Chapter 9
The Construct of Decentering in Research on Mathematics Learning and Teaching

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According to Zuberbühler (2018), human communication is a social activity that requires the combined effort of at least two participants who consciously and intentionally cooperate to construct the meaning of their interaction. Humans utilize multiple modalities during communication, including the spoken word, drawings, gestures, written symbols, etc. A key feature of intentional communication is that the person acting attempts to draw another's attention to what they consider relevant entities, both real and imagined. The intentionality of the one doing the communication is to convey their meanings through speaking and actions. A goal of the one receiving the communication then is to understand the meanings being conveyed. The mental processes involved in understanding the speaking and actions of another person from the perspective of that person involve imagining how they might have been thinking. Steffe and Thompson (2000) classified actions to build a mental model of another's thinking from the perspective of the other as a form of reflection they called *decentering*.

Characterizing communication involves describing the model each person constructs of the other's thinking and the assumption that the speaker utters phrases that are coherent and convey the meaning that the speaker intends. Musgrave and Carlson

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(2017), in a study of precalculus instructors' meanings[1] for the idea of average rate of change, reported that instructors with weak meanings for the idea of average rate of change had difficulty expressing a meaning that was coherent and clear enough for others to understand. Other studies (Baş-Ader & Carlson, 2022; Carlson & Baş-Ader, 2019; Rocha, 2021, 2022; Rocha & Carlson, 2020; 2021, 2022 Teuscher et al., 1982) report data that illustrate how a teacher's image of understanding and learning an idea influences a teacher's actions, including how they respond to students' statements and questions. Efforts to study and characterize communication between an instructor and a student should then consider how a teacher's meaning for the idea under discussion influences a teacher's actions. Making sense of a student's thinking will also require that the instructor put their own understanding aside and try to understand how the student might be thinking to have said or written what they did. Such an instructor would be classified as engaging in decentering or acting in a decentered way. In contrast, if an instructor acts as if students' thinking is identical to their own or discounts students' contributions to interactions, then we say the instructor is acting in a non-decentered way (Steffe & Thompson, 2000). It is also important to consider if researchers studying teaching or student learning are engaging in decentering. Are they trying to model their subjects' thinking, or are they only focused on judging the degree to which their subjects' meanings align with their own?

This chapter traces the historical roots of the construct of decentering. We follow by providing an overview of the construct of decentering and its theoretical framing in the context of mathematics education research. We then discuss the uses of decentering for studying student thinking and illustrate the symbiotic relationship between a researcher's mathematical meanings for understanding and learning an idea and a researcher's decentering actions. We further describe what our data has revealed about the role of decentering and the process by which a teacher might develop *ways of thinking for teaching an idea*. We introduce a framework that illustrates the process of acquiring *ways of thinking for teaching* an idea. We follow by discussing our studies of teacher decentering and elaborate five levels of mental actions that provide a more fine-grained description of the mental actions teachers exhibited in our investigations of teachers' decentering when interacting with one another.

Theoretical Background, Framing, and Connections

Decentering's Origins and Adaptation for Use in Mathematics Education Research

In this section, we provide an overview of how the idea of decentering was originally observed in Piaget's early work and how the construct was adapted for use in mathematics education research.

[1] We adopt Thompson et al.'s (2019) use of the word "meaning" in place of "understanding" to refer to the "space of implications of an understanding," including the images and actions that are available, and thus influence the possible ways in which the individual might act.

The Origins of Decentering in Piaget's Genetic Epistemology

Piaget introduced the construct of *decentering* when studying children's cognitive, affective, and social development[2] as they progressed through three developmental stages. In his theory, decentering referred to a child's gradual progress away from egocentrism and their growing ability to coordinate multiple aspects of an object or situation simultaneously (multiple *centrations*) (Piaget, 1965, 1995). During the sensory-motor stage of development (the period between birth to approximately 2 years), the child interacts directly with people and objects that are physically present. The child is unable to mentally construct an image of another when they aren't physically present since the child lacks the semiotic function (the ability to create representations) in this stage. However, mental development during this stage provides a basis for the child's future intellectual development (Piaget, 1965, 1995; Piaget & Inhelder, 1966, 1973). In particular, through a general decentering process, the child structures his universe during this stage. As Piaget and Inhelder (1966, 1973) explained:

> In the course of the first eighteen months, however, there occurs a kind of Copernican revolution, or, more simply, a kind of general decentering process whereby the child eventually comes to regard himself as an object among others in a universe that is made up of permanent objects (that is, structured in a spatio-temporal manner) and in which there is at work a causality that is both localized in space and objectified in things. (p. 13)

As children transition to the pre-operational stage (approximately ages 2 to 7 years), they can conceive of objects outside of their immediate experience (von Glasersfeld, 1995). Piaget's experiments (e.g., viewing a mountain from another position) further revealed that it is during the pre-operational stage that children develop the semiotic function and thus begin to construct mental representations or representational thought and progress to using symbols and signs to evoke objects when they are not physically present (Piaget & Inhelder, 1966, 1973). Prior to their transition to a decentered state, children are egocentric (they struggle to distinguish others' viewpoints, perspectives, and perceptions from their own – their world is strictly about themselves), their reasoning is dominated by figurative, perceptual features of situations (e.g., they do not understand how properties of an object like its volume might be preserved even if its appearance changes), and they tend to focus only on one aspect of an object or situation at a time (Piaget, 1965, 1995; Piaget & Inhelder, 1966, 1973).

As children decenter, they begin to understand and appreciate that others perceive the world in unique ways and have independent feelings, motivations, and opinions. It is through decentration that the child introduces reciprocity among the diversity of points of view along with differentiating among them. Decentration also allows for the ability to consider their actions as reversible and to understand that some properties of objects are conserved under transformations. Children's acquisition of notions of conservation exemplifies this sort of constructive processes

[2] In Piaget's theory of constructivism, cognitive, affective, and social development of a cognizing subject are inseparable (Piaget & Inhelder, 1966, 1973).

(Ginsburg & Opper, 1988; Piaget, 1945, 1952). In an experiment about the conservation of continuous quantity, the liquid in container A is poured into a narrower container B or wider container C (Piaget & Inhelder, 1966, 1973). Younger children describe the volume of liquid in container A increasing when poured into container B or decreasing when poured into container C. This is because they focus only on the figurative image of the liquid's height in each container. They do not conceptualize their actions of pouring as transformations of the state of the liquid nor as an action that could be reversed (Ginsburg & Opper, 1988; Piaget & Inhelder, 1966, 1973). Older children, however, understand that when the liquid is poured from container A into container B it could simply be poured back into container A and that these actions do not alter the liquid's volume. The child decenters these transformations from the child's actions, and thus the child is no longer constrained by pre-operational thought nor dependent on specific figurative representations (Piaget & Inhelder, 1966, 1973).

Adapting Decentering for Use in Mathematics Education Research

Many researchers (e.g., Baş-Ader & Carlson, 2022; Carlson et al., 2007; Steffe & Thompson, 2000; Thompson, 2013), including us, have adopted the term "decentering" as a construct for studying mathematics teaching and for researching student learning that extends its meaning as "a shift away from egocentrism" and "the increasing ability to coordinate centrations in one's reasoning" that occurs during childhood development. Earlier in the paper, we defined "decentering" as attempting to understand another's actions from the perspective of the other by imagining how the other person might have been thinking. On the other hand, acting in a non-decentered way is characterized by imputing one's own reasoning, goals, beliefs, worldview, or understandings onto another person to explain their actions.

Thus, it is useful to understand our use of "decentering" to be synonymous with the notion of modeling another person's thinking in such a way as to hypothesize meanings, beliefs, goals, and so on that would logically result in that person's observable actions even if (or especially if) those meanings, beliefs, goals, and so on differ from our own. Teachers decenter when they work to understand a student's meaning and reflect on what meaning a student constructed and then leverage their image of a student's meaning to inform their subsequent instructional actions. A researcher's decentering actions are similar, although the researcher typically has more robust schemes for the idea under investigation and thus is better equipped to understand and characterize a student's thinking. They are also typically more singularly focused on building a model of a student's thinking and/or testing a hypothesis for advancing a student's meaning for an idea that is a focus of a study. We will continue to clarify the uses of decentering by providing concrete examples of how teachers and researchers have used decentering in their work.

Connections Between Decentering and Other Theoretical Constructs Used in Mathematics Education

In this section, we describe the theoretical foundations and connections between the decentering construct as we define it and Piaget's constructs of reflecting and reflected abstraction, first- and second-order models for individuals' reasoning, and mathematical knowledge for teaching (MKT).

Reflecting and Reflected Abstraction and Their Connection to Decentering in the Context of Teaching

According to von Glasersfeld (1991), reflection "allows us to step out of the stream of direct experience, to re-present a chunk of it, and to look at it as though it were direct experience while remaining aware of the fact that it is not" (p. 90). It follows from this conceptualization that the reflection process has two main components: representing and abstraction. The mental act of considering a chunk of experience and isolating it from what came before and what follows is a simple kind of abstraction, and representing is the mental act of treating the abstraction as an object in and of itself. Representation of a previous experience then refers to the mental actions that reconstruct the experience, thus making it conscious for the individual (von Glasersfeld, 1991).

Piaget adopted and extended the idea of reflection in his theory of abstraction (Simon et al., 2004; Tallman, 2021; von Glasersfeld, 1995) Among the four kinds of abstraction (von Glasersfeld, 1991), reflecting and reflected abstraction are particularly relevant to decentering.[3] Reflecting abstraction is the process that "involves an individual's reconstruction on a higher cognitive level of the coordination of actions from a lower level" (Tallman, 2021, Reflecting Abstraction section, para. 12), such as the reconstruction of physical actions at the level of mental representations of those actions. This reconstruction requires isolating those actions from their effects so that the actions can be generalized, and this process is the source for the construction of new knowledge or conceptions (Simon et al., 2004; Tallman, 2018). That is, the result of reflecting abstraction is a scheme of meanings.[4] It is worth noting that an individual can have a scheme of meanings but lacks conscious awareness of the scheme to the extent that they are unable to explain it or discuss it in general terms or consciously manipulate symbolic representations of the scheme's actions without needing to

[3] See Ellis, Lockwood, and Paoletti (Chap. 6) and Tallman and O'Bryan (Chap. 8) for a more detailed discussion of reflecting and reflected abstraction.

[4] We often find it useful to remind readers that all types of abstractions in Piaget's theory have the ability to "go wrong." That is, depending on what an individual attends to and takes away from experiences dictates what becomes abstracted, and it is quite common for someone to develop a scheme that, from an observer's perspective, contains faulty reasoning and incorrect information or is incomplete.

reimagine the coordinated actions they represent (Piaget, 1977, 2001). Conscious awareness of one's own reasoning comes with *reflected abstraction*. Tallman and O'Bryan (Chap. 8) describe reflected abstraction as follows:

> Thus, reflected abstractions enable an individual to explicitly formulate the results of prior reflecting abstractions. Reflected abstraction, then, describes a scheme constructed at the reflected level of thought through reflection on the results of prior reflecting abstractions. These reflected schemes possess an essential characteristic (cognizance of projected action coordinations, decontextualized and applicable to a generalized class of objects) that endows the knower with the capacity to explicitly formulate meanings and strategically apply them in a range of novel contexts. Indeed, Piaget (1977, 2001) most often associated reflected abstraction with behavioral expressions of such capacities. For example, in describing reflected abstraction as a "detailed *description* of characteristic actions" (p. 167), Piaget associated reflected schemes with the aptitude to explicitly formulate meanings entailed and ways of reasoning supported by the scheme. (p. 249)

Reflecting and reflected abstraction are thus critical constructs in decentering in mathematics education research and with respect to decentering during teaching. It is first necessary that an individual has constructed schemes for the mathematical ideas at hand via reflecting abstraction. The individual then must have constructed a reflected scheme as a reorganization of previously constructed schemes so that their mathematical meanings for the ideas are coherent, organized, and refined and have been brought into the individual's conscious awareness. This involves reflecting on one's own reasoning, comparing one's reasoning in one context to the reasoning in another context, considering additional implications of that reasoning, etc. The process of comparing the characteristics of one's reasoning to the characteristics of another's reasoning necessarily requires that the individual has developed or is developing conscious awareness of their own reasoning [i.e., requires reflected abstraction].

First- and Second-Order Models and Their Connection to Decentering

Thompson (2002) summarized the important distinctions between first- and second-order observers and between first- and second-order models initially proposed by Steffe et al. (2009). To first understand these distinctions, Thompson emphasizes that "any description of affairs must be done at a level of monitoring that is above what is being described" (p. 303). For example, a researcher describing children's thinking within a research study is "not just acting from an image of how students think. She is monitoring how she understands their thinking, and she has connected it with ways of thinking about [related mathematical ideas]" (p. 303). Researchers and teachers working with students could be actors in the situation, or they could be acting as observers even while being part of the interaction:

> If [researchers/teachers] do not reflect on aspects of an interaction that are contributed by students, if they are fused with the situation as they have constituted it, then they are actors in the interaction, not observers. To be an observer of another while involved in interaction,

> one necessarily moves to a level of reflection. When this happens, it adds a dimension to the social interaction — the observer is now acting purposefully and thoughtfully, *using* the social interaction instrumentally. (Thompson, 2002, p. 303)

Assuming a researcher or teacher is acting as an observer (i.e., they are reflecting on students' contributions), they can be acting as first- or second-order observers. First-order observers acknowledge that students can have different reasoning from the observer and are oriented to develop an image of that reasoning. Second-order observers are additionally oriented to consider the implications of the student's reasoning and to consider if other ways of reasoning might be more beneficial. Put another way, "first-order observers address what someone understands, while second-order observers address what *they* understand about what the other person *could* understand" (Thompson, 2002, p. 303).

