrari-Escolá, Martinez-Sierra, & Méndez-Guevara, 2016; Gruver, 2018), changing the notation used to represent the value of a logarithm (Hammack & Lyons, 1995), and approximating the value of a logarithm with repeated division (Vos & Espedal, 2016), yet researchers continue to report that many students struggle to develop coherent understandings for logarithmic notation and properties and the logarithmic function (Berezovski, 2008; Chua & Wood, 2005; Gol Tabaghi, 2007; Kenney, 2005; Strom, 2006; Weber, 2002). As a response, this study investigated strategies for helping students develop productive meanings for exponents, exponential functions and other concepts foundational to the idea of logarithm that might support students in understanding logarithms and their properties.

Developing students' understanding of the idea of logarithm requires much more than being introduced to and applying Euler's definition, "if $x > 0$, the logarithm of x to base a ($a > 0$, $a \neq 1$), is the real number y such that $a^y = x$ and is symbolized with $y = \log_a(x)$" (Euler, 1770/1984Euler, /, 1984Euler, 1770/1984). To support students in understanding the idea of logarithm, we focus on the meaning conveyed by the logarithmic function in contrast to introducing and applying Euler's definition, the approach demonstrated in traditional United States curriculum. To illustrate our approach consider the equation $2^3 = 8$ and its equivalent statement $\log_2(8) = 3$ using the logarithmic function. We begin by orienting the student to what the exponent on 2 represents—the number of elapsed doubling periods and say that doubling a quantity 3 times is equivalent to octupling the quantity once. When

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piloting our tasks to support student understanding of logarithm we observed that students were fluent in using colloquial terms such as doubling, tripling, quadrupling, etc. This led to our decision to leverage students' informal intuitions by introducing the term *b*-tupling to represent a quantity becoming *b* times as large or increased by a factor of *b*. We can then maintain this consistency for large and non-integer values of *b*—instead of saying 2 increased by a factor of 1.45, we can say that 2 experienced a 1.45-tupling, or 1.45-tupled.

Since the logarithmic function, $\log_b(x)$, relates an *x*-tupling (the function's independent quantity) to the number of *b*-tupling periods needed to *x*-tuple (the function's dependent quantity) it is useful for students to first recognize that the logarithmic function relates two quantities' values as their values vary together. When using the logarithmic function in an applied context students need to first conceptualize the quantities to be related and then consider how they are related, also known as quantitative reasoning (Smith & Thompson, 2007) prior to attempting to construct function formulas that are meaningful to the student (Castillo-Garsow, 2010; Ellis, 2007; Hackenberg, 2010; Moore, 2010; Moore & Carlson, 2012; Saldanha & Thompson, 1998; Thompson, 1993, 1994a, 1994b). As students continue to consider how the values of the two quantities are changing together to further convey more global function behavior of a logarithmic function, they are said to be engaging in covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). If an instructional goal is for students to be able to access and use the idea of logarithm to model quantitative relationships students will benefit if they are supported in conceptualizing quantities, their relationships and how they vary together.

While a number of studies have examined students' difficulties in understanding the idea of logarithm and the effectiveness of non-traditional interventions, the study of the understandings students develop while participating in conceptually oriented exponential and logarithmic lessons is an emerging area of research. The primary question motivating this investigation was: What understandings and reasoning abilities do students develop and use during an exponential and logarithmic instructional sequence aimed at supporting students in acquiring a strong meaning for the idea of logarithm? To address this question, we conducted two consecutive teaching experiments (Steffe & Thompson, 2000) that focused on characterizing and advancing students' ways of thinking as they worked through lessons that were designed to support their developing an understanding of the idea of logarithm.

This paper characterizes conceptions that are productive for students to acquire in introductory lessons on exponential growth that we conjectured to support their understanding logarithmic growth. We also describe two undergraduate precalculus students' understandings and learning as they individually worked through exploratory lessons for understanding logarithmic growth and associated introductory ideas for understanding the idea of logarithm. The data from our first teaching experiment revealed two understandings that we characterized as foundational for constructing a productive meaning for the idea of logarithm. We consider an understanding to be foundational for learning an idea (or ideas) if it has been documented through research to be both a conceptual barrier and critical for understanding that idea (or ideas). As one example, our pilot of tasks aimed at developing students' understanding of the idea of logarithm revealed that students' inability to conceptualize the overall growth of a doubling followed by a tripling as a 6-tupling, was an obstacle to students' understanding the idea of logarithm and some logarithmic properties. Our identification of foundational ideas for understanding the idea of logarithm led to our creating tasks we hypothesized would support students in developing these understandings. We used these tasks in a subsequent teaching episode during our first teaching experiment, and again with our second subject during the second teaching experiment. Our data revealed that these tasks were effective in developing these understandings in both subjects. When discussing the results we illustrate the role of these foundational understandings when constructing logarithmic expressions, logarithmic properties, and logarithmic functions. We conclude by elaborating the importance of these two essential understandings when learning and using the idea of logarithm.

2. Literature review

Logarithmic and exponential growth are often discussed separately, but are closely related. With exponential growth, there is often a focus on growth factors (or *b*-tuplings) provided a number of *b*-tupling periods. While with logarithmic growth, there is a focus on the number of *b*-tupling periods it takes for some quantity to grow by a given factor. In this section, we review literature regarding student learning of both exponential and logarithmic ideas. We also acknowledge the studies that we leveraged when describing our learning trajectory and lessons for our teaching experiment.

2.1. Research literature on students' understandings of exponents and exponential functions

A student who conceptualizes exponentiation only as repeated multiplication will likely be limited to interpreting natural number exponents. In cases when an exponent is a non-natural real number, say 0, –2, ¾, or $-\pi$, the interpretation of exponentiation as repeated multiplication is ineffective. While some researchers advocate a repeated multiplication approach (e.g. Goldin & Herscovics, 1991; Weber, 2002), others believe this approach limits students (e.g. Confrey & Smith, 1995; Davis, 2009; Ellis, Özgür, Kulow, Williams, & Amidon, 2015). In particular, Confrey and Smith (1995) argue that the standard way of teaching multiplication through repeated addition is inadequate for describing a variety of situations such as magnification, multiplicative parts (i.e. finding a fraction of a split), reinitializing and creating an array. Weber (2002) proposed that students first understand exponentiation as a process (in terms of APOS theory) before viewing exponential and logarithmic expressions as the result of applying the process. A student with a process conception of exponent will be able to generalize her understanding to cases in which the exponent is a non-natural number. Specifically, Weber had his students complete a variety of exponential activities using the definition that "b^x represents the number that is the product of *x* many factors of *b*." With this conception, we can describe $9^{2.5}$ to be the number that is the product of two and a half factors of 9, while under the view of repeated multiplication, a student might write "9 · 9 ·?" . If a coherent understanding of exponential functions (and later logarithmic functions) is desired of our students, it is imperative that they have productive meanings for exponents.

Confrey and Smith (1995) claimed that exponential function learning involves mental actions of splitting and covariation. The authors describe splitting as a multiplicative operation different from repeated addition that arises in situations involving magnification, similarity and sharing, for example. Direction in the splitting structure suggests either multiplication or division (doubling vs. halving, etc.). The authors provided empirical evidence (students utilize the idea of halving to determine the area per child on a playground) that they claim suggests that splitting is an intuitive construct for multiplication and division. Confrey and Smith described the covariation approach as considering two sets of data and the relationship between the sets. That is, this approach encourages the description of how one quantity varies in relation to another and allows for the discussion of rates of change, differences, and accumulation. In particular, exponential functions can be characterized as having constant multiplicative rates of change (Ellis et al., 2015). Confrey and Smith described how to produce exponential functions using splitting and covariation and concluded that the use of covariation, splitting and the idea of the isomorphism between the additive and multiplicative worlds helps avoid concealing the relevant splitting unit/base that relates to the functional situation and helps avoid an overreliance on algebraic representation.

This study's intervention expanded on Ellis, Özgür, Kulow, Williams, and Amidon's (2012, 2015) small-scale teaching experiment that examined continuous quantities covarying exponentially. The three middle school participants in Ellis et al.'s study were asked to consider a scenario of a cactus named Jactus whose height doubled every week. Eventually, the initial height, weekly growth factor and amount of time needed to grow by the provided factor were altered to provide variety. The authors noticed three significant shifts in the students' thinking over the course of the study. At first, the students attended only to Jactus' height and concluded he grew by means of repeated multiplication. Eventually, the students began to coordinate this repeated multiplication with the corresponding changes in the amount of time that elapsed. The second shift consisted of students determining the factor by which Jactus' height grew for varying changes in the number of weeks by means of calculating the ratio of two heights. Finally, the third shift involved the students generalizing the reasoning noted in the second shift to include non-natural exponents (i.e., to determine the 1-day growth factor). The authors noted that a student's ability to coordinate the growth factor (or ratio of height values) with the changes in elapsed time contributed to the student successfully defining the relationship between the elapsed time and Jactus' height. This study leveraged findings from Ellis et al.'s study of Jactus the Cactus to promote more meaningful discussions on logarithms; we explain this expansion in more detail in the Methodology section.

