A RESEARCH BASED APPROACH TO DEVELOPING, REFINING, AND ASSESSING STUDENT LEARNING IN AN ONLINE PRECALCULUS COURSE

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Abstract

Post-secondary student enrollment in online courses is increasing, yet we know little about the mathematical meanings students construct while engaged in online coursework. Prior research tested the impact of online courses on measures such as student retention rates, satisfaction scores, and GPA, but did not necessarily question the mathematical meanings students developed while engaging with course materials. This paper describes our design process for creating a student-centered online course and discusses the focal points of the course based on decades of mathematics education research. I also share promising quantitative data from the first four semesters of piloting and conclude by discussing the next steps for our research and development.

Keywords: Online learning, calculus preparation, curriculum design.

1 INTRODUCTION

Studying online learning is increasingly important considering the growing prevalence of online courses in post-secondary education. Since 2012, U.S. student enrollment in post-secondary courses declined steadily, yet enrollment in distance-learning courses (primarily delivered online) increased each year. In Fall 2017 nearly 34% of all post-secondary students in the U.S. enrolled in at least one distance-learning course and almost 16% enrolled exclusively in distance-learning courses [1]. These numbers are likely to increase considering that about 71% of academic leaders say that online learning represents a key part of their long-term growth strategies because they create a larger potential population from which to recruit students and provide additional pathways for students to complete degree requirements [2].

Prior research in online learning, regardless of the subject area, can mostly be grouped into four broad categories (with some overlap). Studies that:

1. examined the impact of online learning on student engagement and affect due to the strong link between student affect and success [e.g., 2-6]. These studies often use surveys to identify sources of positive and negative emotional reactions to online courses.
2. described key characteristics of students who are successful in online courses [e.g., 7-9]. These studies showed that characteristics such as self-motivation, technology skills, and good time management were linked with higher student success rates.
3. compared student performance in online vs. face-to-face courses. Some studies found positive impacts of online courses on student satisfaction and performance [e.g., 3, 10]. Other studies found neutral or negative impacts of online courses on student grades, satisfaction, and completion rates [e.g., 5, 9, 11-14]. [15-16] completed large-scale studies on the effects of online courses and found negative impacts on completion rates and student GPA and showed that the impacts were amplified for students from disadvantaged populations.
4. investigated and prescribed best practices for creating online courses [e.g., 4, 17-19]. Their advice focused on issues such as ease of navigation, clarity in deadlines and course requirements, and encouraging social engagement among students in online courses.

These studies used experimental design methodology to investigate administrator-level concerns, such as maintaining or improving passing rates, GPA, and retention of students in an online course as compared to students in an in-person course. The content, assignments, assessments, and lessons in the online course were typically identical to those used in the in-person course. As a result, issues in the underlying face-to-face course (such as assessments that are not validated relative to predicting students’ success in future math courses or potential incoherence in lesson or unit design) are reproduced without question in the online course. Results then presented often take the form of an "either-or" analysis where the conclusions suggest one of the two approaches is better when instead it could be that neither course meets objective goals of high persistence rates and quality preparation for future courses. This leaves critical research questions unaddressed. (a) Are students learning mathematics worth knowing? (b) Are they constructing deep, flexible, and coherent meanings for the targeted ideas? (c) What aspects of the online lesson support students' construction of important targeted meanings? As Thompson [20 p. 61] points out, "if we intend that students develop mathematical understandings that will serve them as creative and spontaneous thinkers outside of school, then issues of meaning are paramount."