First-order models are "models the observed subject constructs to order, comprehend, and control his or her experience (i.e., the subject's 'knowledge')" (Steffe et al., 2009, p. xvi). That is, first-order models can be thought of as someone's personal understanding of a mathematical idea, problem solution, scenario, etc. It describes how that person thinks about the mathematics at hand. Second-order models are "models observers may construct of the subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (Steffe et al., 2009, p. xvi). In constructing a second-order model, a teacher or researcher "[puts themself] into the position of the student and attempts to examine the operations that [they] (the researcher) would need and the constraints [they] would have to operate under in order to (logically) behave as the student did" (Thompson, 1988, p. 159). The result of this construction is the researcher's/teacher's second-order model of the subject's thinking that explains the researcher's/teacher's observations of the student's states and activities.

These distinctions are important for understanding whether teachers' actions indicate them operating in decentered or non-decentered ways and, if evidence of decentering exists, how to characterize the nature of the decentering and its possible implications for supporting students' learning. Teachers acting in non-decentered ways (i.e., operating solely as actors in a situation) are not acting reflectively and rely solely on their first-order models (their meanings) to guide pedagogical decisions (Teuscher et al., 1982). "For instance, a teacher who plans a lesson based on how she learned an idea is using her first-order model to plan the lesson" (Baş-Ader & Carlson, 2022, p. 101). It is also possible for researchers to judge data based on their first-order models (their meanings for ideas and methods for solving problems). For example, Lobato and her colleagues (Lobato, 2006, 2008, 2012, 2012; Lobato et al., 2017; Lobato & Siebert, 2002) point out that traditional learning transfer studies often fail to detect transfer precisely because they rely on students seeing the same connections between problem situations that the researchers see, while variations in students' prior experiences and how they conceptualize what *they* see get overlooked. Teachers or researchers operating from first-order models are constrained to using their own thinking and meanings to make sense of students' activities.[5]

On the other hand, decentering requires that the individual is acting reflectively (i.e., acting as a first- or second-order observer in a situation and considering students' contributions to the interaction). Decentering actions are then driven by the propensity to construct second-order models of students' thinking and, ideally, to use these models to inform a teacher's or researcher's actions. The nature of the decentering actions and their usefulness, however, depend on whether the teacher/researcher positions himself as a first- or second-order observer.

As necessary as they are, second-order models of knowing made by a first-order observer provide only weak guidance to a teacher, developer, or researcher. The only thing they can draw from them is that what students end up knowing comes out of the sense they make of teaching and bears no necessary relationship with what the teacher, developer, or researcher intended. Second-order models of knowing made by a second-order observer, however, can provide strong guidance for the teacher, developer, or researcher who has developed a vocabulary and system of constructs to describe students' conceptual schemes together with transformations, reorganization, or other modifications in them (Thompson, 2002, p. 304).

A teacher or researcher constructs a second-order model through decentering actions if they wonder, "How might the student be thinking to act as they did?" which necessarily involves interpreting a student's actions through a model of the student rather than their (the teacher's/researcher's) own cognitive schemes. However, the ultimate usefulness of this model depends on the degree to which the observer possesses a rich, coherent system of meanings and ways of thinking such that the teacher or researcher can reflect not just on how a student understands an idea but also on the implications and usefulness of that understanding for the student's future mathematical experiences. In other words, the nature of the teacher's/researcher's first-order model matters in providing capacity for constructing second-order models of others' meanings and considering the implications of those meanings. We will demonstrate examples of this in later sections.

Later in this chapter, we discuss a framework developed by Baş-Ader and Carlson (2022) that characterizes five levels of decentering actions exhibited by teachers when interacting with students when teaching. We hypothesize and describe connections between these decentering actions and teachers' mathematical meanings for teaching (MMT), their level of awareness of their own mathematical meanings, and how they position themselves as observers while interacting with their students.

[5] We want to emphasize that we do not intend to convey that first-order models are not useful or important for decentering. In later sections we discuss the nature of a teacher's/researcher's first-order models that are necessary for and support constructing second-order models. The teacher's/researcher's first-order model is used to comprehend and control their experiences (including experiences of interacting with students or research subjects). A person cannot abandon their first-order model because it enables and constrains what they can observe, notice, and reflect on when engaging with students or research subjects. We however caution assuming that one's first-order model characterizes how others think.

Mathematical Knowledge for Teaching and Its Connection to Decentering

After having introduced the idea of teachers acting as first- or second-order observers in an interaction and the importance of constructing second-order models of students' thinking, it is natural to wonder what enables researchers and teachers to construct these models. In this section, we discuss the kind of specialized knowledge necessary for constructing second-order models of students' thinking and positioning oneself as a second-order observer while doing so.

Key Developmental Understanding (KDU) and Pedagogical Understanding

Silverman and Thompson (2008) "propose a theoretical framework that extends a constructivist perspective to include the development of MKT [Mathematical Knowledge for Teaching]" to address the need for a theoretical foundation for studying teaching and teacher development (p. 501). Silverman and Thompson argue that content knowledge alone is insufficient for ensuring high-quality teaching. Instead, a teacher's MKT is based first in what Simon (2006) called a *key development understanding* (KDU) or "a way to think about understandings that are powerful springboards for learning, and hence are useful goals of mathematics instruction" (Silverman & Thompson, 2008, p. 502). Developing a KDU involves reflective abstraction whereby new knowledge is constructed and connected to existing knowledge through learning experiences designed to promote those understandings (perhaps in teacher preparation courses or professional development settings). Having a KDU is not enough, however. MKT develops when a teacher conceptualizes the pedagogical power of their KDU.

As a teacher thinks about the content to be taught, she envisions a student (other than the teacher) working through the material, easing through some problems and stumbling over others. The entire time, the teacher must ask herself "What must a student understand to create the understanding that I envision?" and "What kinds of conversations might position one to develop such understandings?" The prospective teacher must put herself in the place of a student and attempt to examine the operations that a student would need and the constraints the student would have to operate under in order to (logically) behave as the prospective teacher wishes a student to do. This is reflective abstraction [in particular, reflected abstraction].

A KDU might be viewed as a pedagogical action, where *action* is used in the Piagetian sense. Teachers are engaged in pedagogical actions when they wonder, "What might I do to help students think like what I have in mind?" Their question is posed in a domain-specific manner, such as "How might I help my students think about logarithms as an accelerated condensing and recoding of the number line?" The development of MKT involves separating one's own understanding from the hypothetical understanding of the learner (Steffe, 1983). When a person views a pedagogical action as if she is not an actor in the situation (even though she is), and when the person can separate herself from the action (and thereby reflect on it), the pedagogical action has been transformed into a pedagogical understanding (Silverman & Thompson, 2008, p. 508).

There are a few important points of emphasis from this passage connecting MKT, reflective abstraction, and decentering. Silverman and Thompson (2008) describe how a KDU with pedagogical potential gains pedagogical power when a teacher reflects on their students' current meanings while hypothesizing activities and interventions they conjecture for advancing students' meanings. A teacher is decentering as they "separate one's own understanding from the hypothetical understanding of the learner," but it differs from decentering actions that involve a teacher constructing a second-order model of a student's thinking while directly interacting with a student.[6] A teacher's images of students' thinking relative to an idea emerge from one-on-one interactions with individual students. As a teacher (or researcher) engages with different students, she might emerge with images of multiple *hypothetical students*. Thus, when we talk about decentering in this paper, we are not talking solely about decentering actions that one engages in when interacting directly with another. Decentering can occur in all phases of planning for, engaging in, and reflecting on the result of instruction (or research) with students. Silverman and Thompson also emphasize that the development of MKT involves teachers engaging in reflecting abstraction on their KDU (the product of which is a reflected abstraction or a reflected scheme) and that this process is motivated by attempts at decentering.

Epistemic Students Emerge from Conceptual Analysis and Second-Order Models

Conceptual analysis, described by von Glasersfeld (1991) and elaborated by Steffe (1983) and Thompson (2002, 2011), addresses teachers'/researchers' "need to describe what students might understand when they know a particular idea in various ways" (Thompson, 2011, p. 43). The goal of conceptual analysis is to hypothesize the mental operations a person engages in as they reason about a situation based on their personal image of that situation. von Glasersfeld used conceptual analysis both to generate models of how others understand certain ideas and to generate meanings for ideas that would be beneficial for students in reasoning about those ideas in powerful ways (and why those meanings are especially beneficial). Thompson (2011) writes that:

> Finally, as illustrated in this papers [*sic*] first part, conceptual analysis can be employed to describe ways of understanding ideas that have the potential of becoming goals of instruction or of being guides for curricular development. It is in this regard that conceptual

[6] Note, however, that images of these hypothetical students are often formed from past experiences working with actual students in addition to engaging in *conceptual analysis*. We will discuss these connections in the next section.

analysis provides a method by which to construct and test a foundation of mathematics education in the same way that people created a foundation of mathematics.

In summary, conceptual analysis can be used in four ways:

(1) in building models of what students actually know at some specific time and what they comprehend in specific situations,
(2) in describing ways of knowing that might be propitious for students' mathematical learning, and
(3) in describing ways of knowing that might be deleterious to students' understanding of important ideas and in describing ways of knowing that might be problematic in specific situations.
(4) in analyzing the coherence, or fit, of various ways of understanding a body of ideas. Each is described in terms of their meanings, and their meanings can then be inspected in regard to their mutual compatibility and mutual support.

I find that conceptual analysis, as exemplified here and practiced by Glasersfeld, provides mathematics educators with an extremely powerful tool. It orients us to providing imagistically-grounded descriptions of mathematical cognition that capture the dynamic aspects of knowing and comprehending without committing us to the epistemological quagmire that comes with low-level information processing models of cognition (Cobb, 1987; Thompson, 1990). Conceptual analysis provides a technique for making concrete examples, potentially understandable by teachers, of the learning trajectories that Simon (1995) calls for in his re-conceptualization of teaching from a constructivist perspective and which Cobb and his colleagues employ in their studies of emerging classroom mathematical practices (Cobb,2000; Gravemeijer, 1994; Gravemeijer et al., 2014). In addition, when conceptual analysis is employed by a teacher who is skilled at it, we obtain important examples of how mathematically substantive, conceptually-grounded conversations can be held with students (Thompson, 2011, p. 44–45)

We include this entire quote because Thompson's description of conceptual analysis and its uses captures its relationship with decentering. Conceptual analysis is the mechanism by which teachers/researchers engage in aspects of decentering (even when no students are present) and involves the same mental activity involved in decentering in the moment of interaction with students/subjects.

A primary goal of mathematics education researchers employing a radical constructivist's[7] theoretical perspective is to model students' thinking, usually about particular mathematical ideas. To do so, these researchers often go through a multi-step process. The early phases involve synthesizing related literature on previous studies, reflecting on past work with students, and performing aspects of conceptual analysis about the idea (focusing on uses 2, 3, and 4 from Thompson's (2011) list). What emerge from this work are generalized models of students' thinking called *epistemic students* (Thompson, 2002).[8] An epistemic student is a description of a

[7] Mathematics educators leveraging radical constructivism as a background theory take seriously that (1) students' mathematical reality is distinct from the researchers' and (2) researchers have no direct access to students' mathematical reality.

[8] Thompson (2002, 2008) discussed the notion of an *epistemic person (student)* as a way of thinking about a particular idea. It is built from Piaget's (1981,1987) notion of an *epistemic subject*.

way of thinking about a particular idea that we could hypothetically attribute to a person in the way we might say, "Imagine that a student thinks about angle measure as reporting the area of a region bounded by an angle and a circle centered at the angle's vertex." The researcher constructs images of multiple epistemic students, considers the implications of each way of thinking about the idea and what observable behaviors might indicate that someone possesses a similar way of thinking about the idea, and, depending on the research study, may go on to consider how to scaffold learning experiences that support students in having the opportunity to construct a particular way of thinking about the idea the researcher believes to be powerful. Engaging in this activity involves a type of decentering activity. The epistemic students generated depend to a large degree on the researcher's own first-order model of the mathematics at hand and represent the researcher's initial attempts to generate models of others' thinking related to the idea.

Researchers then typically engage in work with actual students, such as conducting clinical interviews (Clement, 2000) or teaching experiments (Steffe & Thompson, 2000). While interacting with subjects in these settings, the researcher's goal is to generate conjectures about subjects' observable actions (utterances, drawings, gestures, etc.) and the nature of the subjects' meanings that might explain these actions. These second-order models that emerge are the result of the researcher's decentering actions and inform subsequent conjectures of the student's thinking. As researchers repeatedly decenter when interacting with students in the context of their learning or using an idea (e.g., average rate of change, exponential growth), the researcher's models of students' thinking become more viable, and their images of students' ways of thinking (epistemic students) become more refined. It is through the successive refinements of a theoretical model that it becomes more stable, although, as von Glasersfeld and Steffe (1991) have noted, we cannot arrive at absolute certainties, and "The most one can hope for is that the model *fits* whatever observations one has made and, more importantly, that it remains viable in the face of new observations" (p. 98).