2.2. Research literature on students' understandings of logarithms

There have been a number of studies that have examined students' difficulties in understanding the idea of logarithm (i.e. logarithmic notation, logarithmic properties, the logarithmic function) and the effectiveness of non-traditional interventions. However, few studies have focused on the developmental trajectory of understandings students need when learning exponential growth that are essential for understanding the idea of logarithm. Furthermore, little information has been reported about the understandings foundational to the idea of logarithm students must develop to understand the idea of logarithm well.

The difficulties students have with developing coherent understandings of the idea of logarithm are likely multidimensional. In a typical precalculus course, logarithmic functions are the first function family students encounter that is not explicitly defined by an algebraic formula, leaving students with no direction on how to determine the value of $\log_b(m)$ given values of b and m. Instead, students are expected to either apply their understandings of the idea of logarithm, exponents and powers to approximate the value of a logarithm for some input value, or, more commonly, use technology to calculate its value. Logarithmic functions are also the first function family that students encounter in which the function name is not a single letter. This may introduce added complexity for students who already struggle in using function notation (Musgrave & Thompson, 2014; Thompson, 2013). Additionally, aspects of logarithmic notation have a dual nature to them (Kenney, 2005). For example, in the equation $\log_b(x) = y$, b, x, and y may take on a variety of meanings to an individual – b often takes on the form of a parameter (staying consistent within the context of a problem, but varying from problem to problem), x serves as the input variable to the logarithmic function and is a tupling,[1] and y serves as the output variable to the logarithmic function and is the number of b-tupling periods needed to x-tuple.

In addition to these unavoidable complexities, studies have shown that students struggle to understand, explain and apply the properties of logarithms (Berezovski, 2008; Chua & Wood, 2005; Gol Tabaghi, 2007; Kenney, 2005; Weber, 2002). Some students in Kenney's (2005) study experienced difficulties simplifying equations involving at least two logarithmic expressions shortly after correctly applying Euler's definition to simplify single logarithmic expressions. This suggests that the standard approach to introduce students to logarithms by giving them the statement that $\log_b(x) = y \leftrightarrow b^y = x$, which does not directly state the relationship between two or more logarithmic statements, is an insufficient foundation for students trying to develop an understanding of the properties of logarithms. Therefore, it seems reasonable to consider that if students continue to have difficulties in understanding the idea of logarithm (i.e., logarithmic notation, logarithmic properties, the logarithmic function), they may still need to develop some understanding(s) foundational to the topic. This study was designed to shed light on conceptions that build the foundation for the idea of logarithm. Identifying and describing these understandings can inform curriculum design and instructional decisions to make them more coherent and effective in supporting student understanding and using the idea of logarithm.

[1] Recall: A b-tupling occurs when a quantity becomes b times as large. Therefore, a b-tupling period is the amount of change in one quantity (typically time) needed for a second quantity to become b times as large. We say that the second quantity has *b-tupled* over some interval of change of the first quantity.

3. Theoretical perspective

The theoretical perspective used for this study is radical constructivism (Glasersfeld, 1995), which proposes that knowledge is constructed in the mind of an individual and therefore cannot be directly accessed by anyone else. The intention is not to classify how every student will come to learn the idea of logarithm, but rather to model the mathematical realities of individual students. Doing so will initiate a conversation of epistemic students one might encounter while teaching the idea of logarithm. Under this perspective, researchers can, at best, attempt to form a model of students' thinking (Steffe & Thompson, 2000). A model is considered reliable when the student's utterances, written work, and movements are in alignment with the model and does not necessarily have to be mathematically correct. That is, if the subject responds in a way that is mathematically incorrect, the researcher will be interested in modeling how the student was thinking for his claims to make sense to him. When a researcher develops such models, she is trying to model the student's cognitive structures that comprise knowledge, known as schemes. These structures are organizations of mental actions or mental operations (reversible actions) and may even be complex and contain other schemes (Piaget, 2001). An action is "all movement, all thought, or all emotion – [that] responds to a need" (Piaget, 1967, p. 6). Researchers rely on a student's observable actions when attempting to form models of his schemes, such as utterances, written work, movements or body language.

Researchers who are interested in using the teaching experiment methodology to model student learning (i.e. cognitive structuring and restructuring) should make sure to provide students with opportunities for reflection (Derry, 2009). The goal of our study was to model our subjects' knowledge development of the idea of logarithm as each subject completed lessons in a teaching experiment designed to advance her meanings. Our data collection and analysis focused on understanding and characterizing the meanings students constructed as they engaged in tasks and responded to questions that provided opportunities for reflection.

4. Conceptual analysis

In this section, we present the conceptual analysis that guided the design of our intervention and goals for student learning of the idea of logarithm. According to Glasersfeld (1995) a conceptual analysis is used to describe the mental operations that might explain why people think the way they do. Thompson (2008) elaborated uses of conceptual analysis that include articulating productive ways of thinking for learning an idea. Our conceptual analysis of the idea of logarithm describes what we have observed and hypothesize to be productive ways of thinking for learning the idea of logarithm.

As a student processes the statement $\log_b(m)$ it is helpful to recognize the word "log" as the name of a function that determines the number of b-tupling periods needed for an m-tupling to occur. To illustrate the usefulness of this way of thinking we describe this usage in the context of the starfish problem (Fig. 1).

When reading the problem statement it is helpful if students recognize that the function f is relating the quantities, number of years since 1990, t, and the number of starfish in Hawaii, $f(t)$. It is also productive to view the expression $1500(1.2)^t$ as the function rule for determining the values of $f(t)$ (number of starfish) for specific values of t, where 1500 is the number of starfish in 1990 and 1.2 is the factor by which the number of star fish increases each year. The formula (statement 1) provides a general description of how the values of t and $f(t)$ vary together. If prompted to determine how long it will take for the population of starfish in Hawaii to reach 3480, the problem can be conceptualized as determining the specific value of t for which the starfish population, $f(t)$, has a value of 3480. When solving this equation (statement 2) for t it is reasonable to first divide both sides of the equation by 1500 to obtain statement (3). It is productive to think about $\frac{3480}{1500} = 2.32$ as the factor by which the initial number of starfish, 1500, grows. We can also say that the initial population of starfish 2.32-tupled. When solving for t we are trying to find the number of years (1.2-tupling periods) needed for the starfish to 2.32-tuple. Using logarithmic notation, we can represent this value as $\log_{1.2}(2.32)$. With use of technology we can determine that $\log_{1.2}(2.32) \approx 4.6$, and conclude that after approximately 4.6 1.2-tupling periods, or years, the starfish population will 2.32 tuple to a population of 3480 starfish. This conception of the output or value of a logarithmic function is meaningful for this problem and lays an important conceptual foundation for understanding logarithmic properties.

Use of the tupling language allows one to easily think about and represent non-natural number bases and/or the number of base-tupling periods. For example, suppose the 2-tupling (doubling) period for some quantity is $+1$ weeks. Then, the quantity will ½-tuple in –1 weeks. Symbolically, we represent this number of 2-tupling periods needed to ½-tuple as $\log_2(\frac{1}{2}) = -1$. We elaborate (Table 1)

The starfish population in Hawaii has increased 20% per year since 1990 and is modeled by the function $f(t) = 1500(1.2)^t$, with t representing the number of years since 1990. Determine how long it will take for the population to reach 3480 starfish.

(1) $f(t) = 1500(1.2)^t$

(2) $3480 = 1500(1.2)^t$

(3) $\frac{3480}{1500} = (1.2)^t$

(4) $2.32 = (1.2)^t$

(5) $t = \log_{1.2}(2.32)$

(6) $t \approx 4.6$ years

Fig. 1. The starfish problem: The output of a logarithmic function.

Table 1
Taxonomy of the idea of logarithm.

Hypothetical order of instruction	Topics or ideas foundational for learning the idea of logarithm	Desired student understanding
1	Division as measurement	To measure a value of Quantity A in terms of a value of Quantity B, we write $\frac{\text{Value of Quantity A}}{\text{Value of Quantity B}}$. If $\frac{\text{Value of Quantity A}}{\text{Value of Quantity B}} = m$, we say Quantity A is m times as large as Quantity B.
2	n-unit growth factor	When coordinating the values of two quantities, if the value of the first quantity increases by n-units while the next value of the second quantity is m times as large as its current value, then the n-unit growth factor is m.
3a	To m-tuple (VERB)	If the value of a quantity becomes m times as large, we say the quantity's value m-tuples.
3b	m-tupling (NOUN)	An m-tupling is the event in which the value of a quantity becomes m times as large.
4	m-tupling period	An m-tupling period is the amount of change in the independent quantity of an exponential function needed for the dependent quantity of the exponential function to become m times as large.
5	Exponential growth	When relating two continuous quantities, Quantity A and Quantity B, if for equal changes in Quantity A, Quantity B grows by a constant factor, then the two quantities are related exponentially.
6a	Exponent (on a value, b)	The number of elapsed b-tupling periods. Written b^x where x is the number of elapsed b-tupling periods.
6b	b^x- A growth factor	The factor by which a quantity will grow over x, b-tupling periods is b^x.
7	Growth factor conversion	If $c^1 = b^x$, then one c-tupling period is the same as x b-tupling periods.
8a	Equivalent growth factors and the exponential function	Suppose the two points (0, a) and (x, $f(x)$) satisfy an exponential relationship. Also suppose b represents the 1-unit growth factor. Then for any change of x in the independent quantity, the dependent quantity will become $\frac{f(x)}{a}$ or b^{x-0} times as large. Therefore, $\frac{f(x)}{a} = b^{x-0}$, or $f(x) = ab^x$. This function relates two quantities varying in such a way that every value of x determines exactly one value of $f(x)$.
8b	Logarithmic notation and the logarithmic function.	$\log_b(x)$ represents the number of b-tupling periods it takes (the initial value of an exponential function) to result in an x-tupling. This notation portrays a covarying relationship between an x-tupling and the number of b-tupling periods needed to experience an x-tupling ($\log_b(x)$). These two quantities vary in such a way that every value of the x-tupling determines exactly one value of the number of b-tupling periods needed to experience an x-tupling.

productive meanings for our construct of tupling and describe specific ideas we conjecture to be foundational for understanding exponential and logarithmic functions. Our hypothetical ordering of the ideas (Table 1) that are foundational to learning the idea of logarithm is based in Simon's idea of hypothetical learning trajectory (HLT). According to Simon (1995) and Simon and Tzur (2004) a hypothetical learning trajectory (HLT) includes a list of learning goals for students (Table 1), tasks intended to promote these learning goals (see Appendix A), and hypotheses about student learning as they respond to specific prompts and tasks. We describe desirable student understanding in Table 1, column 3, and elaborate our hypotheses of intended student learning for introducing the tupling language in Table 2. Our results section discusses intended student learning for other tasks used in the interviews.

