Journal of Mathematical Behavior 31 (2012) 48–59

Contents lists available at SciVerse ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Students' images of problem contexts when solving applied problems

Kevin C. Moore[a,*], Marilyn P. Carlson[b]

[a] Department of Mathematics and Science Education, University of Georgia, 105G Aderhold Hall, Athens, GA 30602-7124, United States
[b] School of Mathematical & Statistical Sciences, Arizona State University, P.O. 87804, Tempe, AZ 85287-1804, United States

A R T I C L E I N F O

Available online 7 October 2011

Keywords:
Quantitative reasoning
Student reasoning
Precalculus
Word problems
Problem solving

A B S T R A C T

This article reports findings from an investigation of precalculus students' approaches to solving novel problems. We characterize the images that students constructed during their solution attempts and describe the degree to which they were successful in imagining how the quantities in a problem's context change together. Our analyses revealed that students who mentally constructed a robust structure of the related quantities were able to produce meaningful and correct solutions. In contrast, students who provided incorrect solutions consistently constructed an image of the problem's context that was misaligned with the intent of the problem. We also observed that students who caught errors in their solutions did so by refining their image of how the quantities in a problem's context are related. These findings suggest that it is critical that students first engage in mental activity to visualize a situation and construct relevant quantitative relationships prior to determining formulas or graphs.

1. Introduction

It is well documented that even high performing precalculus and calculus students have difficulty solving problems in which they are asked to reason about relationships between quantities that have continuously changing values (e.g., Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Monk & Nemirovsky, 1994; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994b). These studies suggest that secondary mathematics curriculum and instruction needs an increased focus on developing students' abilities to construct and formalize (e.g., determining formulas and graphs) relationships between quantities that vary in tandem. Contributing to the call for improving secondary instruction, the National Council of Teachers of Mathematics (NCTM) *Standards* advocated that students should acquire understandings and reasoning abilities that lead to their using mathematical formulas and graphs to represent how quantities in novel problems are related and change together (2000). The NCTM *Standards* also called problem solving an "integral part of all mathematics learning" (2000, p. 51) and suggested that a majority of high school mathematics topics can be introduced through novel problems involving applied contexts, thus setting the expectation that students gain confidence and proficiency in reasoning about relationships between quantities.

In an attempt to gain insights into the difficulties students encounter when modeling relationships between two dynamically varying quantities, we explored undergraduate precalculus students' reasoning as they attempted to solve novel

* Corresponding author. Tel.: +1 706 542 3211; fax: +1 706 542 4511.
E-mail addresses: kvcmoore@uga.edu (K.C. Moore), marilyn.carlson@asu.edu (M.P. Carlson).

0732-3123/$ – see front matter
doi:10.1016/j.jmathb.2011.09.001

applied problems.[1] The goals of this study were to characterize students' images of problem contexts and determine how these images influenced the students' ability to create meaningful and correct formulas and graphs.[2]

2. Background

Several studies have provided insights into the various ways that a student reasons about relationships between the values of two quantities (e.g., Carlson, 1998; Carlson et al., 2002; Harel & Dubinsky, 1992; Saldanha & Thompson, 1998). In one study, covariational reasoning, described as the "cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354), was found to be a critical reasoning ability for thinking about and representing the quantitative relationships described in a problem statement. In another study, Dubinsky and Harel (1992) reported that a *process* conception of function, described as "a dynamic transformation of quantities according to some repeatable means" (p. 85), is critical for performing and applying basic operations such as function composition and determining a function's inverse. Understanding and applying such operations is commonly needed when modeling relationships between quantities in an applied context (Carlson, 1998; Kaput, 1992).

The aforementioned studies provide valuable ways to classify students' reasoning about two quantities, but they do not describe the quantities that the students under study initially conceptualized when processing the words of a problem statement. They also do not characterize how a student's understanding of a problem's context evolves during her solution attempt. A study of mathematicians' problem solving approaches revealed that the mental actions performed when attempting to understand a problem's context are critical because they provide the basis for future constructions and reflection during the problem solving process (Carlson & Bloom, 2005). This finding highlights the need for future investigations of how an individual's understanding of a problem's context influences her solution to the problem.