It is worth noting that a teacher's decentering actions can similarly lead to the teacher constructing images of epistemic students. In Silverman and Thompson's (2008) description of how a teacher's KDU gains pedagogical power, they are describing the same mental actions involved in constructing images of epistemic students. These images provide the foundation for instructional design and guidance for leading classroom conversations. When engaged in instruction, the teacher's epistemic students help guide the teacher's decision-making (such as what questions to ask or how to follow up on student errors). However, if teachers are acting as first- or second-order observers while engaged in teaching, it is likely that they will routinely recognize novel student reasoning or encounter unexpected difficulty in supporting students' learning of some mathematical idea. The second-order model a teacher constructs when teaching and reflecting on specific classroom

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Piaget distinguished three constructs: the individual (an actual person), the psychological subject (an abstract person described in terms of psychological traits), and the epistemic subject (a coherent collection of ways of knowing).

interactions can lead to the teacher refining and adapting their images of students' thinking relative to an idea and when interacting with specific tasks related to that idea. These reflections may lead to the teacher adjusting her teaching practices and lessons according to these updated images of students' ways of thinking about an idea. We further claim that a teacher's first-order model of the mathematical idea under consideration/discussion influences the nature of the teacher's image, the usefulness of those images, and their ability to notice and reflect on elements of student's reasoning that can feed into the kinds of refinements in their teaching practice we have described. We will return to these ideas later in this chapter and propose a framework that describes the uses of decentering during teaching and its role in a teacher's development of *mathematical meanings for teaching* and *mathematical ways of thinking for teaching specific ideas*.

Clinical Interview Methodology

Mathematics education researchers have devised data collection methods for generating artifacts to inform their construction of a model of a student's thinking. One data collection method is clinical interviews, first used by Piaget in his early experiments investigating children's cognitive development. This method was later expanded to include open-ended interviews and think-aloud protocols (Clement, 2000).

The Role of Decentering in Clinical Interview Data Collection

The over-arching goal of a clinical interview is to reveal information about a subject's mental processes "at the level of a [subject's] authentic ideas and meanings" (Clement, 2000, p. 547). However, as researchers' knowledge base and inquiries have evolved, so have the field's data collection methods. As one example, Clement (2000) formalized two general categories of clinical interviews, generative and convergent. The goal of a generative clinical interview is to produce theoretical descriptions of mental structures and processes that reveal nuanced and diverse ways of thinking. Researchers in mathematics education commonly use generative clinical interviews in the early phases of exploring student thinking relative to a specific mathematical idea (e.g., rate of change, angle measure, function composition, accumulation). Their analysis of the clinical interview data generates a conjectured description of a subject's thinking (a second-order model developed via conceptual analysis). A researcher's coding activity involves their use of (or development of) theoretical constructs to label the subject's utterances, explanations, and actions as a basis for their claims about what the student might have been thinking. Convergent clinical interviews seek to apply the theoretical models developed from generative clinical interviews to code new or larger data sets. While the goal in convergent clinical interview studies is to have codes developed that allow high levels of agreement across independent coders, decentering does come into play when certain subjects or data sets turn out to be difficult to code reliably. Such instances indicate a need to revisit or conduct new generative studies to update models of student thinking (which necessarily involves conceptual analysis and decentering).

Clement's (2000) discussion of clinical interviews includes detailed descriptions of the various types of clinical interviews and their purposes. He also notes that the theoretical descriptions generated by a researcher's clinical interview data analysis are influenced by the researcher's current conceptions and theories. However, descriptions of methods for collecting clinical interview data rarely discuss how a researcher determines the interview task(s) or develops the line of questioning used during data collection and, more generally, how a researcher's theoretical perspective on what is entailed in learning an idea impacts the model that the researcher constructs of the subject's thinking. This observation led us to ponder the understandings and thinking that led to Piaget's task design and his line of probing that produced novel insights into children's thinking. More generally, we invite the reader to consider the mechanisms for building a model of another's thinking – what is the nature of the researcher's meanings and ways of thinking relative to the idea being studied that enables a researcher to notice and model important aspects of a research subject's thinking? What thinking is involved in putting aside one's own way of thinking about an idea and viewing a student's actions in responding to a problem from the student's perspective (decentering)?

Examples of Decentering in Mathematics Education Research

In this section, we provide two examples of how researchers' decentering actions and engaging in conceptual analysis influenced their work, including what they noticed, what they reported, and the connection between their first-order knowledge and the second-order models they constructed of subjects' thinking. As we discuss these examples, we illustrate connections among and uses of the various theoretical constructs already discussed.

Example 1: How a Researcher's Meaning for Rate of Change Informed Data Collection and Data Analysis

Thompson (2016) describes his observation of a ninth-grade teacher as she is teaching a lesson on the point-slope and point-point formulas for linear functions.[9] Thompson's characterization of the teacher's interactions reveals the role of his

[9] Thompson conveyed that the teacher had attended a professional development workshop he led, and the teacher had subsequently decided to use a task similar to one that he had used during the

meaning for constant rate of change and his use of decentering in describing the teacher's actions.

Thompson (2013) reports that the teacher first plotted the two points (3, 1) and (7, 4) on a coordinate axis. This action was followed by the teacher sketching line segments to illustrate her claim that "the function goes over 4 and up 3." Thompson further conveyed that the teacher attempted to locate the *y*-intercept of the line by moving to the left four units from the point (3, 1). He then noted that while the teacher was constructing the line (see Fig. 9.1), she stopped mid-sentence after uttering the phrase, "If we go 4 to the left…." After a long pause, the teacher stopped her lesson and proceeded to discuss the homework assignment for the next day.

Thompson (2013) provided his fine-grained description of the teacher's actions (including utterances and drawings) used to characterize her thinking. According to Thompson:

> She saw the change in *x* as a chunk. This was unproblematic in the case of one point. However, her chunk in this problem did not place her at $x = 0$ as she wished. Second, her meaning for slope was "rise over run", where rise and run were both chunks. Third, her computation of slope, not evident in this excerpt but made clear later, was of a procedure that produced a number that is an index of a line's "slantiness". Division did not produce a quotient that has the meaning that the dividend is so many times as large as the divisor—3/4 as a slope was not a number that gave a rate of change. It gave a "slantiness". Fourth, her meaning for rate of change entailed neither smooth variation nor proportionality. It was more akin to her meaning for slope—two things changing in chunks. These meanings not

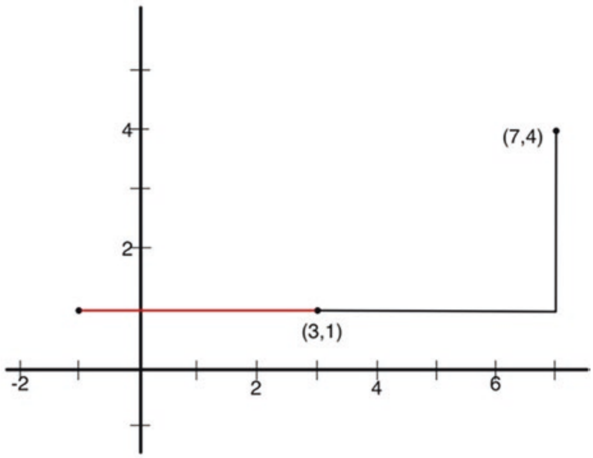

Fig. 9.1 An example provided by Thompson (2016, p. 84)

workshop (P. Thompson, personal communication). She decided to use the task the day he was visiting their school so he could observe her lesson. Note that Thompson (2016) explains that his intention with this task in the workshop was for teachers to determine the function's constant rate of change and use this to coordinate proportional changes in *x* and *y* to determine the function's vertical intercept. Such a solution method requires multiplicative reasoning, which contrasts with the additive "rise over run" reasoning many teachers were employing.

> only failed to provide Sandra a connection between her current setting (two points) and prior method, they led her down the dead-end path she followed. (Thompson, 2013, pp. 81–82)

We claim that Thompson's (2013) ability to characterize the teacher's thinking (construct the second-order model he described) was based on both the actions and utterances that Thompson observed and his interpretation of these actions, and both of these were influenced by his first-order model of what is entailed in understanding and learning the idea of constant rate of change (including ideas about variation, proportionality, slope and rate of change, the meaning of division, and so on).[10] Furthermore, it was similar to second-order models of teachers' thinking in the professional development workshop that prompted him to create the task in the first place (P. Thompson, personal communication). Thompson described that many of the teachers' images of slope entailed an amount of rise and an amount of run, where rise and run represented two things changing in chunks, and they were therefore relying on additive reasoning to think about slope. Thus, Thompson deliberately created a scenario such that teachers attempting to leverage this image would encounter difficulties completing the task. This suggests that Thompson was alert to the limitations of this way of thinking and was intentional in designing the task to reveal and support students and teachers in confronting this conception of slope.

Thompson (2013) went on to describe a way of thinking about constant rate of change that would have been more useful for Sandra: "Had Sandra reasoned proportionally and with smooth continuous variation, she might have said '… over 3/4 of 4 and up 3/4 of 3'. That would have given her the graph's *y*-intercept" (p. 81). Note that Thompson's analysis involves multiple aspects of conceptual analysis (building models of a subject's thinking and describing ways of knowing a mathematical idea that might be either powerful or problematic) and indicates that he positioned himself as a second-order observer while constructing his second-order model of the teacher's thinking (focused not just on how her thinking was different from his but also the implications of that thinking). His suggestion of a more powerful way of thinking about constant rate of change was based on the thinking he considered at the time to be productive, including an image of the two quantities varying together smoothly and continuously and the recognition of a proportional relationship between the changes in the two quantities and all that proportionality entails. Finally, we point out that Thompson's first-order model of

[10] We don't have the space to fully unpack all that Thompson and others have written about students' and teachers' meaning for these ideas (e.g., Byerley et al., 2012; Castillo-Garsow, 2010; Johnson, 2015, 2018; Thompson, 1994b, 1994, 2000, 2011; Thompson & Carlson, 2014; Thompson & Thompson, 2007, 1994). However, Thompson's interpretations of the teacher's actions revealed aspects of his image (at the time of reporting) of the ways of thinking that he deemed to be most productive (and less productive). For example, his conceptualization of the difference between thinking about variations happening in chunks compared to smooth transitions where the varying quantities takes on all possible values within an interval oriented his attention to how the teacher was conceptualizing variation and the implications of her "chunky thinking."

constant rate of change influenced the data he presented to make arguments in the paper about the importance of MKT for creating mathematical experiences where students have a chance to learn mathematical ideas that are powerful and will serve them well in the future.We invite the reader to consider questions such as the following. What led to Thompson's (2013) construction of the task and characterization of the teacher's thinking? What meaning of constant rate of change did Thompson deem to be productive? In particular, what was Thompson's first-order knowledge of constant rate of change that resulted in him being alert to the "chunky thinking" that he observed? How did Thompson's first-order knowledge influence the second-order model he built of the teacher's thinking? We further invite the reader to reflect on what guides your own task design and data collection. How does your first-order model for understanding and learning an idea influence what questions you ask, what you notice, and how you interpret and report your data?

Example 2: Researcher Decentering in a Teaching Experiment on Logarithms

In this section, we provide an example of researchers' iterative efforts to understand and characterize what is entailed in understanding the idea of logarithm, and we highlight how the products of the researchers' decentering actions advanced their first-order models of what is entailed in understanding and learning the idea of logarithm. We also illustrate how the researchers' first-order models for teaching the idea of logarithm evolved during, and as a result of, their decentering actions. Their conceptual analysis included fine-grained descriptions of students' thinking relative to understanding the idea of logarithm and then shaped the researchers' hypothetical learning trajectory and subsequent teaching experiment design and data analysis. Their first-order models (including images of students' thinking) also influenced their subsequent decentering actions, including the questions they posed and the models they built. We begin by discussing the purpose and design of teaching experiments in cognitively focused mathematics education research.

Iterative Models of a Student's Thinking Informs Teaching Experiment Design

A teaching experiment involves a series of teaching episodes that comprise a teaching agent, one or more students, a witness, and a recording method. Teaching experiments differ from clinical interviews in that they involve experimentation with the ways and mechanisms of influencing students' mathematical understandings (Steffe & Thompson, 2000). Teaching experiments allow a teacher-researcher to generate situations of learning systematically and to test conjectures and local hypotheses about students' mathematical meaning for a specific idea and how the student's meaning is constructed (Steffe & Thompson, 2000). The design of a teaching experiment begins with the researcher describing the mental actions and ways of thinking she conjectures to be essential for understanding the idea that is the focus of the study (i.e., performing a conceptual analysis; Thompson, 2011). A researcher might generate her initial conceptual analysis by reviewing the research literature related to understanding and learning the idea while leveraging her prior decentering actions and images of students' thinking when learning that idea. In subsequent studies focused on this idea, the researcher generates new images of students' thinking that lead to modifications of her conceptual analysis. If little is known about what is entailed in understanding and learning the idea that is the focus of the investigation, it is common for researchers to conduct a sequence of small-scale investigations (e.g., exploratory teaching interviews[11]) with several students prior to conjecturing a trajectory and activities to support students in learning the idea in ways the researcher hypothesizes will be useful for them.