Simon and Tzur (2004) stressed that student learning happens when the student reflects on both his actions and the effects of those actions when completing a task. It is in this reflection of the action-effect patterns that the student begins to develop new ways of thinking and (hopefully) achieves the goals set for him by the lesson's creator. We designed task sequences for each learning goal to promote student reflection of the action-effect patterns. Similar to what was advocated by Simon and Tzur (2004) these task sequences typically progressed through four stages: (1) Activity Problem – engaged student in making sense of problem context by conceptualizing quantities in the problem context and how they are related, (2) Optional Activity Problem – encouraged student to consider relationships between quantities and effects of previous actions in the first stage but can still be verified by engaging with the activity, (3) Non-activity Problem – encouraged student to reflect on his thinking as he engaged with the previous problems while considering relationships between quantities, (4) Abstract Problem – encouraged student to generalize through reflection on activity-effect relationships of the three previous stages. This progression was specifically designed to provide the student opportunities to advance and strengthen his thinking while reflecting on the preceding questions. Consider the following sequence of tasks (Table 2) used to introduce and teach the tupling language.

The first task in the four phase task sequencing introduces the tupling language and its usefulness by engaging the student in the activity of constructing a cactus and then using that cactus' height to construct one that is two times as tall and another that is three times as tall. The second task prompted the student to reflect on the effects of drawing eight copies of the initial cactus for the purpose of recognizing that the resulting cactus is eight times as tall as the initial cactus; further reflection is promoted by considering how the height of a cactus that 4.5-tuples in size compares with the cactus' original height. The third task is framed to support students in reflecting on the thinking they used in tasks 1 and 2 to observe that a 57-tupling of a cactus results in a cactus that is 57 times as tall as the cactus' initial height. The final task prompts students to generalize the idea of tupling, by reflecting on the actions and results of the three prior tasks, to describe an arbitrary number of tuples.

Table 2
Sequence of tasks for learning the tupling language.

Stage	Task	Desired answer	Rationale for task design and sequencing
1	Draw a cactus. Suppose this cactus doubles, which we refer to as 2-tuples, in height. Draw the resulting cactus after this growth. The resulting cactus is how many times as large as the starting cactus?	*The student might measure the height of the cactus the student initially drew and use that height to draw a new cactus that is as tall as two copies of the initial cactus*	This task introduces the new tupling language while also using the colloquial term "doubles." Students are encouraged to participate in the activity of drawing the initial and resulting cactus.
	Also consider: 3-tuples, or triples	The resulting cactus is 2 times as large as the initial cactus.	The effect of the activity that students may notice is that the resulting cactus is 2 times as tall as the initial cactus.
2	Suppose the cactus you drew 8-tuples (octuples) in height. If needed, draw the resulting cactus after this growth. The resulting cactus is how many times as large as the starting cactus?	*The student might measure the height of the initial cactus drawn and draw a new cactus that is as tall as eight copies of the initial cactus. Or, the student might reflect on his thinking in the previous task and answer the question without needing to draw the resulting cactus.*	This task uses the new tupling language while also providing the colloquial term "octuples." The student may choose to participate in the activity of drawing the initial and resulting cactus if he experiences difficulties reflecting on the activity-effect relationship from the previous task. However, participating in the activity is optional.
	Also consider: 4.5-tuples	The resulting cactus is 8 times as large as the initial cactus.	A student that reflects on the activity-effect relationship might imagine drawing the new cactus and conclude that the new cactus will be 8 times as tall as the initial cactus.
3	Suppose a cactus 57-tuples in height. If you were to draw the resulting cactus, what would we observe? (Hint: The resulting cactus is how many times as large as the starting cactus?)	*The student might describe a process of measuring 57 copies of the original cactus to arrive at the resulting cactus' height.*	This is the first task in the sequence where participating in the activity of drawing the resulting cactus would be too tedious. Instead, we designed this task to perturb students who relied on the activity aspect of the task to reflect on the activity-effect relationships of the previous tasks and imagine the result of the activity as if the actions were performed.
	Also consider: 1000-tuples	The resulting cactus will be 57 times as large as the initial cactus.	
4	Suppose the cactus provided x-tuples in height. If you were to draw the resulting cactus, what would we observe? (Hint: The resulting cactus is how many times as large as the starting cactus?)	The resulting cactus will be x times as large as the initial cactus.	This task does not provide a numerical value for the student to work with. Rather, the student is expected to make a generalization using the tupling language to describe an arbitrary number of tuples. Students who struggle to make this generalization may need to attempt more tasks like those in Stages 2 and 3 to engage in reflection of the activity-effect relationship.

5. Methodology

For this study, we conducted two consecutive teaching experiments (Steffe & Thompson, 2000) over two semesters that focused on advancing and characterizing students' ways of thinking as they completed lessons that were designed to support their understanding of the idea of logarithm. We recruited two precalculus undergraduate students, Lexi and Aaliyah (both pseudonyms), from a large southwestern university to participate in the teaching experiments. Lexi was chosen from a class of 35 students in the first semester to participate in four 1.5 -h teaching episodes over the course of a three-week period instead of attending her precalculus class sessions on exponential and logarithmic ideas. Her grade in the class at the start of the interviews was a D-. We selected Lexi because she was motivated and was verbal, and we were interested in identifying ways of thinking that might benefit all students in learning the idea of logarithm. During the first teaching experiment, we observed unexpected weaknesses in Lexi's understanding of foundational ideas (e.g., multiplicative growth) that resulted in our adapting our HLT and intervention going forward. More specifically, we designed tasks to assess and advance students' understandings of ideas, provided they were deemed to be critical for understanding exponential or logarithmic growth. Leveraging our findings from the first teaching experiment, we developed an assessment that contained 10 multiple choice and 2 free response questions (Appendix B) that assessed students' understanding of foundational ideas that we conjectured to be critical for understanding the idea of logarithm after completing the first teaching experiment. This assessment was administered to 124 precalculus students at the start of the second semester and then scored to determine the percent of students who provided a correct answer to each item. This data provided insights about how widespread specific conceptions were among this population of students. In addition we used this data to select a second subject, Aaliyah, who exhibited weak understandings of ideas that were revealed to be problematic and critical for advancing Lexi's understandings. We wanted a student who exhibited similar misconceptions as Lexi so we could gain new information about our revised HLT and lessons aimed at advancing students' meanings of ideas we had conjectured to be critical for learning and understanding the idea of logarithm. We engaged Aaliyah in 7 teaching episodes over the course of a 3.5-week period. Each teaching episode lasted for approximately 1.5 h. Aaliyah's grade in the class at the start of the interviews was an A.

Prior to the start of each teaching experiment we updated our hypothetical learning trajectory (Simon & Tzur, 2004; Simon, 1995)

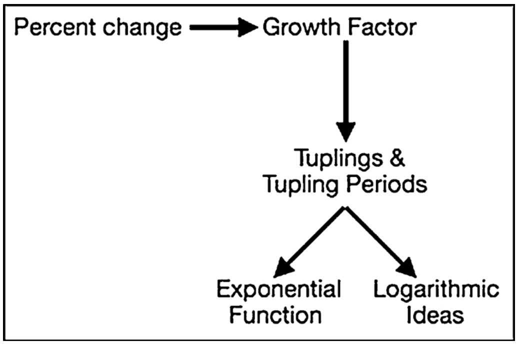

Fig. 2. General flow of the teaching experiments.

for the idea of logarithm. This included updating our learning goals and hypotheses for student learning, and subsequently modifiying our tasks for supporting our updated learning goals and hypotheses for student learning. The instructional sequence designed for this study used the conceptually-based exponential situation designed by Ellis et al. (2012), 2015) entitled Jactus the Cactus – that engaged students in examining the growth of a mystical cactus whose height doubled in size each week. We extended the context and Geogebra applet to engage students in examining tupling periods, a key idea in our developmental trajectory, and referred to Jactus as "Sparky the Saguaro." We also designed tasks to engage students in using this reasoning in applied contexts. The overarching goal for all tasks used in this study was to support students in learning the ideas of exponential functions and to help them develop an intuitive contextual interpretation of the idea of logarithm before introducing a formal statement using logarithmic notation. Fig. 2 provides an overview of the flow of both teaching experiments. Our characterization of what is involved in learning the idea of logarithm is centered on the ideas of tuplings and tupling periods. These ideas stem from a discussion on growth factors. In exponential lessons, however, information about the growth factor is often provided through a discussion on a set percent change, or by representing the percent a new output value is of a current output value in an exponential function. Therefore, we began the teaching experiments with a brief discussion on determining percentages of values and progressed to determine the corresponding growth factor given a quantity's growth expressed as a percent. Due to the adaptive nature of teaching experiments, we do not examine each and every task prepared for and/or used in this study in this section (see Appendix A for planned tasks for Teaching Experiment #2). However, throughout the presentation of results, we describe tasks that best reveal the student learning that the tasks were designed to address.