2 THEORETICAL PERSPECTIVE

Our work is grounded in *radical constructivism* [21] as a background theory and *quantitative reasoning* [22-24] as an orienting framework. Radical constructivism guides our underlying assumptions that a student's mathematical meanings (a) are entirely internal to the student and (b) refer to the complex web of understandings, imagery, connections, etc. that make up the scheme to which a student assimilates a given stimulus (such as a given graph, word, or animation). A student's meanings are never directly accessible to any other person. A researcher interested in a student's mathematical meanings attempts to model them by reflecting on the student's observable behaviors and theorizing a set of meanings that, if the student possessed them, would best explain those behaviors [20]. This perspective influences our choices in instructional design because we assume that knowledge cannot be transmitted directly from one person to another. Rather, all individuals construct their own meanings from their experiences. Thus, instructional activities should be designed to entice and support students in constructing meanings that have been found to be essential for continuing their study of mathematics. This perspective on learning supports instruction that (a) provides students an intellectual need for learning ideas, (b) engages students in using essential reasoning abilities (e.g., proportional reasoning, quantitative reasoning) as opposed to engaging exclusively in repeated practice (e.g., methods for solving an equation), (c) confronts problematic limitations in their current meanings so that they have opportunities to construct productive meanings for key mathematical ideas, and (d) includes course lessons and units that promote coherent ways of thinking about key ideas by emphasizing connections across topics [25]. These theoretical foundations inform the design of our lessons and provide a lens for analyzing our data.

3 COURSE DESIGN

3.1 Background on course design

This paper focuses on the design process for creating a research-based online course, and initial data to evaluate its effectiveness in supporting student learning in precalculus. For the last several decades some researchers [e.g., 26] have investigated student learning and reasoning for curriculum to use in face-to-face instructional settings. One approach for developing research-based curriculum is to use a design-based research [27]. Starting with theories of how students learn (such as radical constructivism as a background theory coupled with more domain-specific theories related to mathematics and particular mathematical ideas [28]), researchers generate interventions to enact in specific learning environments. Study of student learning in the context of the intervention produces insights for modifying their learning theories and intervention, and then repeat the research and refinement cycle anew. Figure 1 summarizes design-based research's continuous development cycle.

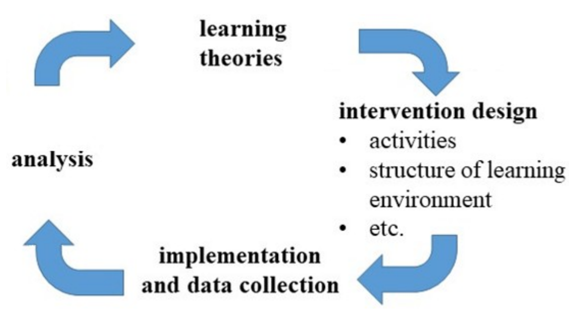

Figure 1. Design-based research's continuous development cycle [27].

3.2 *Pathways Precalculus* in-class materials: Development and success

The original motivation for creating curriculum materials involved two observations. Our prior work to improve precalculus teaching engaged teachers in professional development focused on their conceptions of precalculus ideas. Our study of this intervention revealed significant improvements in the teachers' understandings of the ideas central to their lessons, while their instructional focus remained tightly linked to their textbooks, and thus their students' learning and performance did not improve. These results suggested that improvements in teaching and student learning must be accompanied by improved instructional materials. The second motivation was the persistently low passing rates (proportion of students earning an A, B, or C) at U.S. universities for introductory mathematics courses like college algebra and precalculus. It's common to see overall passing rates between 40% and 60% in U.S. university precalculus courses, which creates a significant barrier to student entry and success in STEM fields.

The face-to-face curriculum materials our research team designed were built on the principles outlined in the theoretical perspective section with *quantitative reasoning* (students' ability to conceptualize objects' measurable attributes, measurement schemes to produce numerical values that describe the attributes' sizes, and relationships between pairs of attributes) [22-24], and *covariational reasoning* (students' ability to connect pairs of covarying quantities in a way that allows them to track patterns in how they change together) [29-31] being the key mathematical learning goals providing coherence to lessons and units across the courses. Our team has been developing, testing, and refining the face-to-face algebra and precalculus curriculum materials [26, 32-33] and accompanying professional development training for instructors. The materials contain rich conceptual and contextual contexts, careful scaffolding of questions to support student-centered activity, applets to focus in-class discussions, a focus on connections across topics, and an emphasis on mathematical ideas proven vital to students' success in later math and other STEM courses.