Exploratory Teaching Interviews Lead to Advancements in a Researcher's First-Order Model

Weber (2002) demonstrated that stating and attempting to explain the common conversion equation that links a logarithmic statement and its equivalent exponential statement, $\log_b(x) = y \leftrightarrow b^y = x$, were not effective in supporting students in understanding of what the statement, $y = \log_b(x)$, conveys—in particular, that y represents the number of factors of b in the product that equals the value of x. Kuper (2020) leveraged these findings in their design of tasks for a series of initial exploratory teaching interviews aimed at producing models of students' meanings for exponential relationships and logarithms that might inform her first-order model for students' learning of the idea of logarithmic function and understanding logarithmic properties conceptually. Kuper began by probing students' thinking when responding to the bacteria growth task, which states: "The population of bacteria in a culture t days since the bacteria started growing can be determined by the formula $P(t) = 7(2)^t$. How many days have passed when the bacteria population is 224?".

Results from performing a retroactive conceptual analysis on the data led Kuper and Carlson (2020) to identify common ways of thinking used by multiple students. Some students viewed $P(t)$ as the name of the function instead of a representation of the bacteria's population and the expression $7(2)^t$ as instructions for calculating a value. These conceptualizations did not support the students in viewing the expression $7(2)^t$ as defining the co-varying relationship between the population's bacteria $P(t)$ and the corresponding values of t.

[11] See Steffe and Thompson (2000) and Castillo-Garsow (2010) for a description of exploratory teaching interviews.

Kuper and Carlson (2020) further report that it was common for students to exhibit a calculational conception of solving the eq. $224 = 7(2)^t$. That is, they described their solution process as following a "solving for" algorithm of isolating a variable. Consequently, the students did not attend to the quantitative meaning of the calculations and expressions produced when constructing their solution. For example, their first step, dividing 224 by 7 to produce a result of 32, was viewed as a calculation disconnected from the context (i.e., students could not explain what 32 represented in the context). We can contrast this with a quantitatively focused solution approach where the steps and expressions emerge through a process of determining the values of important quantities in the situation.[12] This way of thinking would develop the ratio 224/7 as a relative size comparison of the final number of bacteria and the initial number of bacteria (with the result, 32, being the measure of that relative size). From this conceptualization, writing the eq. $32 = 2^t$ could be explained as representing the fact that the number of bacteria must grow to become 32 times as large as the original number in two ways: (i) as the result of making a relative size measurement and (ii) as the result of doubling the initial number of bacteria some number of times (in this case five times). (It is worth noting that Kuper (2020) gained clarity in the utility of this way of thinking about the meaning of the output of a logarithmic function, which impacted her first-order model of exponential functions and logarithms, during analysis of her interview data.) Since most students did not exhibit this quantitative approach when constructing their solution, their justifications for the solution included statements like, "$t = 5$ is the exponent on 2 that gives an answer of 32" and the answer 5 is the result of "taking the log of both sides." They also observed students conceptualizing the solution ($t = 5$) as being the unknown value that satisfies the original statement $224 = 7(2)^t$ because it produces the correct numerical value. We contrast this rather superficial meaning for the equation's solution with other possible meanings, such as seeing t as representing a varying number of days since the first bacteria population measurement was taken and imagining the value of t varying with the bacteria population; then viewing the solution to $224 = 7(2)^t$ as the value of t that corresponds to the number of bacteria reaching 224.

While Kuper (2018) had initial hypotheses about meanings for logarithms that would be productive for students understanding the idea of logarithm, her decentering actions led to her constructing new images of students' thinking (epistemic students). In particular, she gained clarity about students' conceptions of function and function notation and their conception of what the output value of a logarithmic function represents. It is also noteworthy that the researcher's decentering actions, including her reflecting on a student's thinking and efforts to create tasks to advance

[12] See O'Bryan (2018, 2020) and Carlson et al. (2003) for more information about *emergent symbol meaning*, a way of thinking about calculations, expressions, and formulas where the order of operations and structure emerges from the way the individual has conceptualized relationships between quantities in the situation as opposed to viewing them merely as steps or parts of a calculation algorithm.

a student's thinking, led to adaptations to her first-order model and subsequent refinements of her conceptual analysis and instructional materials.

After their analysis of this initial interview data and after reflecting on ways of thinking they conjectured would advance students' thinking, Kuper and Carlson (2020) conducted additional exploratory teaching interviews to test their hypotheses. They devised tasks to support students in understanding a ratio as a relative size, a quotient as a measure of the relative size of a ratio's numerator and the denominator, and the usefulness of these meanings on a student's conception of the expression $\log_2(32)$ as expressing the number of times some value needed to be doubled to increase that value by a factor of 32. The researchers' analysis of the exploratory teaching interviews with the bacteria task revealed that, as students shifted to interpret $t = 5$ as representing the number of doublings required for the initial number of bacteria to grow by a factor of 32, students began to coordinate the number of doubling periods (5) and the relative size of the final number of bacteria and the initial number of bacteria. Put another way, if some value is doubled every day for 5 days, then the period over which the doubling occurred is 1 day, and the result of doubling a value each day for 5 days would produce a new value that is 2^5 (or 32) times as large as the original value.

To illustrate another foundational understanding, after exploring students' conceptions of repeatedly doubling and repeatedly tripling some value, including the ability to explain what the results of the repeated doubling or repeated tripling represented, Kuper and Carlson (2020) observed that the fluency students acquired did not extend to determining and representing repeatedly increasing some value by other factors (e.g., 1.5 or 17). Kuper and Carlson's analysis of these interactions led to their hypothesizing that introducing the term *tupling* would lead to students' generalizing their fluency in reasoning about doubling and tripling to non-integer numbers and result in their conceptualizing and speaking about tupling a number 1.25 times.[13] This observation was not only transformational relative to their hypotheses about productive meanings for students. It also became a new way of thinking about exponentiation for the researchers because it provided clarity in their own reasoning about exponentiation and productive ways of thinking about the connections between exponential expressions and logarithmic expressions (i.e., it profoundly changed their first-order models of exponential and logarithmic functions).[14]

We have provided these descriptions of two exploratory teaching interviews to highlight the subtle and novel insights the researchers gained about students'

[13] In a later section, we described a hypothetical trajectory for introducing the idea of tupling.

[14] See Kuper and Carlson (2020) for a detailed analysis of the development of the *tupling* idea and term and a conceptual analysis of its relationship to how the researchers thought about related mathematical ideas. For purposes of this chapter, it is enough to note the following descriptions. "If the value of a quantity becomes m times as large, we say the quantity's value m-tuples…An m-tupling is an event in which the value of a quantity becomes m times as large…An m-tupling period is the amount of change in the independent quantity of an exponential function needed for the dependent quantity of the exponential function to become m times as large…[An] Exponent (on a value, b)…[is] [t]he number of elapsed b-tupling periods" (Conceptual Analysis section, Table 1).

thinking and learning that were only possible by the researchers taking seriously their commitment to model students' thinking (i.e., generating second-order models of students' thinking and reflecting on these models to inform modifications to a researcher's first-order model).

First-Order Models, Decentering, Second-Order Models, and Conceptual Analysis Inform Task Design

A researcher's models of students' thinking of an idea guide the researcher's task development and conceptual scaffolding. In the prior section, we described the researcher's conception of a student's impoverished conception of ratio and quotient (the researcher's second-order model) and noted that subsequent exploratory teaching interviews revealed students not initially conceptualizing a quotient as a measure of the relative size of a ratio's numerator and denominator (Kuper & Carlson, 2020). The researchers' awareness of students' calculational orientation and its role in obstructing advances in students' understanding of what expressions like $\log_2(32)$ represent (e.g., the number of doublings that results in an initial value growing by a factor of 32) informed the design of new tasks aimed at fostering student reasoning about the relative size of two quantities' values and for representing the repeated tupling of some initial value. As the researchers reflected on the thinking they had observed, in relation to the thinking they deemed to be productive, we claim they were engaged in decentering – they were bringing to mind students' thinking and deciding on specific actions they hypothesized (e.g., showing an animation and deciding on specific questions to pose) for supporting advances in students' reasoning.

As an example of the impact of this reflection on decentering (meta-reflection), Kuper and Carlson (2020) proposed to initially ask students to construct a cactus. This was followed by a request to construct a second cactus to represent a doubling of the height of the first cactus (Table 9.1). The researchers also elected to introduce the term two-tuple to refer to a double in a cactus height. The rationale for this choice was to support students in developing a meaning for the idea of tupling in a familiar context as a foundation for future engagement in activities to generalize this meaning to conceptualize an 8-tupling, 4.5-tupling, and eventually an x-tupling. We claim that the decentering actions of the researchers contributed to their constructing *ways of thinking about teaching the idea of logarithm*. These included the researchers conceptualizing specific approaches (see Task Column in Table 9.1) for engaging students in conceptualizing the relative size of two quantities' values as a foundational way of thinking for conceptualizing tupling and considering how many two-tuplings of some value would result in a final value of some quantity becoming 32 times as large as its initial value. We further claim that the products of the researchers' decentering actions led to advances in their second-order models of specific students' thinking and their resulting epistemic students (generalized images of students' ways of thinking) related to learning the idea of logarithm. The researchers' updated hypothetical learning trajectory was similarly refined to

Table 9.1 Sequence of tasks for learning the tupling language

Stage	Task	Possible responses	Rationale for task design and sequencing
1	Draw a cactus. Suppose this cactus doubles, which we refer to as two-tuples, in height. Draw the resulting cactus after this growth. The resulting cactus is how many times as tall as the starting cactus? Also consider: three-tuples or triples	*The student might measure the height of the cactus the student initially drew and use that height to draw a new cactus that is as tall as two copies of the initial cactus* The resulting cactus is two times as large as the initial cactus	This task introduces the new tupling language while also using the colloquial term “doubles.” Students are encouraged to participate in the activity of drawing the initial and resulting cactus The effect of the activity that students may notice is that the resulting cactus is two times as tall as the initial cactus
2	Suppose the cactus you drew is eight-tuples (octuples) in height. If needed, draw the resulting cactus after this growth. The resulting cactus is how many times as large as the starting cactus? Also consider: 4.5-tuples	*The student might measure the height of the initial cactus drawn and draw a new cactus that is as tall as eight copies of the initial cactus. Or the student might reflect on his thinking in the previous task and answer the question without needing to draw the resulting cactus* The resulting cactus is eight times as large as the initial cactus	This task uses the new tupling language while also providing the colloquial term “octuples.” The student may choose to participate in the activity of drawing the initial and resulting cactus if he experiences difficulties A student that reflects on the activity-effect relationship might imagine drawing the new cactus and conclude that the new cactus will be eight times as tall as the initial cactus
3	Suppose a cactus is 57-tuples in height. If you were to draw the resulting cactus, what would we observe? (Hint: The resulting cactus is how many times as large as the starting cactus?) Also consider: 1000-tuples	*The student might describe a process of measuring 57 copies of the original cactus to arrive at the resulting cactus' height* The resulting cactus will be 57 times as large as the initial cactus	This is the first task in the sequence where participating in the activity of drawing the resulting cactus would be too tedious. Instead, we designed this task to perturb students who relied on the activity aspect of the task to reflect on the activity-effect relationships of the previous tasks and imagine the result of the activity as if the actions were performed
4	Suppose the cactus provided x-tuples in height. If you were to draw the resulting cactus, what would we observe? (Hint: The resulting cactus is how many times as large as the starting cactus?)	The resulting cactus will be x times as large as the initial cactus	This task does not provide a numerical value for the student to work with. Rather, the student is expected to make a generalization using the tupling language to describe an arbitrary number of tuples Students who struggle to make this generalization may need to attempt more tasks like those in Stages 2 and 3 to engage in reflection of the activity-effect relationship

include anticipated ways of thinking observed in prior exploratory teaching interviews. These updates resulted from advances in the researchers' first-order model for understanding and learning the idea of logarithm. These advances in the researchers' first-order model also led to more effective decentering (see Table 9.1, column 2) and more targeted probing (see Table 9.1, column 1) during subsequent exploratory teaching interviews.

Advances in the researchers' image of student thinking led to advances in the researchers' trajectory for reasoning about the relative size of two quantities' values (see Table 9.1, column 3)—in particular, the researchers were more anticipatory of student thinking and thus better prepared to respond productively to thinking that students presented. To view the tasks and ordering for other ideas in the hypothetical learning trajectory for supporting students' understanding of the idea of tupling, and gaining fluency in using the idea to describe what the dependent quantity of a logarithmic function represents, see Kuper and Carlson (2020).