Throughout the teaching experiment, the first author, whom we will refer to as the interviewer, prompted the students to compare Sparky's height at two different instances of time. To accompany the Sparky the Saguaro tasks, we designed a Geogebra applet that displayed a dynamic image of Sparky's height and time elapsed since January 1st. The applet was designed to provide a variety of viewing options. The students could view Sparky grow as if watching a time-lapse video, observe his height above the corresponding elapsed time since his purchase, document his height every m weeks, and document his height for any m-week change with the additional option of displaying "measuring lines" to help determine Sparky's current height in terms of "how many times as tall Sparky is as compared to his previous height" (Fig. 3). Throughout the teaching experiments we used these displays to explore and advance our subject's understanding of tuplings and tupling periods.

Following each teaching episode, we conducted a retrospective analysis (Steffe & Thompson, 2000) and analyzed the students' actions (verbal, written, and motions) following an open, axial and selective coding approach (Strauss & Corbin, 1998) in an attempt to develop models of student thinking and to inform future sessions. As an example, we considered the students' use and explanation of the Geogebra applet images in the context of their solutions to gain insights into their conceptions of the covarying quantities in the situation. During this analysis stage, we watched the recordings of each interview and made note of behaviors that revealed student's thinking, shifts in the student's thinking or moments when the student made an essential mistake (Steffe & Thompson, 2000). In the subsequent episodes we tested our hypotheses by posing prepared questions and attending to student utterances. Based on the student's actions, we modified our claims as needed, and asked questions we thought would support our subject in confronting problematic and developing desirable conceptions and ways of thinking (as described in our conceptual analysis). We interpreted our classifications as accurate if the student acted in a way consistent with the behaviors described in our taxonomy (Table 1). Subsequent to our analysis of student thinking during each teaching experiment we reviewed our findings for each teaching episode and refined our categorizations. We also updated our HLT to include novel insights that were revealed about student thinking as each subject engaged with tasks to support their learning the idea of logarithm. The subjects were not asked to complete assignments between teaching episodes. The results describe the thinking that our subjects exhibited as they engaged with tasks designed to support their learning and understanding the idea of logarithm.

6. Results

This section presents results from analyzing video data of Lexi and Aaliyah as they independently completed tasks to reveal their thinking and advance their understanding of exponential and logarithmic functions. Each section describes a specific finding and discusses our analysis and characterization of the subject's thinking that led to this finding. Since our teaching experiments were

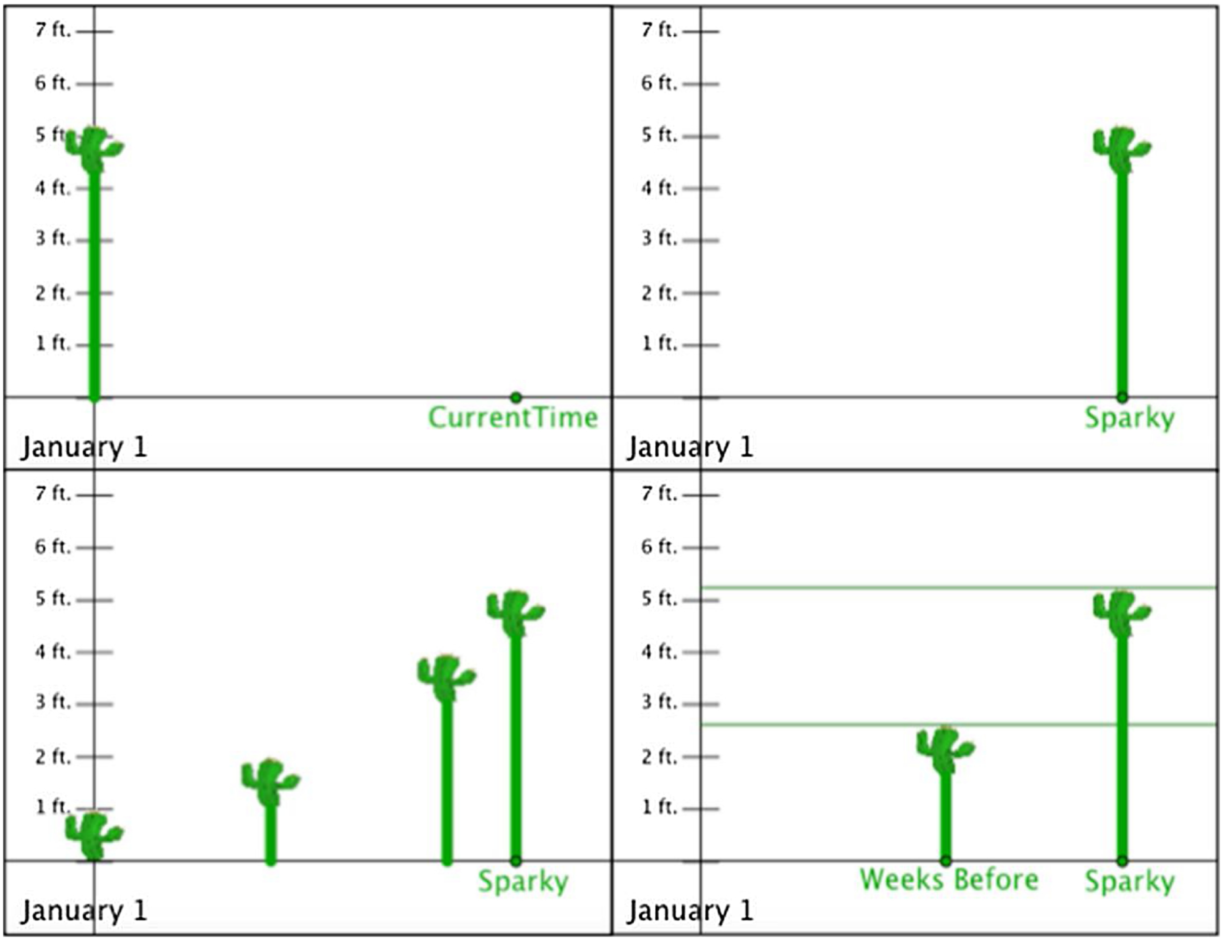

Fig. 3. Different display options in Sparky the Saguaro Geogebra applet.

conducted in succession we present the data and characterize the meanings expressed by Lexi relative to lessons focused on a specific understanding. We follow this by discussing our analysis and characterization of Aaliyah's thinking on the exponential and logarithmic lessons.[2]

6.1. Foundational understanding #1: an A-tupling followed by a B-tupling is equivalent to an AB-tupling

This section presents data from our teaching episodes focused on understanding and advancing Lexi and Aaliyah's conceptions of the overall effect of an *A*-tupling followed by a *B*-tupling. Our data revealed initial obstacles and shifts in Lexi and Aaliyah's conceptions of the overall effect of an *A*-tupling followed by a *B*-tupling as being the same as an *AB*-tupling. We also describe our subjects' thinking as they completed tasks that we designed to support them in constructing a productive meaning for multiplicative growth.

6.1.1. Teaching experiment #1: Lexi's experiences involving foundational understanding #1

The teaching experiment began with a prompt for Lexi to compare Sparky's height at different moments. This was followed by a request for her to determine different growth factors to represent Sparky the Saguaro's growth[3] over different periods of elapsed time. In an attempt to determine the 3-week growth factor, Lexi began by noting Sparky's initial height of one foot at week zero and then claimed, "three time(s)– no, every week it's doubling, or times two for the height. So to get to week three, you'd say it's like, you wouldn't say 6 times as large – that wouldn't make sense. I feel like you would say 3 times as large – that doesn't make sense either." This response suggests that Lexi first considered multiplying the 1-week growth factor (2) by the number of elapsed weeks (3) to calculate the 3-week growth factor. However, she quickly ruled out that option and looked to other values appearing in the situation.

[2] Lessons focused on a specific understanding might be updated after a teaching episode with Lexi and before engaging Aaliyah in a similar lesson.

[3] Sparky the Saguaro is the mystical cactus whose height doubles in size each week. For this study, Sparky's initial height on January 1st was 1 foot.

Lexi then appeared to observe the height of the cactus three weeks after its purchase and eventually concluded that at the end of week 3 Sparky would be 8 times as large as the initial Sparky. However, there was no evidence to suggest that Lexi had contemplated the relationship between the 1-week growth factor (2) and the number of weeks elapsed (3), as a means to obtain the 3-week growth factor (8). In particular, although Lexi noted that Sparky was doubling in height every week, her responses and attention to the heights of the cacti suggest that she had not yet conceptualized that doubling Sparky's height three weeks in a row is the same as growing by a factor of 2^3, or 8.