Our in-class materials have been successful in improving passing rates in precalculus and in preparing students for future math courses. For example, at one U.S. university the precalculus passing rates were approximately 35 to 40% per semester before our intervention. After implementing the *Pathways Precalculus* in-class materials, in conjunction with professional development training and on-site extensions such as including clicker questions during lectures, passing rates increased to between 70 to 81% per semester [34]. And, even when taking into account the larger number of students passing precalculus, students who used *Pathways* materials in precalculus had 8% higher passing rates in calculus than students from the alternative prerequisite course (a precalculus/ trigonometry course covering similar content using traditional textbooks and lectures). Furthermore, students using these materials at 11 U.S. universities demonstrate higher mean scores and gains on the Precalculus Concept Assessment (PCA) [35] than students at other universities or at the same university using other curriculum materials. This assessment tests students' understanding of the ideas most important for students' success in calculus and is a good predictor of student passing rates. Out of 25 questions, students who correctly answer 13 or more questions have a 77% passing rate in calculus while students correctly answering 12 or fewer questions have a passing rate of 40%. Precalculus students' pre-test scores on the PCA generally average from 6 to 9 correct answers, and this is remarkably stable across universities and settings. In addition, PCA gains at the end of precalculus courses are quite flat regardless of the university or curriculum students use (the best mean post-test score observed in non-*Pathways* classes is 9.2). Students using *Pathways Precalculus* materials commonly average 14 to 17 correct answers with mean pre-test to post-test gains of 6 to 12.

3.3 Creating an online *Pathways Precalculus* course

In addition to the challenges that motivated our team to create the in-class *Pathways* materials, increasing demands for online courses coupled with the questionable success of current online offerings, motivated us to apply what we learned when studying and refining our successful face-to-face materials when creating our online precalculus course. The widespread success and fairly dramatic variability in the context and instructor effectiveness when using these materials revealed that our consistent focus on targeted reasoning abilities [e.g., 36] contributed most to student learning and success in later math courses. This finding led to our rewriting the courses from scratch to create lessons where the students were actively engaged in constructing meaning *throughout* each lesson with the assistance of programmed questions and interactive features. While continuing to target quantitative and covariational reasoning as key learning goals, along with specific meanings for ideas such as constant rate of change and exponential growth developed in our in-class materials, we programmed lessons using iMathAS [37] with a focus on student engagement. Typical lessons begin with students completing review tasks based on ideas linked to the current lesson, then completing introductory tasks that build on these ideas in new ways. Short text, a short video, and/or interactive applets and diagrams follow with students answering questions to provide feedback on whether they are making sense of the mathematical ideas being developed, while also contributing to students' advancing their meanings. This is achieved by students completing tasks that foster their engagement in the reasoning abilities that lead to their mathematical learning. Throughout the lessons students are repeatedly prompted to engage with applets, complete tasks, and respond to questions that are scaffold to promote fluency in evoking productive reasoning abilities and conceptions. Each lesson has associated homework tasks that reinforce the ideas developed in the lesson, and after about five lessons students complete a short quiz. Students complete unit exams after every (approximately) 10 lessons. The course has a built-in forum where students can post specific questions and engage in discussions about tasks within the lesson or homework. Past lessons and assignments are available for students to review at all times (except during exams).

4 METHODOLOGY

For the initial pilot semester (Spring 2018), two of the course designers (including the author) co-taught a precalculus section of 68 students at a U.S. university. Students were not specially selected for this study and represented a similar population compared to all other precalculus sections at the university. For the pilot study, students completed all work in the online environment but completed tests in class every two weeks. During the semester students could receive help via online forums, email, video conferencing, or in face-to-face tutoring time offered by the instructors on campus. At the beginning of the semester, students completed a demographic form, a survey designed to assess their beliefs about mathematics, self confidence in their mathematical abilities, and opinions about the course. They also completed a multiple-choice concept inventory focusing on quantitative and covariational reasoning, rate of change and linear functions, and percent change and exponential growth. Students again completed the beliefs survey and concept inventory at the end of the course.