The summary of the researchers' hypothetical learning trajectory in Table 9.1 reflects how the researchers came to think about exponential growth as a tupling process, a logarithmic expression as representing a number of tupling periods for a quantity to become some number of times as large as its original value, and the meaning of the relationship between exponential and logarithmic functions that follows from these understandings. The researchers then considered their images of epistemic students, grounded in second-order models of students they had worked with, and imagined the kinds of tasks, questions, and interventions necessary for perturbing and advancing the thinking of students who present these hypothetical ways of thinking. For example, the learning trajectory in Table 9.1 shows the researchers' expectation that drawing diagrams representing the relative sizes of the cactus before and after growing, connected to common language such as "doubling" and "tripling," would provide objects of reflection for making sense of the tupling process and developing a meaning for *m*-tupling as a general idea. Their images of epistemic students also guided them to imagine some students remaining reliant on figurative activity, such as drawing diagrams and not transitioning to coordinating the multiplicative growth of the cactus height and elapsed time more generally. This led the researchers to design tasks that would become increasingly difficult without abstracting the process of tupling to construct a mental image of the general relationship between elapsed time and the cactus's height. The learning trajectory in Table 9.1 became the foundation for a teaching experiment with one student.

Modeling Student Thinking in the Context of a Teaching Experiment

Kuper and Carlson (2020) implemented their learning trajectory in a subsequent teaching experiment conducted with one student. In the initial phase of their teaching experiment, Kuper and Carlson aimed to support the student in conceiving of logarithms as functions that determine the number of *b*-tupling periods needed for some value to grow by a factor of *m* (*m*-tuple) (e.g., $\log_2(8)$ represents the number of doubling periods needed for a value to eight-tuple or become eight times as large). To facilitate this, Kuper and Carlson engaged the student in a series of tasks that prompted her to examine the growth of a mystical cactus, “Sparky the Saguaro.” During each teaching episode, the researchers continually updated their model of the student’s thinking as they interacted with the student. They included the following vignette from an interaction with the subject, Lexi.:

In an attempt to determine the 3-week growth factor, Lexi began by noting Sparky’s initial height of one foot at week zero and then claimed, “three time(s)– no, every week it’s doubling, or times two for the height. So to get to week three, you’d say it’s like, you wouldn’t say 6 times as large – that wouldn’t make sense. I feel like you would say 3 times as large – that doesn’t make sense either.” This response suggests that Lexi first considered multiplying the 1-week growth factor (2) by the number of elapsed weeks (3) to calculate the 3-week growth factor. However, she quickly ruled out that option and looked to other values appearing in the situation. Lexi then appeared to observe the height of the cactus 3 weeks after its purchase and eventually concluded that at the end of week 3, Sparky would be eight times as tall as the initial Sparky. However, there was no evidence to suggest that Lexi had contemplated the relationship between the 1-week growth factor (2) and the number of weeks elapsed (3) as an approach for conceptualizing the 3-week growth factor (8). In particular, although Lexi noted that Sparky was doubling in height every week, her responses and attention to the heights of the cacti suggest that she had not yet conceptualized that doubling Sparky’s height 3 weeks in a row is the same as growing by a factor of 2^3 or 8 (Kuper & Carlson, 2020, "Teaching Experiment #1" section, para. 27).

Through a combination of decentering in the moment and performing a conceptual analysis of the interview data after the fact, Kuper and Carlson (2020) constructed an initial model of Lexi’s thinking and the implications of her thinking through the lens of the constructs characterized in their conceptual analysis. This led to follow-up work with the student to establish a more fine-grained description of her meanings that would inform modifications to the researchers’ first-order models, their images of epistemic students, and their hypothetical learning trajectory. In the next teaching episode, Lexi was able to note that the doubling period for the cactus’s height was 1 week, the four-tupling period was 2 weeks, and the eight-tupling period was 3 weeks, but when asked to determine the three-tupling period, Lexi predicted it to be 1.5 weeks (halfway between 1 and 2 weeks since three is halfway between 2 and 4). To determine how Lexi thought about her answer, they asked her to determine the nine-tupling period (which, if her answer was correct and she understood that her answer provided the growth factor for a change in time elapse of 1.5 weeks, would mean that the nine-tupling period was 3 weeks). Rather than build off of her previous work, Lexi predicted that the nine-tupling period would just be a bit longer than the eight-tupling period, first guessing 3.5 weeks and then revising to 3.25 weeks. Kuper and Carlson write that “During the retrospective analysis of the third teaching episode, we hypothesized that Lexi’s difficulties with the aforementioned ideas were due to her not understanding that an *A*-tupling followed by a *B*-tupling had the same overall effect as an *AB*-tupling” ("Teaching Experiment #1" section, para. 29). This was an important additional insight that, once again, contributed to modifications to the researchers' first-order models of the ideas under study, a refinement of their images of epistemic students, and contributed to revisions in their hypothetical learning trajectory. As a final comment, Lexi finally developed a meaning for the idea that a quantity that *m*-tuples and then *n*-tuples grows by a factor of *mn* and this "was essential for Lexi as she attempted questions requiring the use of the first logarithmic property [$\log_b(m) + \log_b(n) = \log_b(mn)$]" ("Teaching Experiment #1" section, para. 31).

Comments on Examples 1 and 2

Examples 1 and 2 provide illustrations of several key ideas related to decentering. In Thompson's (2016) report, the task he designed in the professional development workshop, which the teacher opted to use with her students, was in response to decentering in the moment of working with the teachers. The second-order models he constructed of the teachers' meanings aligned with images of epistemic students he possessed from prior research, reading, and conceptual analysis. He designed the task such that it would be particularly challenging for teachers with the ways of thinking about constant rate of change he hypothesized the teachers possessed, thus introducing a perturbation that would necessitate advances in the teachers' meanings. As he observed the teacher with her students, his first-order model influenced

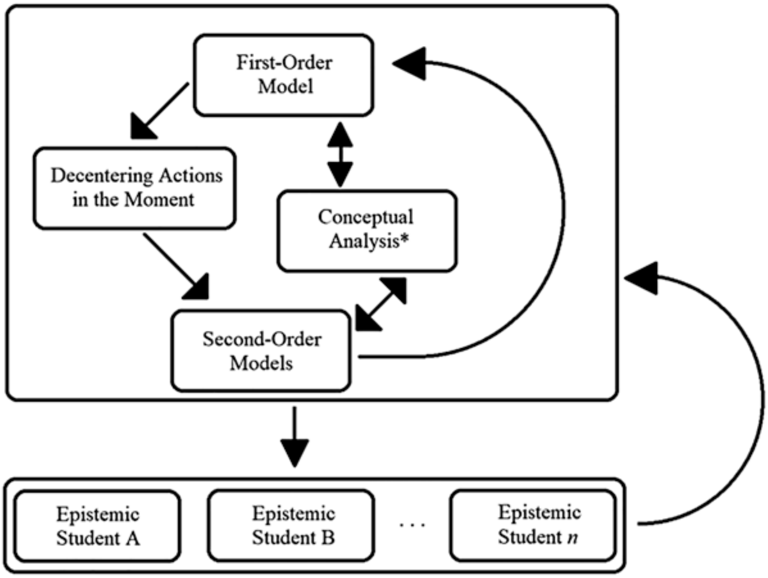

Fig. 9.2 The relationship between first-order models, decentering, conceptual analysis, second-order models, and epistemic students. (Note: *Recall that conceptual analysis can be used in multiple ways, each contributing differently to this process)

both his decentering in the moment (what he noticed and how he responded to what he noticed) and during his retrospective conceptual analysis of the teaching episode and discussing the implications of the teacher's meanings.

Kuper's (2020) and Kuper and Carlson's (2020) work illustrates how products of researchers' decentering actions (second-order models) advanced their first-order model of what is entailed in understanding and learning the idea of logarithm. Kuper's decentering actions when conducting a sequence of exploratory teaching interviews with different students were instrumental in the construction of epistemic students (various ways of thinking about logarithms). They further illustrated how the insights gained from these investigations led to updates in the researchers' first-order model to include (i) ways of thinking about learning the idea of logarithm that were novel to them, (ii) new images of students' thinking when interacting with tasks designed to perturb their thinking, and (iii) new ways of thinking about teaching the idea of logarithm meaningfully to students. They claim that these insights led to refinements of the first author's subsequent decentering actions, including the questions she posed and the nature of the models she built.

Figure 9.2 represents our image of the relationship between many of the constructs and processes we have discussed and exemplified thus far in the chapter.

Due to the cyclical and interdependent nature of many of the constructs and processes, there is no single "entry-point" to thinking about the relationship between first-order models, decentering, conceptual analysis, second-order models, and epistemic students. However, we will start our discussion with first-order models (an individual's meanings and ways of thinking about a mathematical idea). As we have discussed and Thompson (2016) demonstrated, a first-order model impacts what someone is primed to notice and thus the nature of their decentering actions. Someone's first-order model also supports the multiple facets of conceptual analysis (modeling a student's/subject's thinking and considering the implications of particular meanings), and conceptual analysis can produce insights that feed back into and influence the first-order model. Reflection on the results of decentering actions and conceptual analysis can both support the construction of second-order models of students'/subjects' thinking, which, as demonstrated by Kuper (2020) and Kuper and Carlson (2020), can also influence a researcher's first-order models. Engaging in decentering, conceptual analysis, and model-building all feeds into the development and refinement of images of epistemic students (the researcher's image of generalized ways that students think about an idea). Images of epistemic students become incorporated in the teacher's/researcher's first-order model and are helpful for performing conceptual analysis and constructing second-order models because they are familiar ways of thinking a teacher/researcher might recognize and be prepared to leverage during subsequent exploratory teaching interviews.

Uses of Decentering When Studying Teachers and Teaching

In the prior section, we illustrated how a researcher's first-order model of an idea can influence a researcher's decentering actions (to create second-order models of students' thinking, perform conceptual analysis, and construct images of epistemic students) and how a researcher's decentering actions in the context of interacting with a subject about an idea can influence the researcher's first-order model of learning that idea. In this section, we make a case that advancing an instructor's teaching practice to be more meaningful, coherent, and responsive to students' thinking necessarily requires a similar symbiotic interaction between an instructor's mathematical meanings and decentering actions (including decentering in the moment or in retrospect to generate second-order models of students' thinking, when performing conceptual analysis and constructing images of epistemic students). We then illustrate uses of decentering for studying teacher-student interactions and conclude by providing an overview of a framework that emerged from studying teachers in the context of their interacting with students in a classroom setting.

Elaborating Our Meaning for MMT and Its Symbiotic Relationship with Decentering

In our descriptions of a teacher's mathematical knowledge for teaching (MKT), we adopt Thompson's (2016) use of the term *mathematical meaning* when referencing the organization of an individual's actions, images, and other meanings associated with an idea, also referred to as a scheme.[15] It is through repeated reasoning, reflection, and reconstruction that an individual constructs schemes to organize experiences in an internally consistent way (Piaget & Garcia, 1987, 1991; Thompson, 2016; Thompson et al., 2019). Thompson extended the construct of *mathematical meaning* when defining *mathematical meanings for teaching* (MMT) by including characterizations of a teacher's actions that are related to teaching. Our characterization of a teacher's MMT for an idea enabled us to make inferences about how a teacher organizes her experiences with an idea that might reveal the teacher's image of what is entailed in understanding, learning, and teaching an idea.

Silverman and Thompson (2008), as discussed earlier in this chapter, claim that:

> A teacher has developed knowledge that supports conceptual teaching of a particular mathematical topic when he or she (1) has developed a KDU within which that topic exists [e.g., proportionality as a way of thinking for understanding constant rate of change; covariational reasoning as a way of thinking for understanding growth patterns in quantitative relationships], (2) has constructed models of the variety of ways students *may* understand the content (decentering) [through decentering and performing a conceptual analysis]; (3)

[15] Tillema and Gatza (Chap. 3) provide a detailed discussion of *scheme*.

> has an image of how someone else might come to think of the mathematical idea in a similar way; (4) has an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea; (5) has an image of how students who have come to think about the mathematical idea in the specified way are empowered to learn other, related mathematical ideas. (Silverman & Thompson, 2008, p. 508)

We view these phases as a general framing of the processes and products of developing MMT as a specific idea.