During a subsequent teaching episode, the interviewer and Lexi discussed the biconditional nature between statements involving growth factors and tupling periods. During the discussion the interviewer conveyed that we say the n-unit growth factor is b if and only if the b-tupling period is n-units. The interviewer referenced the Sparky context and highlighted that, since the 1-week growth factor is 2, the 2-tupling (or doubling) period is 1 week. When Lexi was probed to determined the 2- and 4-tupling periods while observing Sparky's growth each week she had no difficulty providing a correct response. However, she struggled to explain n-tupling periods when n was not a power of 2. When Lexi was asked to approximate the 3-tupling (or tripling) period, she claimed that it would be 1.5 weeks (so that the three foot Sparky would lie halfway between the 2 foot and 4 foot Sparky). Under the assumption that Sparky was three feet tall after 1.5 weeks, Lexi was prompted to determine the number of weeks it would take Sparky to 9-tuple (or to determine the total amount of elapsed time if Sparky 3-tupled in height again). Lexi's first response did not build off her answer to the 3-tupling period. Instead, she treated the task as a new problem and claimed the 9-tupling period would be 3.5 weeks and then modified her response to be 3.25 weeks (so that the 9 foot tall Sparky would lie closer to the 8 foot tall Sparky). Lexi's response suggests she did not use the understanding that if Sparky's height 3-tupled (or tripled) two times in a row, his resulting height would be 9 times as large as his initial height. Furthermore, the 9-tupling period would be (1.5×2) 3 weeks (based on her first response). However, this is impossible because Sparky becomes 8 times as large during a 3-week period. After the interviewer explained why this was an incorrect amount of time, Lexi stated that the 3-tupling period would have to be less than 1.5 weeks. Again, if this was the case, the 9 foot tall Sparky would appear before the 8 foot tall Sparky – a contradiction! Lexi's second incorrect response suggests that she had still not distinguished the relationship between the 3-tupling and the 9-tupling periods (specifically that 3-tupling twice is equivalent to 9-tupling once). For the remaining portion of the teaching session, Lexi continued to struggle with the idea that if Sparky's height first m-tupled and then n-tupled, we could describe his total growth as growing by a factor of mn.

During the retrospective analysis of the third teaching episode, we hypothesized that Lexi's difficulties with the aforementioned ideas were due to her not understanding that an A-tupling followed by a B-tupling had the same overall effect as an AB-tupling. This finding led to a modification of our HLT. This resulted in our designing a task (Fig. 4) aimed at supporting Lexi in conceptualizing this foundational understanding to be used during the fourth teaching episode.

During the fourth teaching episode, we presented Lexi with the task and she drew Sparky (B) with ease. Using a straightedge, she marked Sparky (A)'s height and drew a new Sparky that was 2 times as tall as the first. However, Lexi was unable to construct Sparky (C)'s height accurately. At first, she drew a cactus that was 2 times as tall as Sparky (B). It appeared as though Lexi interpreted the tupling language to mean doubling. After the interviewer clarified that Sparky (C) should be 4 times as tall as Sparky (B), Lexi drew the correct Sparky (C) by using her straightedge, documenting Sparky (B)'s height and constructing a length that is 4 times as tall as Sparky (B)'s height. Afterwards, the interviewer (INT) and Lexi had the following discussion:

INT: Sparky (C) is how many times as large as Sparky (A)?

Lexi: Um, wouldn't it be like 6 times as large?

INT: OK, can you verify that?

Lexi: Sure (reaching for straightedge)

INT: And as you are marking that off, can you explain how you concluded it should be 6?

Lexi: Um, well I figured that it would be 6 times as tall because right here this is two times so then that 2 plus that 4 would be 6. (Uses the straightedge to measure how many Sparky (A)'s fit into Sparky (C)) Oh so maybe I was wrong. OK, wait, so it's 8 because is it because it's 4 times 2? Would you multiply those instead of adding them?

INT: Mhmm

Lexi: OK

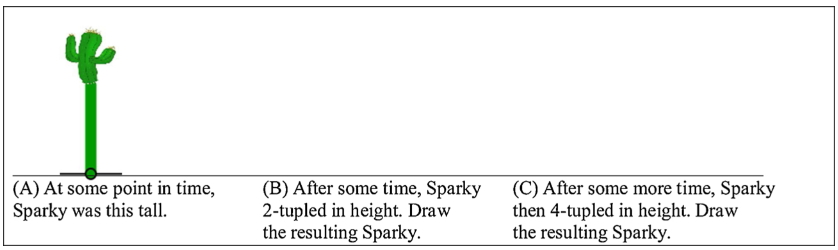

Fig. 4. Task to address foundational understanding #1.

INT: But can you, can you think about, um, instead of just saying "We're going to multiply instead of add," can you think about why it is multiplication?

Lexi: Um, I guess that would make sense because right here, if you're like doubling it in height, you're multiplying it by two. And then if you're 4-tupling it I guess you are going to increase it by like another factor of 4. So instead of adding the factors you would need to multiply them.

Following this first activity, Lexi correctly completed and interpreted two similar tasks – one where Sparky tripled and then doubled in height, and another where Sparky tripled twice in a row. During the remaining time in this teaching episode, Lexi consistently applied similar reasoning, with one exception. In this instance she failed to make sense of the quantities in the situation and expressed that she felt lost in the numbers. However, as soon as the interviewer helped her refocus her attention on the relevant quantities, she began to reason in a productive manner. This finding supports our claim and others' findings that conceptualizing the quantities in a problem context is necessary for constructing meaningful function formula. In the discussion section of this paper we elaborate how the development of Lexi's ability to conceptualize and represent an *A*-tupling followed by a *B*-tupling as an *AB*-tupling was essential for Lexi as she attempted questions requiring use of the first logarithmic property. In particular, we discuss the importance of first conceptualizing the relationship between the tuplings (i.e., Foundational Understanding #1) before discussing the relationship between their corresponding tupling periods (i.e., the first logarithmic property).

6.1.2. Teaching experiment #2: Aaliyah's experiences involving foundational understanding #1

During Aaliyah's exploratory interview, the interviewer prompted her to respond to tasks aimed at revealing her conceptions about growth factors, exponents and logarithmic ideas before beginning the teaching sessions. Two of the tasks (Fig. 5), inspired by our findings in the first teaching experiment with Lexi, focused on the understanding that an *A*-tupling followed by a *B*-tupling has the same effect as an *AB*-tupling. Both tasks examine the height of a cactus experiencing two multiplicative growth spurts; the first task includes an initial height for the cactus, while the second task does not. We further note that these tasks used the colloquial terms "doubles" and "triples" instead of "2-tuples" and "3-tuples". This allowed us to address Aaliyah's meaning for doubling followed by tripling, without the added complexity involved in interpreting the tupling language.

Initially, Aaliyah approached the first task by multiplying the initial height by 2, and then she took the resulting value, 5, and multiplied it by 3 to arrive at 15. The interviewer prompted her to confirm that her calculation was her answer to the first question. She replied by saying, "Yes." When asked if there was a difference between the questions, "by what overall factor did the cactus grow" and "how tall is the resulting cactus", Aaliyah responded by saying the two phrases mean the same thing to her. We decided to discuss Aaliyah's thinking regarding the second question before trying to advance her thinking on the understanding that an *A*-tupling followed by a *B*-tupling is equivalent to an *AB*-tupling. For the second task, Aaliyah claimed that there was not enough information to answer the question because the initial value was not provided. To challenge her thinking, the interviewer drew an initial cactus with no specified height and asked if she could identify the height of the cactus after it quadrupled in size. The following dialogue ensued.

Aaliyah: It's possible but you wouldn't have a number estimate of what it would be.

INT: OK, but even if we don't know the number, there would still be a height right?

Aaliyah: Mhm.

INT: Could you go ahead and um at least identify…how tall the resulting cactus would be after it quadruples in height?

Aaliyah: *Draws a tick mark at the top of the initial cactus' height and measures the cactus' height using two fingers. She keeps her fingers spaced that far apart and marks off four total tick marks spaced apart by that length creating an invisible segment that is four times as large*

1. Suppose you purchased a saguaro cactus that was 2.5 feet tall. If the measure of a saguaro cactus's height doubles (grows by a factor of 2) and *then immediately* triples (grows by a factor of 3), by what overall factor did the cactus grow?
A. 5
B. 6
C. 12.5
D. 15
E. The answer would vary based on the units chosen to measure initial height of the cactus (e.g. inches, feet, meters, etc.).

2. If the measure of a saguaro cactus's height quadruples (grows by a factor of 4) and *then immediately* triples (grows by a factor of 3), by what overall factor did the cactus grow?
A. 3.5
B. 7
C. 12
D. There is not enough information – you need to know the initial height of the cactus.
E. The answer would vary based on the units chosen to measure initial height of the cactus (e.g. inches, feet, meters, etc.).

Fig. 5. Two tasks examining the first foundational understanding.

as the initial cactus.

INT: So this is like the resulting spot right here?

Aaliyah: Yeah

INT: Can you describe what you were doing with your fingers as you were trying to measure that out?