During the course we recruited six students for clinical interviews [38]. These students were selected from a pool of volunteers so that the interviewees represented a variety of majors, prior math course experiences, and GPAs. Each student participated in a pre-interview focusing on select questions from the concept inventory (taking place within a week of completing the concept inventory for the first time) and a post-interview at the end of the course (after completing the concept inventory for the second time). During the interviews we asked students to complete the tasks and discuss their thinking without having the multiple-choice options available to them. A subset of these students also agreed to allow us to use screen-capture to record them completing select lessons and participated in debriefing interviews at the end of these lessons. We compensated each student US$20 per hour for their time. Finally, we collected interaction data for all lessons and assessments including all student responses, average time spent on each task, and average number of attempts per student.

For later semesters (Fall 2018 through Fall 2019) the course was taught by instructors not involved in its development at three different U.S. universities. Because of the various settings and instructors, the course took on three different formats as outlined in the next section. We continued to collect data on passing and completion rates, pre- and post-course beliefs survey data, and in select sections collected PCA results, conducted clinical interviews and screen captures of students completing online lessons, and continued to collect all interaction data for students in the courses.

During data analysis I have based my conclusions primarily on retrospective analysis [39] to characterize changes in students' mathematical meanings throughout the intervention. Our conceptual analysis of the targeted mathematical ideas and our hypotheses informing the learning trajectories in lessons and units provided the initial analysis framework. However, during analysis my goal was to develop the best explanations for observed student behaviors, including creating models for alternative meanings students might possess that were not represented our intended learning goals. My final analysis produced insights about productive and unproductive student meanings, how students' meanings changed during the course, and how features of the course design may have contributed to the results.

5 RESULTS

5.1 Passing and persistence rates using our online curriculum

We have been piloting our online version of *Pathways Precalculus* for five semesters in three different formats: (a) a fully online course with no face-to-face class meetings, (b) a "flipped" format where students complete lessons and assignments outside of class but have optional face-to-face class time scheduled each week for tutoring from the instructor, and (c) the same "flipped" format but face-to-face class sessions are mandatory and the instructor supplements online lesson with mini lectures, extensions, and other activities. The results are shown in Table 1.

Table 1. Passing rates for Pathways Precalculus Online or Hybrid Online.

Type	Percent of students starting the course who complete the course (took the final exam)	Percent of students who started the course who pass the course (grade of A, B, or C)	Percent of students who complete the course who pass the course (grade of A, B, or C)
Fully online (a)	90%	77%	86%
Hybrid (b)	86%	75%	87%
Hybrid (c)	95%	93%	97%

These persistence and passing rates are relatively high compared to similar courses at these universities, suggesting that the courses could be a solution to low passing rates in algebra and precalculus courses at similar universities. Although we have only tested the hybrid (c) model for one semester, it shows particular promise in supporting students in successfully completing prerequisite courses for calculus.

5.2 Impact on students' affective characteristics

We administered a Likert scale beliefs survey to students in the original pilot section and in Fall 2019 in two sections of our fully online course. In the pilot course only about 60% of students completed the post-survey, rendering the data unreliable. However, in our recent online sections the survey completion rate was close to 100%. For these fully online courses, students generally self-reported shifts towards more productive beliefs and higher self-confidence. We consider higher persistence, lower frustration, confidence in mathematical abilities, and enjoyment to be productive outlooks. The shifts from the pre- to the post-course survey results are summarized below.

- Small, general shifts towards more productive beliefs relative to enjoyment of mathematics, confidence in mathematical abilities and the ability to successfully complete word problems, and the need to make unsuccessful attempts during the learning process.
- Generally lower reported frustration and higher persistence when asked to complete challenging problems.
- More students claiming that they would be interested in taking additional math courses.

An example of the scale of positive shifts is shown in Figure 2.

Given that several studies [e.g., 40-41] highlight the important role of affective measures on student success, the generally positive shifts in student confidence and enjoyment in completing mathematical tasks is encouraging and suggests that the course may be setting students up for future success in mathematics and STEM courses in ways that go beyond just content-focused preparation.