Smith and Thompson (2008) and Thompson (1989) have examined the thinking involved in conceptualizing both a situation (e.g., a runner in a race) and the quantities in a situation (e.g., time elapsed or distance ran since beginning the race). They describe these mental actions as quantitative reasoning and suggest that a student who constructs a robust system of quantitative relationships can be supported in creating meaningful formulas to represent how values of specific quantities are related.

According to Smith and Thompson's (2008) theory of quantitative reasoning, a *quantity* is a conceived attribute of an object such that the individual envisions that the attribute admits a measurement process. A quantity is cognitively constructed—the quantities that people know are the quantities they have conceptualized. This statement is consistent with our stance that an individual's experience and knowledge is unique to the individual. For instance, an individual could conceive of a quantity in a way that does or does not include imagining the quantity's measure varying with respect to variations in another quantity's measure.

Thompson (2011) further defines *quantification* to be the process by which one assigns, or imagines assigning, numerical values to measurable attributes. In order for an individual to comprehend a quantity, the individual must have a mental image of an object and attributes of this object that can be measured (e.g., the distance of a car from the starting line, the time since the car left the starting line). The individual must implicitly or explicitly imagine that an object's attribute *has a measure*, whether stated or not and whether known or not (e.g., imagining a car being a number of miles from a starting line). *Values* result from doing or envisioning a measurement process, but specific values (e.g., 0.5 miles, 0.785 miles, 1 mile) are not required to reason about a quantity's measure. Quantification is then the dialectic among conceiving the object and its attribute, conceiving a unit in which the attribute is measured, and conceiving possible processes by which a measure might actually be obtained (Thompson, 2011).

Thompson (1989) defines a *quantitative operation* as the conception of a quantity as being produced by relating two other quantities. For instance, making a multiplicative comparison between a traversed distance and an elapsed time produces an average speed. With these definitions in place, *quantitative reasoning* is the process of analyzing a situation in terms of quantities and relationships among them. As the structure of a student's conceptualization of a situation develops into a network of quantities and relationships among them, we say that the student's quantitative structure is *emergent*. This network of quantitative relationships will support students' explorations of how pairs of quantities change together. The structure allows a student to have the sense that the situation is "staying the same" even though quantities' values in the situation are changing. The structure is invariant; the quantities' values vary.

In order to describe a student's understanding of a problem's context, we use *image of a problem's context* to refer to how a student imagines the situation described in a problem. We also distinguish between the situation as written (e.g., two runners run a 100-m race) and the imagery formed by a student reading that text (e.g., (a) imagining two runners moving their arms and legs furiously, eyeing each other as they approach the finish line, or (b) imagining two distances, each growing in length and each measured in meters, with one clock timing each runner from the beginning).

[1] For the purpose of this work, we consider a *novel applied problem* to be a task such that (a) the solver cannot apply a memorized procedure and (b) correctly solving the problem requires reasoning about quantities that vary in tandem.

[2] Meaningful formulas and graphs are viewed as representations of how two quantities change together, and they reflect reasoning about changing values of these two quantities.

A student's image of a problem's context also entails a quantitative structure that is shaped by how the student imagines the written situation. The imagery and structure might be static or dynamic, and when dynamic it might contain invariant relationships. As an example, consider a problem that describes a ladder leaning against a wall, with a prompt for students to determine a formula to represent the distance of the top of the ladder from the floor in terms of the distance of the bottom of the ladder from the wall as the top of the ladder slides down the wall. A student might hold any of these images:

- A static image of a ladder leaning against a wall with its top a certain distance from the floor and its bottom a certain distance from the wall.
- A dynamic image of the ladder's bottom sliding away from the wall while its top slides toward the floor; each having the distances described above, but the student imagines that those distances' values vary as the ladder slides. However, the student's image might not contain the constraint that the ladder remains the same length no matter its position.
- A dynamic image as described above, but with the explicit awareness that the ladder is rigid and does not change length even though the distances from the wall and floor vary.

The last image, which includes the constraint that the ladder's length does not change, is necessary for the student to correctly determine how the distance of the top of the ladder from the floor is related to the distance of the bottom of the ladder from the wall. A dynamic image (e.g., an image that includes imagining objects moving) also supports the student in imagining variables as taking on different values as the quantities' magnitudes change (e.g., the number of feet of the top of the ladder from the floor decreasing from 12 to zero feet as the ladder's base is pulled away from the base of the wall).