Aaliyah: Um, basically I took the initial height and then I doubled it and made a mark and then I tripled it and made a mark and so forth.

INT: Ok, so you have four copies of the initial cactus?

Aaliyah: Yes.

This excerpt reveals that Aaliyah was able to represent the height of a new cactus in terms of a cactus with an arbitrary height. The interviewer then asked Aaliyah to determine the height of the resulting cactus if the cactus she just finished drawing tripled in height. Aaliyah then went through the same process as she had before and drew tick marks that were equally spaced according to the height of the quadrupled cactus – ending with a cactus that was three times as tall as the quadrupled cactus' height. With each stage of the cactus' growth documented, the interviewer then asked Aaliyah, "So, what if I asked you, 'How many times as large is this final cactus…compared to the starting cactus?' Would you be able to answer that question?" The following conversation ensued.

Aaliyah: Sort of, kind of. I feel like if we're comparing the big one to the small one it would be…for this one three times as large (pointing to the middle stage).

INT: And how did you get three?

Aaliyah: Because I remember from the initial question that you asked me to make it three times as large, so it's basically taking… um…the biggest size and you kinda divide it by the smaller portion to get an answer.

INT: What about this cactus (points to middle), how many times as large is this cactus compared to the initial one?

Aaliyah: Four times as large.

INT: And, so um, if the biggest cactus is three times as large as the middle one, and the middle one is four times as large as the smallest one, um, did you say that we could determine that the biggest cactus is some number of times as large as the smallest one?

Aaliyah: It's possible, yeah.

INT: And are you saying that it's possible only if we know the numbers, for the like how many feet tall they are?

Aaliyah: Not necessarily because if you wanted to you could take this cactus (middle) and this one (largest) and you can divide it by fours within each section (each third)

INT: And if you did that, how many would you get?

Aaliyah: That would be 12 times as large.

INT: Ok, so this final cactus…should be 12 times as large as this (initial) cactus?

Aaliyah: Mhm.

This excerpt supports that the intervention was successful in supporting Aaliyah to conceptualize the effects of growing by a factor of four and then three. However this exchange does not reveal Aaliyah's conception of how the measure of the initial height of the cactus affects the cactus' growth. The interviewer concluded the discussion of this task by asking Aaliyah if her answer would change if the initial height was different. She replied, "Possibly, I feel like it probably would be. But then again it might not. See here's why I think yes and no: I say no because even if you did have an initial height for the small cactus, you're still times, you know still going by times 12. So, it's technically the same thing but with numbers. And then….I think that was my yes reason. And the other reason was because maybe…what if for the smaller cactus, um, no I'm going to stick with the first theory." Therefore, Aaliyah concluded that the initial height of the cactus would not affect her answer, suggesting that she had conceptualized the final height of the cactus in terms of an arbitrary initial height – in particular, that given any height for the initial value of the cactus, the final cactus' height would be 12 times as large. Her hesitation when answering, however, suggests that this connection was non trivial for her to make. It also reveals that her conception of the meaning of an *A*-tupling followed by a *B*-tupling as an *AB*-tupling may not be stable.[4] We tested this conjecture in the subsequent exchange.

Despite the success of the previous intervention, we were unsure if Aaliyah had considered how the three factors (3, 4, and 12) in the second task were related. We then revisited the first task to see if her thinking had changed. After looking over the problem again, Aaliyah decided she wanted to change her answer:

[4] That is, if asked a similar question in the future she may revert back to her original reasoning pattern.

Aaliyah: So it's growing by a factor of 2 and then 3 so if I wanted to add the two together it's basically growing by 5. So if they just wanted the factor, then it would be A.

INT: And is that consistent with how you thought about number [two] with this drawing out stuff?

Aaliyah: Right, now that I'm looking at it visually, and then reading it, it matches up better.

INT: So how is that reasoning, now that you've had that little aha moment, how is that similar to the stuff that we talked about in number [two]?

Aaliyah: Because with number [two], that's basically saying you quadrupled it and then from there you tripled it. So if you wanted to take the 2.5 and double it, which was 5, and then triple it, which was 15, is basically the same thing. So, if you wanted to know how much it grew, that's saying, "Oh it grew by this, like times this many from the initial height." So that's how I came up with five. Because you doubled it and then you ended up tripling it and then I kinda added the two together rather than timesing it because if you times two by three then it would be six and then that's a completely different answer. So I felt like adding two and three together sounded more logical.

INT: Ok, can you explain why it sounded more logical?

Aaliyah: Because (reaches for calculator) so say if I did 2.5 times 6…(*sees calculation results in 15*)…hmm…and then 2.5 times 5… OOHHH…it's 6! Wait, wait, wait wait. I'm sorry I keep changing my answer!

After some vacillation, Aaliyah concluded that the final cactus would be six times as tall as the initial cactus if the initial cactus' height doubled and then tripled in size. She also verified her answer by drawing a picture that illustrated her reasoning. To challenge her thinking further the interviewer asked, "So what if I had a cactus that doubled in height, then tripled in height, then quadrupled in height?" Aaliyah replied, "That would be 2 times 3 which is 6 times 4 which would be 24," and concluded, "That would be 24 times the initial height…24 would be the growth factor from the initial height because we don't know what it is." Aaliyah's response and attention to her actions suggests that she had conceptualized determining the overall growth factor as multiplying the individual factors together. However, later in the teaching experiment we revisited this foundational understanding and Aaliyah provided stronger reasoning to justify her actions suggesting that the repeated opportunity for her to apply this reasoning with three factors had solidified her reasoning patterns so she was fluent in using this reasoning going forward. In the discussion the interviewer asked Aaliyah what the overall growth factor would be if a cactus' height 3-tupled (tripled) and then from that point 5-tupled in size. Aaliyah responded, "I want to say 15, take 3 times 5, no, mm, wait. If he becomes three times as large – whatever height that may be, and then you take that 3 times as large and you make it into five of them, it's 15, no, it's three times as large, so it's three within each five, it's fifteen." Aaliyah's attention to each of her actions between the initial and resulting stages of the task suggests that she conceptualized the roles of the individual and overall factors in the situation and the relationship between them (either during the interview or some time between the exploratory interview and that teaching session). This data also suggests our intervention for promoting student learning of this first foundational understanding can be successful with students who initially struggle with the idea.

Lexi and Aaliyah were not alone in their difficulties with this foundational understanding. When we recruited subjects for the second teaching experiment, we gave the previously mentioned tasks (Fig. 5) to 124 students. Only 39.5% answered the first question correctly and 52.4% answered the second question correctly. This data provides additional evidence that students might benefit by instruction to support students in conceptualizing an A-tupling followed by a B-tupling as an AB-tupling prior to introducing lessons on exponential and logarithmic growth.

6.2. Foundational understanding #2: the exponent on a growth factor, b, represents the number of elapsed b-tupling periods

This section presents data from the teaching episodes that investigated Lexi's and Aaliyah's meaning for an exponent on a growth factor, b. The goal of the teaching episode was to support students in viewing the exponent on a base b as the number of elapsed b-tupling periods. We report on the students' thinking as they completed tasks that we designed to support students in constructing this meaning.

6.2.1. Teaching experiment #2: Lexi's experiences involving foundational understanding #2

In this section, we present and discuss excerpts from teaching episodes designed to support students in viewing the exponent on a growth factor, b, as representing the number of elapsed b-tupling periods. We conclude this section by discussing how the intervention may have impacted Lexi's thinking.

Although Lexi had claimed during the first interview that any one-week change in the number of weeks would result in Sparky growing by a factor of 2, she did not appear to understand that this 1-week growth factor could be used to represent cases where a doubling in height occurred more than once (i.e., using exponents). For example, as Lexi examined a table (Table 3) she had completed during the first interview, she noted that it was easier to observe that Sparky's height was doubling each week by attending to the values in the Decimal Notation column. It is also noteworthy that she did not describe how doubling was represented in the Exponential Notation column.

As a result, the interviewer decided to reintroduce the idea that the exponents on the 2′s represent the number of doubling periods (weeks) that have elapsed since Sparky's purchase. However, despite the brief discussion, Lexi did not appear to understand that the

Table 3
Differentiating between Product, Exponential, and Decimal Notation.

	Product Notation	Exponential Notation	Decimal Notation
Height at purchase	1	1	1
Height after 1 week	1(2)	$1(2)^1$	2
Height after 2 weeks	1(2)(2)	$1(2)^2$	4
Height after 3 weeks	1(2)(2)(2)	$1(2)^3$	8
Height after 4 weeks	1(2)(2)(2)(2)	$1(2)^4$	16

exponent on a value, *b*, represents the number of *b*-tupling periods. This was apparent as Lexi tried to determine the 1-day, 4-day and 8-day growth factors.