Figure 2. Number of students in Fall 2019 online sections responding with each choice to the statement "I am good at working word problems".

5.3 PCA scores

We administered the PCA in fully online and hybrid (b) courses with post-test mean scores between 11 and 12. This is higher than post-test mean scores for non-*Pathways* courses but falls short of scores for classes using our face-to-face curriculum materials. This seems to be due to two factors. First, our initial iteration of the course omitted some topics either directly or indirectly assessed by the PCA. The decision to omit these topics was based on pacing choices during the initial pilot semester and will be addressed in future iterations of the course. Second, and more importantly, I observed that students' covariational reasoning abilities did not change significantly as a result of the course despite being one of our key learning goals. Throughout student interviews and assessments, we observed persistent challenges for students in making sense of graphical representations of relationships such as seeing graphs as representing constraints on the way that two varying quantities change in tandem. I hypothesize that how the iMathAS system graphing questions operate (students plot a small number of key points and the system completes the graph) reinforce unproductive student meanings relative to graphs. In addition, other graphing-related tasks, such as questions that ask students to manipulate an applet and submit an applet state the meets certain criteria, does not require that students conceptualize the emergent nature of graphs as representing the trace left behind from coordinating covarying quantities. Finally, the nature of our feedback on these graphing tasks appears not to promote reflective activity in students. A major focus of our next iteration of the course will involve exploring ways to better support students' covariational reasoning in an online environment.

5.4 Persistent guessing and *emergent symbol meaning*

One phenomenon I noted was a negative correlation between a student's success in the class and the average number of attempts required to complete lessons tasks. The course was designed so that students often answered questions that were either a review of past ideas, attempts to make sense of new contexts, early attempts to draw conclusions and see patterns in reasoning, and repeated practice for ideas established in a lesson. Therefore, it was not uncommon for certain tasks to have higher average attempt accounts. But through observations and student interviews, we confirmed that students with higher average attempt accounts had the goal of producing correct answers (as opposed to learning concepts). Students with lower average attempt counts interpreted incorrect responses as opportunities to reflect on their understanding and engaged in behaviors that helped them identify the source of their error (drawing pictures, rereading lesson text, reviewing past work, checking for rounding errors, and so on). Students with higher average attempt counts engaged in persistent guessing with a high volume of attempts (sometimes 20 or more) in short periods of time that ended when they landed on the correct answer or gave up and moved on. Not surprisingly, interviews with students who engaged in persistent guessing revealed that they often were unable to explain the mathematical ideas within the lesson in ways consistent with our learning goals.

Further exploration of students' expectations while engaged with the lessons suggested another element of their reasoning. In interviews, students with lower average attempt accounts often explained their calculations and algebraic representations in terms of the quantities they intended to evaluate or represent and they used these expectations during error analysis. When their answers were correct they were able to explain *why* their answers made sense in terms of relationships between quantities in a situation. That is, they had (or developed during the course) an expectation that the calculations they perform and the algebraic representations they create should have quantitative significance. In [42-43] I unpacked and described these expectations, defined the construct *emergent symbol meaning* to refer to this set of expectations, and explored their implications. Students who engaged in persistent guessing tended to perform calculations without explaining or being able to explain the reasoning behind those calculations and tended to judge the accuracy of their final answers by whether the number or form of the answer seemed reasonable (for example, if they were asked to find the value of a quantity after some percentage increase then they deemed their answer correct if it turned out to be a slightly larger value than what they began with) rather than referring back to the context, quantities, or relationships to justify their answers.

Recognizing emergent symbol meaning as a critical component of students' success in the course indicates that we should be more explicit in defining it as a learning goal and targeting its development in lessons. Integrating emergent symbol meaning in our online and face-to-face courses as an explicit learning goal and integrating it into course assessments will be a key part of the next iteration of our courses and research.