What is relevant for our work then is to describe the mechanisms for advancing an instructor's MMT in ways that might lead to the instructor constructing an image of students' thinking and spontaneously leveraging that image when interacting with students, planning a lesson, or reflecting on the impact of their instructional actions on students' thinking. We classify images of teaching that include ways of thinking about students' ways of learning an idea as a *way of thinking about teaching that idea*. A *way of thinking about teaching an idea* then necessarily includes (i) how the teacher reasons about and understands the idea (the teacher's first-order model) and if they have engaged in decentering relative to students' thinking about that idea and (ii) their second-order models of students' thinking about and learning that idea. As a teacher's image of students' mathematics (the teacher's second-order models) becomes more stable through reflecting on acts of decentering, we consider

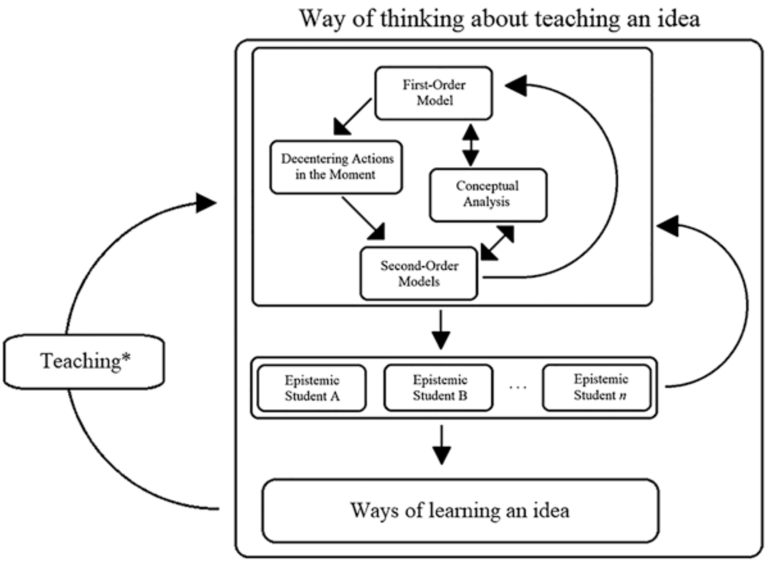

Fig. 9.3 A framework for the development of ways of thinking about teaching an idea. (Note: *The act of teaching provides feedback on ways of thinking about teaching an idea and new opportunities to engage in decentering, conceptual analysis, refine first-order models, and refine images of epistemic students)

these teacher images of students' generalized ways of thinking (epistemic students) as informing (and being assimilated into) a teacher's first-order model or mathematical meaning for teaching that idea (Figs. 9.2 and 9.3) such that the first-order model then contains multiple ways of thinking about the idea.[16] To further clarify, we claim that a teacher's MMT for an idea includes (i) *habitual actions of making sense of a student's thinking when learning an idea*, (ii) *images of students' generalized ways of thinking about that idea (epistemic students)*, and (iii) *images of the effect of prior instructional moves on students' thinking*. We further claim that a teacher's scheme for teaching that idea includes both *ways of thinking about learning that idea* and *ways of thinking about teaching that idea*, in addition to the teacher's stable way of understanding the idea. Through perturbations and accommodations to one's scheme for knowing, learning, and teaching that idea, the teacher constructs more connected *way of thinking about teaching that idea*.

Tallman and Frank (2016) describe how a teacher's attention to constructing models of students' thinking was constrained when the teacher's MKT did not include a more generalized way of thinking about what is entailed in measuring an angle (that any process for measuring an angle involves quantifying the length of the arc subtended by a circle centered at the angle's vertex in any unit proportional to the circle's circumference). They further claim that "the absence of such an intention makes a teacher ill-equipped to recognize the possibility that he or she is promoting students' construction of uncoordinated angle measure schemes" (Tallman & Frank, 2016, p. 92). They illuminate this claim in the data they present to illustrate that a teacher's way of thinking about the idea of angle measure was tied to features of a specific context and was absent of a more general conceptualization of an angle measure as the result of a quantitative comparison. This tendency to employ specific ways of thinking about angle measure based on the context resulted in the teacher not recognizing reasoning used in one context as also being applicable in other contexts.[17] In the moment of teaching, the teacher was focused on determining if students thought about the solution method in the same way as the teacher rather than considering the students' contributions and recognizing the validity of other quantitative comparisons that could produce the correct answer.

Tallman (2018) demonstrated that a teacher's interpretation of and responses to a student's expressed thinking are strongly related to both the teacher's personal mathematical meanings and the learning goals a teacher articulates prior to instruction. This finding further corroborates claims that teachers' mathematical meanings for teaching are a primary source of their instructional decisions and actions (Thompson, 2017). The evidence produced by Tallman is conveyed in his detailed

[16] These ways of thinking about the idea may represent both valid and invalid reasoning and/or may be more or less productive based on the details of a situation.

[17] For example, thinking about angle measure as communicating a fraction of the circumference of all circles centered at the angle's vertex subtended by the angle compared to thinking about angle measure as communicating the relative size of the subtended arc lengths of all circles centered at the angle's vertex and units of measure proportional to the circles' circumferences (such as radius lengths or 1/360th circumference length).

analysis of a teacher's pre-teaching conceptions, classroom instruction, and stated learning goals. As an added methodological consideration, Tallman argues for the value of identifying and characterizing consistencies and discrepancies between teachers' personal and enacted mathematical knowledge. As was demonstrated by Tallman, this approach has the potential to provide insights into "the characteristics of teachers' subject matter knowledge that enable its effective transformation in the context of practice" ("Research on the Enactment" section, para. 8).

The Symbiotic Relationship Between a Teacher's Meaning for an Idea and Her Decentering Actions

Research findings have demonstrated links between a teacher's mathematical meanings and her teaching practices, including her goals for student learning, and how a teacher interprets and responds to a student's questions (Baş-Ader & Carlson, 2022; Heid et al., 2012; Marfai, 1992; Musgrave & Carlson, 2017; O'Bryan & Carlson, 2016; Rocha, 2021; 2022; Rocha & Carlson, 2020; Tallman, 2021, 2018; Uscanga et al., 1995; Zbiek et al., 2014). According to Thompson et al. (1996), and further substantiated with empirical data by Tallman (2018) and Rocha (2022), "If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures" (p. 416). In contrast, if a teacher has a coherent set of mathematical meanings, then it is at least possible, but not sufficient to guarantee, that this teacher will attempt to have her students construct this same set of coherent mathematical meanings (Thompson et al., 1996). This claim leads to the following questions. What processes and support tools are effective for supporting teachers' development of coherent mathematical meanings? How can teachers be supported in noticing and utilizing students' mathematical meanings within lessons designed to facilitate construction of those meanings? Based on our prior discussions of decentering and claim of the symbiotic relationship between a teacher's first-order model and decentering actions, we suggested further exploration of the role of teacher engagement in decentering actions as an approach for fostering ongoing advances in a teacher's first-order model of knowing, learning, and teaching an idea.

In our research, we have found that supporting teachers' development of more coherent meanings for the ideas they are teaching is a critical first step toward teacher's valuing conceptual learning in their students; however, without the support of conceptually scaffolded curricular materials and aligned professional development, the teacher's *ways of thinking about teaching ideas* and their actual teaching practices, including their inquiry into students' thinking, might not advance (Carlson et al., 2010a, b). One benefit of such curriculum materials is how their design supports and encourages students to put their thinking on display, making it more likely that the teacher will begin to notice discrepancies between how students are thinking and how the teacher is thinking. As a teacher begins to puzzle about and make sense of students' thinking, they might also begin to notice commonalities across multiple students' thinking and weaknesses in their own thinking. Both are important in supporting teachers' decentering activity. We provide an example of this in the next section.

Advances in a Teacher's Ways of Thinking About Teaching an Idea

In this section we illustrate how a teacher's interaction with a student in the context of discussing a conceptually focused task in the Pathways materials (Carlson et al., 2021) created a perturbation for the teacher. We further discuss how the teacher's focus on expressing the quantitative meaning for the algebraic representation of the AROC led to advances in the teacher's personal meaning for the idea of average rate of change (AROC), mathematical meaning for teaching the idea of AROC, and the teacher's *ways of thinking about teaching* AROC.[18]

Shifts in a Teacher's Meaning for the Idea of Average Rate of Change (AROC) and Her Ways of Thinking About Teaching the Idea of AROC

Musgrave and Carlson (2017) have reported wide variations in precalculus instructors' conceptions of average rate of change. In their study of 19 mathematics graduate students' mathematical meanings for average rate of change, most expressed a procedural approach that involved their computing $\frac{f(b)-f(a)}{b-a}$ and explaining the result as representing "rise over run" or the slope of a line between two points. After a professional development intervention, the researchers detected shifts in the graduate students' expressed meanings with most associating the idea of average rate of change with a constant rate of change. However, only eight (of the 19) included a description of what the constant rate of change represented, e.g., the constant rate of change over a specified interval of a function's domain that results in the same change in the dependent quantity as the non-linear relationship defined by the function. These findings that even graduate students in mathematics have weak meanings for this foundational idea bring to light the need for professional development for advancing a teacher's meanings for ideas that are the focus of the teacher's instruction (Musgrave & Carlson, 2017).

As a follow-up to analyzing and reporting data on the instructors' conceptions of the idea of AROC, the researchers video-recorded two of the instructors who had

[18] The teacher was using conceptually scaffolded instructional materials that are based in research on learning the course's ideas. She had attended a 3-day workshop prior to teaching with the materials and weekly content-focused professional development for a full year while using the materials.

expressed weak meanings for AROC when teaching this idea in their Pathways Precalculus class. Their analyses revealed one instructor conveying a similar procedural explanation to her students by demonstrating the steps for calculating an average rate of change while describing the result of the calculation as a "constant rate of change from 2 to 7." The results from analyzing a second instructor's interaction with a student revealed how an instructor's decentering actions led to advances in both the instructor's mathematical meaning of the idea of AROC and her *ways of thinking about teaching* the idea of AROC. In what follows, we describe what unfolded during the class interaction between this instructor and a student.

Decentering Actions Lead to Advances in a Teacher's MMT for Teaching AROC

In this example, the instructor began her lesson by showing her students how to determine the value of $\frac{f(7)-f(2)}{7-2}$ for a quadratic function f. After completing the calculation, a student asked the instructor what the answer $\frac{f(7)-f(2)}{7-2}$ meant. (The instructor later indicated that at that moment she was unsure how to answer the student's question. She further conveyed that she had difficulty saying what an AROC represented in prior semesters, even though the curriculum included a conceptual definition and explanation.) Since the curriculum had a consistent focus on representing the meaning of symbols,[19] the instructor decided to follow up by exploring how the student was visualizing the symbols in the AROC definition. She asked the student to draw the graph of f on a whiteboard at the front of the room and then asked the student to locate the points $(2, f(2))$ and $(7, f(7))$ on f's graph and illustrate "where you see $f(7) - f(2)$ and $7 - 2$ on the graph." The student constructed vertical and horizontal lines to illustrate these changes on the graph. With some prompting the student was able to describe the horizontal line's length as representing a change in x from 2 to 7 and $f(7) - f(2)$ as "how much f changed when x increased from 2 to 7." The instructor then asked the student to explain how she saw the expression $\frac{f(7)-f(2)}{7-2}$ on the graph. The student responded by asking, "Isn't this just the change in y over the change in x, which is the slope of a straight line?" The instructor responded by asking the student, "What straight line?" After the student constructed a line segment connecting the points $(2, f(2))$ and $(7, f(7))$, the instructor prompted the student to explain what this slope of the line segment represented. After a long pause, the instructor asked the more specific question, "What constant rate of change does the slope of this line represent?"

[19] For more detail on this focus, see O'Bryan (2018), O'Bryan and Carlson (2016), and Carlson et al. (2003).

In the instructor's description during a follow-up interview, it was during this long pause that she first conceptualized the calculation for the AROC as representing a constant rate of change over a specified interval of a function's domain (e.g., from 2 to 7) that results in the same gain in the function's dependent quantity compared to what was gained by the function on that interval. She stated that prior to this moment she was only trying to recite the textbook description of what an AROC represents and was struggling to repeat the wording. This comment suggests that she might also have been perturbed by her prior inability to provide an adequate explanation and thus had brought the idea to mind at times other than when she was attending professional development or when she was teaching.

After the long pause, the student responded by describing the result of the calculation as the constant rate of change from 2 to 7. The instructor asked the student for clarification by demonstrating on the graph the change in the dependent quantity on that interval of the domain. We conjecture that the shift in the instructor's understanding during the prior interaction is what motivated her to ask this question. The student responded by pointing to the vertical segment connecting $(7, f(2))$ and $(7, f(7))$ to represent the value of $f(7) - f(2)$ and answered by saying the slope of the line segment connecting the two points is the constant rate of change on the interval from 2 to 7 that gives the correct change in the function value on that interval.

We claim that this exchange is an example of a teacher advancing both her MMT for AROC and her *way of thinking about teaching the idea of average rate of change* in the moment of teaching. We claim that this shift occurred when she was attempting to understand her student's thinking. Because of the perturbation, the instructor experienced when she was unable to provide a response to the student's initial question and her subsequent actions to probe the student's meaning for the symbols in the AROC expression, the interaction appeared to advance the instructor's scheme for AROC. In the moment of interacting with the student, the instructor conceptualized how the AROC calculation determined the constant rate of change that achieved the same net change as the function value on the interval of the domain over which she was determining the function's AROC.

The data suggest that the instructor's acts to decenter led to her reorganizing her scheme for AROC. We conjecture that the perturbations she experienced when attempting to explain the idea in prior semesters (as noted above) contributed to her experiencing an accommodation to her scheme that resulted in her advancing her meaning for AROC. In the follow-up interview with this instructor, she revealed that she planned to use a graph and the same line of questioning about what the symbols in the AROC definition represent when teaching the idea of AROC in the future. This comment suggests that she also experienced a shift in her *ways of thinking about teaching AROC*. If the instructor engages in decentering behaviors in future interactions with students when teaching the ideas of AROC, it is likely that both her image of *students' ways of learning the idea of AROC* and her *ways of thinking about teaching the idea of AROC* will continue to advance. In contrast, if the instructor's actions are exclusively focused on conveying her more conceptually oriented *way of thinking*, with no consideration given to her lesson scaffolding, her students' thinking, or how she is being interpreted when teaching, her image of students' thinking and her image of teaching the idea of AROC will not be likely to advance further.