Lexi's initial attempt to determine the 1-day growth factor involved her dividing the 1-week growth factor (2) by the number of days in a week – arriving at 2/7. However, when Lexi calculated 2/7ths in her calculator and observed a value less than 1, she claimed her method would not work because multiplying by 2/7 will "make the value smaller." Had Lexi conceptualized that the exponent on 2 represented the number of doubling periods, or weeks, that have elapsed and that the entire expression represented the growth factor for that specified period of time, we hypothesize that she would have been able to conclude that $2^{1/7}$ was the 1-day growth factor because one day is $1/7^{\text{th}}$ of a 2-tupling period. Although Lexi was confident that a 7 must be involved in calculating the 1-day growth factor using the 1-week growth factor, she did not know how to proceed. Trying a different method, the interviewer used the applet to calculate the 1-day growth factor and asked Lexi to define a function to determine Sparky's height in feet in terms of the number of days since his purchase. The interviewer then asked Lexi to compare the two function definitions (one in terms of weeks and the other in terms of days since Sparky's purchase) taking into account the relationship between weeks and days ($d = 7w$). Unfortunately, the conversation did not appear to have any lasting effect on Lexi's thinking. For example, when Lexi was asked to determine the 4-day growth factor, she said, "So wouldn't we have to just do the same thing, but with a 4 in it?" To represent her answer, she wrote the 1-day growth factor (approximately 1.1) obtained from the applet with an exponent of 4. Lexi's attention to the results of her actions suggests that she viewed finding a new growth factor as a process of substituting in the value representing the designated amount of time as the exponent to the appropriate known growth factor. Similarly, she referred to the expression $2^{1/7}$ from our previous discussion when asked to express the 4-day growth factor in terms of the 1-week growth factor, and wrote $2^{4/7}$. It was unclear as to whether or not Lexi viewed $2^{1/7}$ or $2^{4/7}$ as being equivalent to the 1-day and 4-day growth factors respectively. It appeared that Lexi viewed the numerator of the fraction in the exponent of 2 as representing a number of days and simply replaced the 1 with a 4 instead of reasoning that a 4-day period is $4/7^{\text{ths}}$ of a 1-week period.

Finally, the interviewer asked Lexi to determine the 8-day growth factor given the 4-day growth factor. Lexi referred to her previous answer of 1.1^4 and replaced the 4 with an 8, claiming that the 8-day growth factor was 1.1^8. The interviewer then reposed the question and said, "If I said, '1.485 is the 4-day growth factor' how would you find the 8-day growth factor?" Lexi immediately responded that she would multiply the value by two. Had Lexi conceptualized that the exponent on a value, *b*, represents the number of *b*-tupling periods, she should have been able to conclude that the 8-day growth factor is 1.485^2 because there are two four-day periods that make up an eight-day period. Lexi experienced similar difficulties at the beginning of our third lesson.

In the final teaching episode, exponential notation was not used until we began discussing the major ideas behind the third logarithmic property ($\log_b(x^y) = y\log_b(x)$). The following dialogue demonstrates Lexi's struggle in viewing the exponent on a growth factor, 4, as representing the number of 4-tupling periods that have elapsed.

INT: Given that the quadrupling or 4-tupling period is 2 weeks, describe how you would determine the 4 to the 50th -tupling period. Whatever that is. It is pretty big – I don't want to punch it into my calculator

Lexi: This comes out to be two weeks – this beginning part – doesn't it? Um. I don't know what to do with the 50.

INT: Ok. What does this exponent on the 4 represent?

Lexi: The number of tupling periods it takes to get there.

INT: So the number of 4-tupling periods that have passed. How long is a 4-tupling period?

Lexi: I don't even know. I'm like, I don't remember.

INT: So read the statement again.

Lexi: Oh, it's two weeks.

INT: So a 4-tupling period is 2 weeks and the exponent on 4 represents the number of 4-tupling periods have passed. So how long is the 4 to the 50th -tupling period? How can we calculate that?

Lexi: I don't know what you do with the 50. I understand what it is, I don't know what to do with it though to get this extra value that it's giving us.

At first, it appears as though Lexi viewed the exponent, 50, as representing the number of 4-tupling periods it takes to grow by a factor of 4^{50}. However, she was unable to immediately conclude that if there are 50 4-tupling periods and each 4-tupling period is 2

On Saturday morning, Mary made a batch of bread dough and set the dough in a warm place to rise. Suppose the volume of the dough triples every 2 hours and suppose the starting volume of the dough was 45 cm^3. Which of the following statements best describes what information the 5.5 conveys in the expression: $45(3)^{5.5}$?

A. 5.5 hours have elapsed
B. 5.5 two-hour (tripling) periods have elapsed
C. 3 gets multiplied by itself 5.5 times
D. This expression doesn't make sense – you can't multiply a number by itself 5.5 times.
E. None of the above

Fig. 6. Task examining the second foundational understanding.

weeks long, then 100 weeks have elapsed. This suggests she did not conceptualize the multiplicative relationship between the number of elapsed 4-tupling periods and the number of weeks in a 4-tupling period. In an effort to help her conceptualize that the 4^n -tupling period will be $2n$ weeks, we presented Lexi with related examples where the exponent was a smaller number (e.g. 2, 5, 10). After working through a few of these smaller-exponent examples, Lexi arrived at the correct answer, but seemed to get there by attending to the results of her previous actions rather than conceptualizing what the exponent represented and conveyed in the situation. This data suggests that her difficulty was with her understanding of the meaning of the exponent, and possibly with the tupling language. Either way, this dialogue suggests Lexi began to view the value of an exponent as representing a number of tuplings throughout the course of the instructional sequence on tupling periods.

6.2.2. Teaching experiment #2: Aaliyah's experiences involving foundational understanding #2

Lexi's success with the last task in the first teaching experiment motivated us to modify our tasks to involve more examples where an exponent of one on any given base represents a non-unitary amount of time (i.e. 2 h, etc.). We hypothesized that doing so might help students begin to view the exponent on a value, b, as representing the number of elapsed b-tupling periods. During her exploratory interview, we presented Aaliyah with a task (Fig. 6), inspired by our findings in the first teaching experiment. Initially, Aaliyah chose answer choice A stating, "it made me think of the equation, I don't remember what it's called, but when they give you how many times it triples, per hour with your initial number, it's written out a times b to the whatever power it may be. Oh, and usually when it's to the power, it typically conveys time." Aaliyah's conception that the exponent represented time worked for her when the *base*-tupling period[5] was 1 -h, 1-minute, or 1-second, etc. However, as in the case of this task, Aaliyah experienced difficulties interpreting the meaning of exponents when the provided *base*-tupling period was not a typical unit of measure (e.g. 2h, 6min, etc.).

Throughout the teaching sessions that followed, Aaliyah demonstrated consistent thinking when it came to determining a specific growth factor provided an amount of time. For example, when asked to determine the 6 -h growth factor for the previous task, Aaliyah began with the number of elapsed hours, then divided this value by the number of hours it took to triple, and subsequently used the resulting fraction as the exponent to 3. She appeared to be expressing that if the b-tupling period was n units, and m units had elapsed, the m-unit growth factor would be $b^{m/n}$. Her approach produced the correct answer. However, it is noteworthy that Aaliyah experienced difficulty when asked to describe what the value of the exponent represented. Instead of attending to the entire value of the exponent, she often focused on the numerator of her fraction in the exponent, followed by her claiming that many units had elapsed. This technique proved troublesome when the exponent was simplified or written as a fraction where the value of the denominator was not the same as the number of units needed for the output quantity's value to *base*-tuple.

During the fourth teaching episode, the interviewer asked Aaliyah to determine different growth factors for the following situation: On Saturday morning, Mary made a batch of bread dough and set the dough in a warm place to rise. Initially, at 9am, the dough's volume was 45 cm^3. Suppose the volume of the dough 3-tuples (triples) every 2 h. Aaliyah employed her fractional approach to determine the 1 -h, 2 -h, 25 -h, ½-h and H-h growth factors. However, when the interviewer asked her to interpret the simplified value of the exponent she determined for the ½-h growth factor, she quickly resorted to discussing the values of her original fractional exponent as seen in the following dialogue:

INT: What about if we wanted to find the ½ hour growth factor, or we could say 30 min.

Aaliyah: Then you could do three to the power 30 over 120 if you wanted to make the two hours into minutes.

INT: Ok, so you're, are you saying like 30 over 120 is the 30 min that we're talking about over the 120 min in two hours?

Aaliyah: Yes.

INT: That's how you're getting that fraction?

Aaliyah: Mhm

INT: OK. Um, so 30 divided by 120 simplifies to a fourth, or 0.25. What does that 0.25, or a fourth represent in that context then?

Aaliyah: The growth factor for a 30-minute time change

[5] Recall an m-tupling period is the amount of change in the independent quantity needed for the dependent quantity to become m times as large.

INT: So this, this 0.25 is the growth factor?

Aaliyah: Yes. For the 30 min, well no because you don't have the three. So the 0.25 is another way of saying um, is basically representing the 30 min. But, yeah, representing 30 min within two hours by itself, without the growth factor.

Later in that same interview Aaliyah was asked to revisit this part of the task and stated, "I would take the three to the power of 0.25 divided by two to give me that many times the three can 3-tuple." Her suggestion that we rewrite the growth factor as $3^{0.25/2}$ was an (incorrect) attempt to make the denominator of the exponent be the number of hours it took for the dough's volume to 3-tuple (triple). She made a similar attempt in the fifth teaching episode when trying to interpret the growth factor $4^{3/4}$ in the Sparky situation. Aaliyah rewrote the growth factor to be $4^{0.75/2}$ stating, "I wrote the divided by two because it 4-tuples every 2 weeks and … and then with three fourths after I turned it into a decimal, I was basing the decimal after how many times will it 4-tuple in 2 weeks… but, since that's not the case, … it's probably just 0.75 by itself. But then you don't know, well I don't know if um…if it's the final answer for how many times it can 4-tuple or something else." The interviewer discussed why the rewritten growth factor was not equivalent to the original problem and concluded that instead we could have written $4^{1.5/2}$. The interviewer then asked Aaliyah to interpret the growth factor and she said, "It basically means since, so because um, the cactus 4-tuples every two weeks, we want to figure out how many times it'll 4-tuple within a 1.5 week period." Aaliyah concluded that in 1.5 weeks, the cactus would 4-tuple 0.75 times.