6 CONCLUSIONS

As post-secondary online course enrollment increases in the U.S., students will benefit if (a) the courses are designed to support them in constructing essential meanings, (b) researchers characterize the meanings students do construct while working through the online lessons, and (c) researchers gather data on how lesson features support the construction of productive meanings. Many online courses are based on converting face-to-face courses into the online environment without reconceptualizing the course for the new setting. For example, the activities students complete are often the same and in-person lectures are recorded and posted in online courses. This is unsurprising considering that instructors at most colleges and universities are not required to complete any kind of special training related to online instruction or course design [15]. [25] argued that students interacting with technology provides an excellent opportunity for them to test hypotheses, receive immediate feedback relative to their hypotheses, and engage in reflection about their mathematical understanding. Our research shows that this is certainly a possible outcome, but students' expectations about their responsibility as students and their beliefs about the quantitative significance of their work impacts whether they benefit from these opportunities. It is clear that modifications to the course are necessary to increase the likelihood that this process occurs as intended, but understanding and addressing students' expectations and beliefs is also critical, and this may require efforts beyond the scope of one course to fully address for many students.

Our successful online course demonstrates that it is possible to apply what is learned about creating effective face-to-face mathematics instruction to create effective online mathematics instruction. Our course further illustrates how to adapt conceptually focused in-class instructional materials to utilize diverse technologies to maximize the potential of the online environment to support student learning. This is in stark contrast to creating copies of in-class courses. An online course designed to maximize student engagement, that targets the meanings we know to be important for future courses, and that provides immediate and consistent feedback to students can positively impact students' self-confidence and enjoyment of mathematics and can achieve high persistence and passing rates. However, there are many interesting and important follow-up questions that our team intends to explore as we continue studying these online courses.

- How do students using our online curriculum materials perform in future STEM courses?
- How do shifts in students' affective characteristics during our courses compare to possible shifts in these characteristics in other online mathematics courses? Do shifts in these characteristics correlate with student success in the course and success in future courses?
- How can we modify lessons, tasks, and assessments to better develop students' covariational reasoning skills and understanding of graphs as well as emergent symbol meaning?
- How much repetition is needed within lessons to ensure that students develop the targeted skills and meanings and what might be the effect of varying the amount of practice based on initial performance?

- What types of feedback within tasks, and methods for providing feedback, are most effective at promoting student reflection about the mathematical ideas targeted within a lesson?
- If courses adapt lessons, tasks, or units based on student performance, what is the best way to identify the underlying causes for low performance and to address these deficiencies to avoid only superficial adaptations?
- What kinds of professional development training is necessary to support online instructors in maximizing the success of their students (both in general and specifically in using our materials)?

We believe that answering these questions, along with complementary work by other researchers, can ensure that online students receive the most effective learning experiences that enable rather than constrain their future success in mathematics and STEM courses and degree programs.

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REFERENCES

[1] U.S. Department of Education, *Digest of educational statistics, 2016*, *Table 311.15.* Washington, D.C.: U.S. Department of Education, National Center for Education Statistics, 2018.

[2] I. E. Allen and J. Seaman, Grade level: Tracking online education in the United States, Babson Survey Research Group, 2015. Retrieved from https://www.onlinelearningsurvey.com/reports/gradelevel.pdf.

[3] P. S. D. Chen, A. D. Lambert, and K. R. Guidry, “Engaging online learners: The impact of web-based learning technology on college student engagement,” *Computers & Education*, vol. 54, no. 4, pp. 1222-1232, 2010.

[4] J. Glass and V. Sue, “Student preferences, satisfaction, and perceived learning in an online mathematics class,” *MERLOT Journal of Online Learning and Teaching*, vol 4, no. 3, pp. 325-338, 2008.

[5] C. Kim, S. W. Park and J. Cozart, “Affective and motivational factors of learning in online mathematics courses,” *British Journal of Educational Technology*, vol. 45, no. 1, pp. 171-185, 2014.

[6] L. J. Levine and D. A. Pizarro, “Emotion and memory research: A grumpy overview,” *Social Cognition,* vol. 22, no. 5, pp. 530–554, 2004.