Characterizing Teacher Decentering

Our claim of the symbiotic relationship between a teacher's MMT and her decentering actions extends to our research activity in that we have found that investigations focused on understanding the nature and development of a teacher's MMT have led to advances in our understanding of mechanisms of decentering and vice versa. Thus, our research has sometimes focused on understanding and characterizing decentering while not focusing on a teacher's MMT or how it develops. In this section, we provide an overview of our investigations that studied teacher decentering and share our most recent framework for characterizing teacher decentering behaviors that we observed in teachers during these studies (Baş-Ader & Carlson, 2022; Carlson & Baş-Ader, 2019; Carlson et al., 2015; Rocha & Carlson, 2020; Teuscher et al., 1982).

Decentering as a Construct for Studying Teachers

We first used decentering (Carlson et al., 2007) to study teachers and teaching in the context of a research and development project aimed at supporting secondary precalculus-level teachers in shifting their instruction to be more conceptually oriented (as defined by Thompson et al., 1994), coherent, and supportive in advancing students' mathematical thinking of key ideas of precalculus that were documented to be foundational for learning calculus (e.g., Carlson, 1998; Carlson et al., 2007, 2002, 2001). Our observations of select teachers' classrooms prior to beginning the intervention revealed widespread practices of procedurally oriented classes with procedural explanations, similar to what had been reported in the Trends in International Mathematics and Science Study (TIMSS) (Stigler & Hiebert, 2008).

The initial intervention included weekly 3-hour professional development meetings[20] engaging precalculus teachers in tasks to support their constructing coherent meanings of the key ideas of precalculus (Carlson et al., 2007).[21] Teachers also met twice per week in professional learning communities (PLCs) of four to seven teachers from the same school. Each PLC had a designated facilitator (one of the

[20] The teachers received 3 hours of mathematics education graduate credit and a stipend for attending.

[21] The targeted meanings were based on our current conception of the body of research on knowing and learning key ideas of precalculus (Breidenbach et al., 1992; Carlson, 1998; Carlson et al., 2001, 2002, 2015; Carlson, Oehrtman & Engelke 2010; Carlson, Oehrtmen & Teuscher, 2010; Engelke et al., 2005; Frank, 2017; Kuper, 2020; Kuper & Carlson, 2020; Carlson et al., 2021; Monk, 2010; Moore, 2012; Moore & Carlson, 2012; O'Bryan, 2018; Strom, 2015; Thompson, 1994a, b, 2000</CitationRef>).

teachers) who led the community in discussing the idea (e.g., rate of change, exponential growth) that was the focus of the prior week's professional development meeting. The facilitators received weekly coaching between meetings from a project leader who had reviewed a video recording of their PLC meeting from the prior week to offer suggestions for improving their facilitation abilities, including the advancement of their colleagues' thinking and meanings, with the coach highlighting interactions that probed another's thinking. For example, they posed questions to promote reflection on unproductive PLC interactions (e.g., "Sharon, is there a reason you don't allow Mary to answer questions?"; "Juan, what do you think David meant when he said…?"; "Dan, are you aware of who is doing most of the talking?"),[22] shared video of interactions in which the facilitator posed questions aimed at revealing how another PLC member was thinking, and prompted the community of PLC facilitators to discuss aspects of the interaction.

The results from coding the videos of the professional development and PLC meetings that were collected over that full semester revealed that the teachers generally did not attempt to communicate their thinking when speaking and did not make efforts to understand their colleagues' thinking (Carlson et al., 2007). In particular, the teachers' explanations made vague references to quantities (e.g., "it goes up"), provided variable definitions that lacked precision (e.g., "d = distance"), and explained graphs as static shapes. Their explanations included little or no explanation of the thinking they used to approach a problem or construct their solutions.

These findings of the widespread procedurally oriented patterns of interaction led to our adaption of the norms for the bi-weekly PLC meetings to include a convention that all PLC members attempt to reference quantities when speaking and explain the thinking that led to their constructions and calculations. To raise teachers' consciousness about their patterns of speaking, thus making their speaking an object of reflection, we termed this expectation for their speaking and explaining as *speaking with meaning* (Clark et al., 2008). We further negotiated more general *Rules of Engagement*[23] that included a goal for PLC members to listen to the utterances of their colleagues while attempting to make sense of the meanings they were conveying, and in instances when they were unclear about what was being expressed, we conveyed an expectation that they pose a question that might clarify what they thought was unclear.

The PLC facilitators were charged with taking the lead role in modeling and supporting their colleagues in speaking with meaning and attempting to understand the meanings being expressed by members of their PLC. The study revealed widespread

[22] Over a full year, each of four PLC twice-weekly meetings was video-recorded, with weekly recording coded by two researchers and then shared weekly with research team (including the facilitator coach).

[23] There were four rules of engagement that emerged as explicit conventions for promoting the sociomathematical norms of "speaking with meaning" and attempting to make sense of another's thinking that were adopted by the teachers as explicit conventions for communication within their classrooms. We created posters that listed these conventions and the teachers elected to post them in their classrooms.

advances in the PLC members *speaking with meaning* and substantial advances in three of the PLC facilitators' interest and efforts to make sense of other PLC members' thinking. The various manifestations of decentering we observed are articulated in five decentering moves we identified when analyzing their conversations (Carlson et al., 2007). Their decentering actions ranged from the facilitator showing no interest in understanding the thinking or perspective of another PLC member to the facilitator building a model of another's thinking and through interaction, building a model of how she (the facilitator) might be interpreted, and then using both models to determine her next action.

The analysis of this data further revealed that a facilitator's ability to pose questions that resulted in advances in another's thinking was typically linked to the facilitator's understanding of the idea under discussion. This awareness provided our first insight into the strong link between a teacher's capacity to respond productively to another's thinking and the teacher's personal meaning for the idea under discussion. When PLC facilitators had a weak personal meaning for an idea under discussion, they were not equipped to assess the efficacy of another's thinking or intervene productively to advance another's thinking; nor were they able to provide conceptually oriented explanations. In the authors' reporting (Carlson et al., 2015), they further noted that general inquiries made by a facilitator such as a request for a PLC member to revoice or extend another PLC member's explanations sometimes led to two PLC participants (both who had relatively strong meanings for the idea being discussed) constructing models of each other's thinking. In contrast, the facilitator who showed no interest in understanding her PLC members' thinking "exhibited extreme discomfort with the content" (Carlson et al., 2015, p. 847) and, when probed by the PLC facilitator coach to explain her thinking, consistently responded by explaining how she obtained her answer. On an occasion when the coach encouraged her to ask her colleagues to explain their thinking, she responded by questioning why one would be interested in understanding the thinking of another. She clearly had not adopted the goals of the project. The project leaders later learned that this PLC facilitator was assigned to be the facilitator because she was the most senior member of the precalculus instructional team at that school.[24]

After a year of participating in the intervention, some teachers expressed that their participation in the professional development and PLC meetings had impacted the quality of their interactions with students, in particular their attempts to make sense of their students' explanations and their ability to pose follow-up questions to reveal students' thinking. During the subsequent academic year, the researchers video-recorded and studied the classroom instruction of four PLC members who had demonstrated efforts to make sense of their colleagues' thinking during the PLCs. The researchers' analysis of this data (Teuscher et al., 1982) illustrated how the teacher's decentering actions (or lack of these actions) assisted (or constrained)

[24] We report this result to highlight the importance of initial interest and seeing value in understanding another's thinking as a desirable attribute of a facilitator of other teachers' decentering actions. Going forward in our project, we requested school administrators to identify facilitators who valued the goals of the project.

the teacher in making productive in-the-moment instructional decisions. These studies established that advances in a teacher's ability to build a model of a student's thinking corresponded with advances in a teacher's meanings and that advances toward more conceptually oriented explanations could be fostered by teachers making their speaking and listening an explicit focus of personal and collective reflection. In what follows we describe the products of a third study (Baş-Ader & Carlson, 2022) that built on these findings and produced a more comprehensive *Teacher Decentering Framework*, including specific teacher actions that were associated with five different levels of decentering actions that we observed.

A Decentering Framework for Studying Teaching

Our subsequent professional development work in the Pathways Project continued our focus on promoting reflective discourse (e.g., speaking with meaning and making sense of another's thinking) as a mechanism for advancing a teacher's personal mathematical meanings and teaching practices. In this section, we provide an overview of a framework (Baş-Ader & Carlson, 2022) that resulted from analyzing video data from three 40-student precalculus classes taught by mathematics graduate students using an early version of the Pathways Precalculus research-based conceptually scaffolded curriculum (Carlson et al., 2021) in a university setting. The three subjects had taught with these materials for at least 1 year while attending weekly 90-minute Pathways conceptually focused professional development meetings. In these meetings, the professional development leader held the instructors accountable for attempting to speak with meaning (Clark et al., 2008) and attempting to understand their colleagues' explanations.[25] The choice to collect the data in a context of instructors attending content-focused professional development and using a conceptually focused curriculum provided the possibility for the teacher-student interactions to be coherent and meaningful.

In presenting the data, Baş-Ader and Carlson (2022) described the learning goals for the idea central to the interaction prior to presenting their analysis of the data; doing so enabled the reader to assess the decentering moves they identified relative to their effectiveness in advancing the conversation toward a more productive mathematical way of thinking. Level 0 of the framework (Table 9.2) is characterized as showing no interest in the student's thinking, although the instructor does prompt the student to share her answer. The instructor might show concern for the student getting the correct answer but shows no interest in the thinking the student engaged in to determine her answer. In light of our review of Piaget's construct of intellectual egocentrism (the child unconsciously remains centered on his own actions and own point of view), we extend this construct to

[25] In particular, the professional development leader posed questions to the speaker to call attention to vague descriptions (e.g., "What distance?" or "What does it mean for it to be decreasing?") and regularly asked participants to describe how they had interpreted what a colleague had just described.

Table 9.2 A framework that characterizes a continuum of an instructor's decentering actions (Baş-Ader & Carlson, 2022, pp. 106–107)

Mental actions	Decentering levels	Description of the behaviors
Does not reflect on aspects of the interaction that are contributed by the student Assumes that the student's thinking is identical to her own Uses first-order model during interaction to organize and guide her actions	*Level 0: no interest* Shows no interest in the student's thinking, but shows interest in the student's answer. Makes no attempt to make sense of the student's thinking, but takes actions to get the student to say the correct answer	Asks question to elicit the student's answer Listens to the student's answer Does not pose question aimed at revealing the student's thinking: Poses question that focuses on a procedure [e.g., *What do you do next?*] Poses question to have the student echo key phrase or complete steps to get the correct answer
Does not reflect on aspects of the interaction that are contributed by the student Recognizes that the student's thinking is different than her own but does not intentionally analyze the student's mental actions Uses first-order model during interaction to organize and guide her actions	*Level 1: Interest* Shows interest in the student's thinking, but makes no attempt to make sense of the student's thinking. Attempts to move the student to her way of thinking without trying to understand or build on the expressed thinking and/or perspective of the student	Poses question to reveal the student's thinking [e.g., *Why*? *What does that mean*? *What does that term represent?*], but does not attempt to understand the student's thinking Guides the student toward her way of thinking: Poses question for the purpose of getting the student to adopt her way of thinking or has the student echo key phrase Gives explanation aimed at getting the student to adopt her way of thinking Evaluates how the student's response compares to her way of thinking or approach
Reflects on aspects of the interaction that are contributed by the student Recognizes that the student's thinking is different than her own and attempts to analyze the student's mental actions Builds the second-order model of the student's thinking Uses first-order model during interaction to organize and guide her actions	*Level 2: make sense* Makes an effort to make sense of the student's thinking and perspective but does not use this knowledge in communication	Poses question to reveal the student's thinking and attempts to adopt the student's perspective Does not use the student's thinking for the purpose of advancing the student's current way of thinking: Does not pose question aimed at prompting the student to reflect on or advance her current way of thinking Redirects the conversation toward her way of thinking or approach

(continued)

Table 9.2 (continued)

Mental actions	Decentering levels	Description of the behaviors
Reflects on aspects of the interaction that are contributed by the student Assumes that the student has unique ways of thinking Builds and continually adjusts the second-order model of the student's thinking Operates from the second-order model of the student's thinking to make decisions on how to act	*Level 3: use* Makes sense of the student's thinking and perspective and makes general moves to use the student's thinking when interacting with the student	Prompts the student to explain the student's way of thinking: Poses question to gain insights into the student's way of thinking Leverages the student's expressed thinking to advance the student's understanding of key ideas in the lesson: Poses question or gives explanation that is attentive to the student's thinking and aimed at advancing the student's understanding of the key idea Poses question or gives explanation to support the student in making connections between different viable ways of thinking of the key idea
Reflects on aspects of the interaction that are contributed by the student Assumes that the student has unique ways of thinking Builds and continually adjusts the second-order model of both the student's thinking and how the student might be interpreting her utterance Operates from the second-order model of both the student's thinking and how the student might be interpreting her utterance	*Level 4: use and adjust* Constructs an image of the student's thinking and perspective and then adjusts her action to take into account both the student's thinking and how the student might be interpreting her utterance	Poses question to reveal and understand the student's thinking Follows up on the student's response in order to perturb the student in a way that extends the student's current way of thinking: Poses question informed by the student's current way of thinking and meaning of the key idea Gives explanation informed by the student's current way of thinking and meaning of the key idea

classify an instructor's actions as exhibiting *intellectually egocentric* behavior when they only exhibit Level 0 decentering actions.