The difficulties Aaliyah displayed when attempting to interpret the value of an exponent suggest weaknesses in her conception of the number of elapsed *base*-tupling periods as a new quantity. Therefore when the quantitative relationship changed (i.e., when she was provided with the value of an exponent and asked to find which growth factor was being represented), Aaliyah was forced to make changes to her thinking in order to make sense of the values presented. Both Lexi and Aaliyah were not alone in their difficulties with this foundational understanding. In fact, as we recruited for the second teaching experiment, we gave the previously mentioned task (Fig. 6) to the same 124 students. Only 43.5% answered the question correctly. This evidence suggests that the foundational understanding that the exponent on a growth factor, *b*, represents the number of elapsed *b*-tupling periods is worth discussing in detail prior to lessons on exponential and logarithmic growth.

7. Discussion

In this section, we discuss both understandings we identified for being foundational for understanding ideas of exponential and logarithmic functions. In doing so, we highlight findings that revealed the ways in which these understandings impacted student learning and understanding as they engaged with tasks during the teaching experiment to support their understanding ideas of exponential and logarithmic functions. We also discuss how these understandings may influence student thinking on other logarithmic ideas.

7.1. The relevance of foundational understanding #1

The understanding that an *A*-tupling followed by a *B*-tupling is equivalent to an *AB*-tupling was revealed to be a critical understanding for completing a conceptually focused lesson on exponential and logarithmic functions. Examples of problems that involve such reasoning include: calculating percentages of values, determining partial growth factors, representing, interpreting and calculating logarithmic values, and working with and explaining logarithmic properties. It is possible that students who do not hold this understanding can be successful in answering questions in the aforementioned categories. However, if one has a goal that students develop coherent understandings of exponential and logarithmic functions and other related topics, then addressing this foundational understanding will likely lead to more students obtaining this goal. In the following paragraphs, we describe how this crucial understanding is found in the types of problems listed above and we discuss possible ramifications of not having this understanding.

7.1.1. Determining (partial) growth factors

In the case of Sparky, to determine the 1-day growth factor, we wish to find the number such that when we multiply by this factor 7 times it will have the same effect as multiplying by the 1-week growth factor, 2. Symbolically we write $b^7 = 2$ and then solve for *b* to find the 1-day growth factor. However, a student who does not hold the first foundational understanding may have difficulty setting up this equation and not be able to see why $b \times b \times b \times b \times b \times b \times b = b^7$ or why $b^7 = 2$. When Lexi was presented with the task of determining the 1-day growth factor, she appeared troubled and decided to divide the 1-week growth factor by 7. However, she quickly concluded that her attempt was incorrect after observing that the growth factor was less than 1. Lexi experienced similar struggles when trying to determine the 3-week growth factor. Had Lexi developed this foundational understanding, she would have been able to understand that when Sparky doubles in height three weeks in a row ($2 \times 2 \times 2$), this is the same as growing by a factor of 2^3, or 8. However, at the time, Lexi had not yet developed the foundational understanding that an *A*-tupling followed by a *B*-tupling has the same overall effect as an *AB*-tupling, and thus had to resort to other measures for determining her answer.

7.1.2. Logarithms

Recall $\log_b(m)$ represents the number of *b*-tupling periods needed to *m*-tuple (or grow by a factor of *m*). In other words, this expression represents the number of times a value must *b*-tuple in order to have the same effect as *m*-tupling. In regards to the foundational understanding, *m* takes the role of the *AB*-tupling and the *b* takes the role of the individual factors. If students do not hold the foundational understanding, they may struggle to envision the relationship between b and m.

7.1.3. Logarithmic properties

Foundational understanding #1 is most clearly present in the first logarithmic property, $\log_b(xy) = \log_b(x) + \log_b(y)$, which can be interpreted as, the number of b-tupling periods needed to xy-tuple is equal to the number of b-tupling periods needed to x-tuple plus the number of b-tupling periods needed to y-tuple. To correctly reason through problems involving this property, a student will find it helpful to understand that multiplying by x and then multiplying by y has the same effect as growing by a factor of xy. After developing an understanding of how the tuplings relate, the student will likely be able to see the relationship between the corresponding tupling periods. Specifically, he may conclude that the xy-tupling period will be the same as the sum of the x-tupling period and the y-tupling period. From this point, the student might be able conclude that this relationship will stay consistent as long as the tupling periods are measured using the same unit. Therefore, without this first foundational understanding, it is unlikely that the student will be able to understand the first logarithmic property. Before the intervention in the fourth episode, Lexi struggled to identify the number of weeks it would take for Sparky to grow by a factor of 10 given Sparky's 2-tupling period and its 5-tupling period. However, as Lexi completed the intervention in the fourth episode and attempted a similar task, she conceptualized and related relevant quantities, resulting in her reasoning that if it took Sparky one week to 2-tuple and approximately 1.58 weeks to 3-tuple, then it should take $1 + 1.58 = 2.58$ weeks to 6-tuple. In other words, the number of 2-tupling periods (weeks) needed to 2-tuple plus the number of 2-tupling periods (weeks) needed to 3-tuple is equal to the number of 2-tupling periods (weeks) needed to 6-tuple. Symbolically, $\log_2(2) + \log_2(3) = \log_2(6)$ - a specific case of the first logarithmic property.

7.2. The relevance of foundational understanding #2

The second foundational understanding that is critical for students to develop for a coherent understanding of exponential and logarithmic functions is that the exponent on a growth factor, b, represents the number of b-tupling periods that have elapsed. Students who hold a repeated multiplication view of exponentiation may struggle to interpret the expression $2^{1.5}$ in the Sparky context. However, a student who views the exponent on 2 as the number of doubling periods that have elapsed may interpret this expression to be representing the factor by which Sparky grows over a 1.5 week period. Similarly, such a student should be able to generalize this statement for any number of weeks and therefore be able to meaningfully define the exponential function relating Sparky's height with the number of weeks that have passed since January 1st. Students who have developed this foundational understanding may find it easier to determine and interpret growth factors. For example, if the exponent on 2 represents an elapsed number of weeks and we wish to determine the 1-day growth factor, then since one day is 1/7th of a week, the factor by which Sparky will grow over the course of one day is $2^{1/7}$. Similarly, provided $2^{3/4}$ is a growth factor in the Sparky situation, one may interpret this to be the 3/4-week growth factor. Our analysis of Aaliyah's thinking suggests that this understanding was not necessary for her to be able to determine growth factors. However, when she was asked to interpret the amount of time it would take for the output value to grow by a specific factor, she experienced difficulties when she did not interpret the exponent to be the number of elapsed *base*-tupling periods. This understanding is also foundational for understanding logarithms and logarithmic properties. Even if a student understands $\log_b(m)$ to be the number of b-tupling periods needed to m-tuple (or grow by a factor of m), he may not be able to correctly apply this understanding to solve for the input to an exponential function if he does not see the exponent on b as representing a number of b-tuplings. To compound this issue, exponential notation is utilized in the third logarithmic property, $\log_b(x^y) = y\log_b(x)$, which can be interpreted as, the number of b-tupling periods needed to x-tuple y times is y times as large as the number of b-tupling periods needed to x-tuple once. In order for this property to make sense to the student, it is crucial that he develops the understanding that the exponent on a growth factor, b, represents the number of b-tupling periods that have elapsed (among other understandings).

8. Conclusion

Many studies have examined aspects of logarithms that present difficulties for students, while others have investigated the effectiveness of interventions. In this study, however, we performed a conceptual analysis of what is entailed in understanding the idea of logarithm. This analysis informed our HLT, teaching experiment design, and tasks for specific teaching episodes. We then examined two subjects' thinking as they participated in conceptually based lessons designed to support students in developing strong meanings for exponential and logarithmic functions over consecutive semesters. Our findings revealed two understandings that were initially barriers for our subjects' learning and understanding logarithms; first that an A-tupling followed by a B-tupling has the same effect as an AB-tupling, and second that the exponent on a number b represents the number of elapsed b-tupling periods. We have leveraged these findings to refine our Sparky the Saguaro lesson as we continue our work to support students in developing these foundational understandings.

The results of this study further suggest that students who view the exponent on a base, b, as representing a number of b-tupling periods are better equipped to describe what the output of the logarithmic function represents. Our data also revealed that students benefit by revisiting ideas of doubling, tripling, b-tupling and a b-tupling period throughout an instructional intervention aimed at supporting students in understanding exponential and logarithmic functions and how they are related. We further illustrate how the aforementioned understandings are critical for understanding exponential, and logarithmic functions and their properties. These findings have informed our refinement of the Sparky the Saguaro lesson and should be useful to others who are designing interventions to support students in understanding exponential and logarithmic functions and their properties.

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Declaration of Competing Interest

None.

Appendices A and B. Supplementary data

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