[7] R. Berenson, G. Boyles and A. Weaver, “Emotional intelligence as a predictor for success in online learning,” *International Review of Research in Open & Distance Learning,* vol. 9, no. 2, pp. 1-16, 2008.

[8] N. Dabbagh, “The online learner: characteristics and pedagogical implications,” *Contemporary Issues in Technology and Teacher Education,* vol. 7, no. 3, pp. 217-226, 2007.

[9] J. Park and H. J. Choi, H. J., “Factors influencing adult learners’ decision to drop out or persist in online learning, “ *Journal of Educational Technology & Society,* vol. 12, no. 4, pp. 207-217, 2009.

[10] J. E. Hughes, S. McLeod, R. Brown, Y. Maeda, and J. Choi, “Academic achievement and perceptions of the learning environment in virtual and traditional secondary mathematics classrooms,” *The American Journal of Distance Education*, vol. 21, no. 4, pp. 199-214, 2007.

[11] M. Barbour, R. Brown, L. H. Waters, R. Hoey, J. L. Hunt, K. Kennedy, and T. Trimm, “Online and Blended Learning: A Survey of Policy and Practice from K-12 Schools around the World,” *International Association for K-12 Online Learning*, 2011.

[12] Y. Cao and S. Sakchutchawan, “Online vs. traditional MBA: an empirical study of students’ characteristics, course satisfaction, and overall success,” *Journal of Human Resources and Adult Learning,* vol. 7, no. 2, pp. 1-12, 2011.

[13] B. Means, Y. Toyama, R. Murphy, M. Bakia, and K. Jones, “Evaluation of evidence-based practices in online learning: A meta-analysis and review of online learning studies,” *US Department of Education*, 2009.

[14] T. Tirrell and D. Quick, “Chickering’s seven principles of good practice: Student attrition in community college online courses,” *Community College Journal of Research and Practice,* vol. 36, no. 8, pp. 580-590, 2012.

[15] D. Xu and S. S. Jaggars, “The impact of online learning on students’ course outcomes: Evidence from a large community and technical college system,” *Economics of Education Review*, vol. 37, pp. 46-57, 2013.

[16] D. Xu and S. S. Jaggars, “The effectiveness of distance education across Virginia’s community colleges: Evidence from introductory college-level math and English courses,” *Educational Evaluation and Policy Analysis*, vol. 33, no. 3, pp. 360-377, 2011.

[17] B. Kramarski and M. Gutman, “How can self-regulated learning be supported in mathematical E-learning environments?,” *Journal of Computer Assisted Learning*, vol. 22, no. 1, pp. 24-33, 2006.

[18] S. Ruey, “A case study of constructivist instructional strategies for adult online learning,” *British Journal of Educational Technology*, vol. 41, no. 5, pp. 706-720, 2010.

[19] L. M. Wadsworth, J. Husman, M. A. Duggan, and M. N. Pennington, “Online mathematics achievement: Effects of learning strategies and self-efficacy,” *Journal of Developmental Education*, vol. 30, no. 3, p. 6, 2007.

[20] P.W. Thompson, “In the absence of meaning” in *Vital directions for research in mathematics education* (K. Leatham ed.), pp. 57-93, New York: Springer, 2013.

[21] E. v. Glasersfeld, *Radical constructivism: A way of knowing and learning*, London: Falmer Press, 1995.

[22] P. W. Thompson, “A theoretical model of quantity-based reasoning in arithmetic and algebraic,” *Center for Research in Mathematics & Science Education*, San Diego State University, 1990.

[23] P. W. Thompson, “Quantitative reasoning and mathematical modeling,” in *New perspectives and directions for collaborative research in mathematics education. WISDOMe Mongraphs,* vol. 1 (L. L. Hatfield, S. Chamberlain, and S. Belbase eds.), pp. 33-57, Laramie, WY: University of Wyoming, 2011.