To illustrate this behavior and how we extracted information from our analysis, we present one of the excerpts from Baş-Ader and Carlson (2022) detailing an interaction between an instructor and student in the context of talking about a task that prompted students to describe the relative size of two quantities. As background, the authors conveyed that the students were given two lines of different lengths (60 and 25 inches), and students were expected to perform a multiplicative comparison of the segments' lengths. Baş-Ader and Carlson explain that the curriculum prompted students to determine the length of one segment using the second segment's length as the unit of measurement. The reader is also alerted to the fact that both segments are given in a common unit, so it is possible for the student to determine the relative size of quantity A in units of quantity B by performing the calculation (measure of quantity A in the common unit)/(measure of quantity B in the common unit). Baş-Ader and Carlson explain that the quotient, k = (measure of quantity A in the common unit)/(measure of quantity B in the common unit), conveys that quantity A is k times as large as quantity B. They further note that this pre-algebra-level task is included in Pathways Precalculus curriculum because the idea of relative size is foundational for understanding and representing exponential growth and other ideas that appear in a precalculus course (e.g., rational functions, radian measure).

Excerpt 1, taken from Baş-Ader and Carlson (2022, p. 108), highlights the main point that the instructor's (Karen's) focus was on getting the students to say the correct answer. She told them what to calculate ("quantity A over quantity B") and then prompted them to restate the value of the quotient using the numbers they were given for the two lengths.

Excerpt 1

1 Karen: If we want to see how big A is in terms of B…Remember, so quantity A over quantity B, what is that?
2 Student 1: Twenty-five over sixty?
3 Karen: Twenty-five over sixty, good. So, if we write twenty-five over sixty, what does that represent, again?
4 Student 2: A over B.
5 Karen: A over B, what does that mean like in English sentence?
6 Student 2: How big A is compared to B.
7 Karen: Exactly, how big A is compared to B. So, twenty-five over sixty, we would say A is equal to twenty-five over sixty whatever B's length is, okay? If we write out twenty-five over sixty in a calculator we will get .416 about. So we could say A is .416 times B.

The instructor listened to students' answers, but at no point during the exchange did she ask students to share their thinking. Baş-Ader and Carlson (2022) concluded by pointing out that the instructor appeared satisfied that a student had finally repeated the words she had previously expressed. We could classify this instructor's interaction as intellectually egocentric behavior since there were no instances of her exhibiting behaviors to suggest that she was interested in her students' thinking.

Table 9.2 displays the researchers' characterization of the decentering levels (column 2), the associated mental actions (column 1), and the instructor behaviors identified when coding the videos (column 3). Each row of the table illustrates how these three categories are linked. As an example, the entries in row 2 characterize Level 1 decentering as expressing interest in a student's thinking by asking the student to say what a statement, term, etc. represents. The instructor recognizes that the student's thinking differs from her own, but she takes no action to understand the student's thinking. Instead, the instructor presents her approach and/or thinking and makes moves to get the student to adopt her way of thinking.

An instructor is classified as exhibiting Level 2 decentering actions if she poses questions to reveal a student's thinking and then shows evidence of understanding the student's thinking (builds a second-order model) but does not use the student's thinking during interaction. Even though the instructor has constructed an understanding of the student's perspective/thinking, her subsequent questions, drawings, and explanations do not build from or leverage the thinking the student displayed. Similar to Level 1, the instructor redirects the conversation to the instructor's way of thinking. In our video analyses of teacher-student interactions, our efforts to characterize teacher-student interactions necessitated our including constructs to classify mental actions that would not be considered decentering since the teacher did not attempt to build a model of a student's thinking. These framework levels allow for the classification of initial shifts toward the students' perspectives, such as showing interest in a student's answers (Level 0), showing interest in a student's thinking, and then taking action to move the student to the teacher's way of thinking (Level 1).[26]

A teacher who is classified as exhibiting Level 2 mental actions is inquiring into a student's thinking and engages in mental actions to construct a second-order model of the student's thinking. However, she does not use her image of how the student is thinking to inform future actions. In contrast, both Level 3 and Level 4 classifications characterize a teacher using her second-order model of a student's thinking to inform her instructional actions. In the highest level (Level 4) of the Decentering Framework, the teacher reflects not only on the student's thinking but also on her own thinking in relation to the student's thinking. This mental action aligns with Liang's (2003) theoretical description of "scheme activity" that she associates with what she calls Type 3 decentering, involving the teacher consciously treating her schemes as objects of reflection. If a teacher then reflects on her images of students' *ways of thinking about an idea* (epistemic students), we say the teacher is developing *ways of thinking about teaching an idea*. As such, we consider Level 4 of our framework to involve not only teachers' reflecting (decentering) activity but also their reflections on prior reflecting (decentering) activity. We also note that a teacher's actions to reflect on her decentering actions and their effectiveness in advancing a student's thinking is constructing *ways of thinking about teaching an idea*.

Our experiences of conducting professional development with hundreds of secondary teachers and college instructors (and a retrospective review of our classroom video data) support that while most secondary and university teachers claim to be interested in student thinking, very few in either population initially exhibited even Level 1 decentering behavior (e.g., Carlson et al., 2015); their focus was primarily on supporting students in learning methods for obtaining answers. Moreover, during the first semester of either a secondary teacher or university instructor using the Pathways cognitively scaffolded instructional materials (Carlson et al., 2021) and

[26] For a detailed description of the mathematical contexts, learning goals, tasks, and researchers' methods for constructing this decentering framework, see Baş-Ader and Carlson (2022).

participating in our weekly professional development, we observed only modest shifts in the teachers' interest in their student's thinking. During the first semester of their teaching with the Pathways Precalculus materials, the instructors appeared unable (or unwilling) to consider their students' thinking while consumed by making sense of the course's ideas.[27] After a teacher's personal meanings for the course's ideas became more coherent (Musgrave & Carlson, 2017), professional development leaders were more successful in fostering productive discussions about methods for revealing students' thinking (Rocha, 2021).

Our approaches for fostering higher-level decentering actions include (i) viewing and discussing classroom videos of a teacher's decentering actions, (ii) asking the instructors to share instances from their teaching when they made sense of and leveraged a student's thinking,[28] (iii) designing assessment items that reveal students' thinking, etc. It is not our goal, nor is it realistic, for most teachers to consistently engage in high-level decentering actions when teaching a course with fixed pacing. However, increasing a teacher's interest in students' thinking and fostering growth in their listening and questioning toward understanding students' thinking can create the expectation that students' thinking is relevant. As the teacher operates with an orientation to understand and affect students' thinking, occasional efforts to decenter, particularly in instances when the teacher is not satisfied with her efforts to affect her students' learning, can over time lead to a large cumulative effect on a teacher's MMT and *ways of thinking about teaching* specific ideas.

In one case in which we collected interview and classroom video data from a few teachers over multiple (as many as three) years, we observed a teacher's lectures becoming more coherent and anticipatory of students' thinking (e.g., making drawings and providing explanations focused on engendering productive ways of thinking). We have observed that a teacher can progress to demonstrate higher levels of decentering actions when supported in doing so—e.g., using cognitively scaffolded curriculum focused on advancing and revealing students' thinking, attending weekly PD aimed at supporting teacher reflection on MMT for ideas, how they are learned, and how they can be taught. In this sense, we claim that advancing toward higher-level decentering actions is developmental and emerges in concert with advances in a teacher's meanings for the idea under discussion and opportunities for teachers to reflect on their student's thinking and the effectiveness of their instructional choices. As teachers repeatedly engage in decentering in the context of teaching a specific idea, their *ways of thinking about learning that idea* and *ways of thinking about teaching that idea* can continue to advance. As we continue to inquire into the symbiotic relationship between a teacher's mathematical meanings and decentering actions, we conjecture that studying

[27] Recall that our assessment of graduate students' meaning for precalculus ideas (Musgrave & Carlson, 2017; Carlson & Baş-Ader, 2019) revealed that even graduate students in mathematics have weak meanings of precalculus ideas (average rate of change).

[28] During one semester the professional development leader began each weekly meetings by asking each (of 9 = nine) instructors to share instances when they made sense of how a student was thinking and then made an instructional move that leveraged the model they had built.

multiple teachers in the context of their teaching a specific idea will expand our understanding of this relationship while contributing to our understanding of how *ways of thinking about teaching that idea* develop.

Concluding Remarks

We have illustrated the important role that decentering plays when modeling a student's thinking during both research and teaching and highlight the symbiotic relationship that exists between a teacher's and researcher's mathematical meanings for an idea and their decentering actions. In particular, we presented examples of how a teacher's/researcher's first-order model of the mathematical ideas impacts what they are positioned to notice when decentering, how reflecting on the results of decentering and performing a conceptual analysis can produce second-order models of students' thinking, and how these models might inform the teacher's/researcher's first-order models. We further highlighted how iterating this cycle (Fig. 9.3) can help teachers/researchers construct stable images of epistemic students. Our examples illustrate how repeated efforts to decenter when interacting with students (during teaching or conducting research studies) help support iterative refinements in an individual's meanings for mathematical ideas and images of others' thinking and learning an idea. As further illustration of the nature of decentering, we discussed the Baş-Ader and Carlson's (2022) Teacher Decentering Framework (Table 9.2) that provides a fine-grained characterization of the mental processes exhibited by teachers as they move from an egocentric perspective to building and leveraging their second-order model of a student's thinking during interactions.

It is worth noting that the examples provided in this chapter are from research studies that focus on quantitative reasoning[29] and covariational reasoning[30] as foundational ways of thinking that can provide coherence to students' mathematical experiences. Students who develop the disposition to reason quantitatively (i.e., identify quantities in a situation, and conceptualize how those quantities' values are related and vary in tandem) are better positioned to construct algebraic and graphical representations of mathematical relationships that are personally meaningful to the one constructing them (see Carlson et al., 2003). O'Bryan and Carlson (2016) also demonstrated that teachers who value and are committed to supporting their students in reasoning quantitatively may be more attentive to and interested in the meanings students express. They may also be more likely to intentionally create opportunities to reveal how their students are thinking and consider what meanings

[29] Quantitative reasoning (Thompson, 1988, 1993, 1994a, 1994b, 2012, 2013) describes a way of thinking whereby an individual identifies quantities in a situation (measurable attributes of an object), conceptualizes a measurement scheme for producing values representing the quantity's size, and conceptualizes a structure in how various quantities in a situation are related and interdependent.

[30] Covariational reasoning (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2014) involves the identification of pairs of quantities in a situation that vary and conceptualizing the constraints on how those quantities change in tandem.

motivated a student's construction of specific mathematical representations of quantitative relationships.

The examples we presented also highlight the role of cognitive research on task development (recall the constant rate of change, logarithm, and AROC tasks) and their role in perturbing and revealing students' thinking and advancing a teacher's mathematical meaning for teaching an idea. Since most teachers do not have the time or resources to write curriculum or study student learning, researchers must lead the way in devising and studying processes for supporting shifts in teachers' mathematical meanings, ways of thinking about teaching specific ideas, and their habitually engaging in decentering (when teaching and when planning and reflecting on their teaching). In our research projects (Musgrave & Carlson, 2017; O'Bryan & Carlson, 2016), we found that researcher-developed professional development and aligned research-developed curricula can facilitate teacher decentering and advances in a teacher's mathematical meanings for teaching ideas. We offer our framework (Fig. 9.3) for advancing teachers' *ways of thinking about teaching an idea* as a general theory for others to leverage and refine in the context of devising and studying mechanisms and tools to support shifts in teachers – shifts that lead to teachers enacting and continually advancing their teaching practices to include a focus on understanding and advancing students' thinking and understanding specific ideas.

As a first line of inquiry for future research, we recommend devising and studying professional development experiences for instructors aimed at supporting advances in their (i) mathematical meanings for teaching an idea (first-order knowledge), (ii) decentering actions when planning a lesson and implementing the lesson, and (iii) subsequent meta-reflection on the effectiveness and products of their decentering actions and instructional choices when teaching the lesson. We hypothesize that professional development with this focus will lead to incremental advances in a teacher's instruction to be more supportive of and responsive to students' thinking and learning. We conclude by providing a short list of questions for researchers to consider:

- As teachers' (or researchers') images of epistemic students become more refined, how does this change what they notice, how they plan lessons, and how they respond to students during interactions?
- What factors foster advancements in teachers' interest in and spontaneously inquiring into a student's thinking during teaching and after teaching an idea? What is the interaction between a teacher's/researchers' decentering actions and her mathematical meanings for teaching that idea?
- What factors foster advances in a teacher's reflection on her prior reflections (meta-reflection) when planning a lesson and when considering the effectiveness of a lesson?
- What factors (profession development experiences/tools) support advances in an instructor's reflection on her *ways of thinking about teaching an* idea, including her pedagogical choices (e.g., if, when, and how to use an applet, choice of tasks for students to complete, interactions/questions for students, what to say and write when leading a discussion)?

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