[24] P. W. Thompson, “Advances in research on quantitative reasoning,” in *Quantitative reasoning: Current state of understanding WISDOMe Monographs*, vol. 2 (R. Mayes, R. Bonillia, L. L. Hatfield, and S. Belbase eds.), pp. 143-148, Laramie, WY: University of Wyoming Press, 2012.

[25] P. W. Thompson, “Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula,” in *Teaching and learning mathematical problem solving: Multiple research perspectives* (E. Silver ed.), pp. 189-243, Hillsdale, NJ: Erlbaum, 1985.

[26] M. P. Carlson, M. Oehrtman, and K. Moore, *Precalculus: Pathways to Calculus (7th ed.)*, Phoenix, AZ: Rational Reasoning, 2018.

[27] The Design-Based Research Collective, “Design-based research: An emerging paradigm for educational inquiry,” *Educational Researcher*, pp. 5-8, 2003.

[28] P. Cobb, “Putting philosophy to work,” *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics*, pp. 3-38, 2007.

[29] L. Saldanha and P. W. Thompson, “Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation,” in *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America* (S. B. Berenson and W. N. Coulombe eds.), pp. 298-304, Raleigh, NC: North Carolina State University, 1998.

[30] P. W. Thompson and M. P. Carlson, "Variation, covariation, and functions: Foundational ways of thinking mathematically," in *Compendium for research in mathematics education* (J. Cai ed.), pp. 421-456. Reston, VA: National Council of Teachers of Mathematics, 2017.

[31] M. Carlson, S. Jacobs, E. Coe, S. Larsen, and E. Hsu, "Applying covariational reasoning while modeling dynamic events: A framework and a study," *Journal for Research in Mathematics Education*, pp. 352-378, 2002.

[32] M. P. Carlson, A. E. O'Bryan, and M. Tallman, *Pathways Algebra I: Implementing the Common Core Standards (2nd ed.)*, Phoenix, AZ: Rational Reasoning, 2015.

[33] M. P. Carlson and A. E. O'Bryan, *Pathways Algebra II: Implementing the Common Core Standards (4th ed.)*, Phoenix, AZ: Rational Reasoning, 2017.

[34] T. McNicholl, "Improving student success and supporting student meaning-making in large-lecture precalculus classes," under review.

[35] M. Carlson, M. Oehrtman, and N. Engelke, "The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings," *Cognition and Instruction*, vol. 28, no. 2, pp. 113-145, 2010.

[36] K. C. Moore and M. P. Carlson, "Students' images of problem contexts when solving applied problems," *The Journal of Mathematical Behavior*, vol. 31, no. 1, pp. 48-59, 2012.

[37] D. Lippman, iMathAS, (n.d.). Retrieved from http://imathas.com/.

[38] J. Clement, "Analysis of clinical interviews: Foundations and model viability," *Handbook of Research Design in Mathematics and Science Education*, pp. 547-589, 2000.

[39] M. Simon, L. Saldanha, E. McClintock, G. K. Akar, T. Watanabe, and I. O. Zembat, "A developing approach to studying students' learning through their mathematical activity," *Cognition and Instruction*, vol. 28, no. 1, pp. 70-112, 2010.

[40] J. Ellis, B. K. Fosdick, C. Rasmussen, "Women 1.5 times more likely to leave STEM pipeline after calculus compared to men: Lack of mathematical confidence a potential culprit," *PLoS ONE*, vol. 11, no. 7, 2016.

[41] E. Hsu, T. J. Murphy, and U. Treisman, "Supporting high achievement in introductory mathematics courses: What we have learned from 30 years of the emerging scholars program," in *Making the Connection: Research and Teaching in Undergraduate Mathematics* (M. Carlson and C. Rasmussen eds.), Washington, DC: Mathematical Association of America, 2008.

[42] A. E. O'Bryan, "Quantitative reasoning and symbolization activity: Do individuals expect calculations and expressions to have quantitative significance?," *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education*, Boston, MA: SIGMAA/RUME, 2020 (accepted).

[43] A. E. O'Bryan, *Exponential growth and online learning environments: Designing for and studying the development of student meanings in online courses*, Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University, 2018.