During a solution attempt, a student's image of a problem's context is emergent in that it necessarily changes and evolves as she attempts to make sense of the quantities in a problem and how they change together. As an example, a student might initially construct an image of a situation that lacks a well-developed structure of the quantitative relationships in the situation (e.g., merely imagining a ladder sliding down a wall without explicitly considering the length of the ladder). If the student is active in considering the quantities of the situation and how they are related, the student's image of the problem's context evolves to include quantities and their relationships, constituting a student's quantitative structure of the problem context. It might also be the case that as the student attempts to formalize her image of the context (e.g., through the use of formulas or graphs, or in considering specific pairs of quantities' values), the structure of her conceptualization also changes.

Past research has revealed quantitative reasoning to be central in students developing meaningful and connected mathematical understandings (Castillo-Garsow, 2010; Ellis, 2007; Hackenberg, 2010; Moore, 2010; Saldanha & Thompson, 1998; Thompson, 1993, 1994a). Collectively, these studies suggest that student learning can be fostered by conceiving of situations composed of quantities and then reasoning about how the values of these quantities change together. The findings of these studies highlight the potential benefits of exploring connections between students' solutions to applied problems and students' images of problem contexts.

3. Methods

3.1. Subjects and setting

This study investigated the thinking of nine undergraduate precalculus level students from a large public university in the southwest United States. Student participation in the study was voluntary, with the students receiving monetary compensation for their participation. The age of the nine full-time students ranged from 18 to 25.

The three classrooms from which the students were drawn were part of a design research study[3] where the curriculum and instruction for the course was informed by theory on learning the idea of function (Carlson, 1998) and the processes of engaging in covariational reasoning (Carlson et al., 2002; Saldanha & Thompson, 1998). Select literature on mathematical discourse and problem solving (Carlson & Bloom, 2005; Clark, Moore, & Carlson, 2008) influenced the scaffolding and implementation of in-class activities and homework. The classroom instruction included direct instruction, whole class discussion, and collaborative activity. The authors of the article collaborated in designing the curriculum and served as the instructors of the three classrooms. The curriculum consisted of 7 modules with the design of each module resulting from a conceptual analysis of the mental actions conjectured to be necessary for students to develop conceptual understandings of the module's topics. Specific topics of focus in the 7 modules include rate of change, proportionality, functions, linear functions, exponential functions, logarithmic functions, rational functions, and trigonometric functions.

3.2. Data collection and analysis methods

The students participated in clinical interviews where the interviewer (the first author) asked the students to respond to novel applied problems with protocols designed to reveal information about the students' thinking and their images of

[3] The students' enrollment in the specific classes was voluntary and the students were unaware of the instructional design of the course prior to enrolling.

the problem contexts. The clinical interviews followed the methodology described by Clement's (2000) and Goldin's (2000) principles of structured, task-based interviews (e.g., four stages of free problem-solving, suggesting heuristics minimally, guiding the use of heuristics, and exploratory, metacognitive questions). The interviewer prompted the students to read each problem aloud and explain their thinking when solving each task. This interview approach included open-ended questioning for the purpose of producing insights into the students' mental actions and reasoning processes.

We analyzed the data following an open, axial and selective coding approach (Strauss & Corbin, 1998) that involved examining the interactions between the students and interviewer in an attempt to model and understand the students' thinking. We consider mathematical thinking to be unique to the individual, and our goal was to build a model of the mental images that contributed to the students' solutions. Thompson (2000) has described this approach as performing a conceptual analysis of the data in an effort to accurately portray the thinking that led to the student's observable or verbal products. Specifically, we analyzed the students' actions (verbal, written, and motions) in order to build a model of their images of problem contexts. As an example, we considered the students' use and explanation of diagrams in the context of their solutions to gain insights into their images of the problem contexts.

4. Results

We provide an analysis of student work on two different tasks to characterize their responses and thinking. The presented data illustrates common findings from analyzing the transcripts and video data of all nine subjects. The data is followed by a discussion of the findings from analyzing the students' written and spoken products. Our analyses offer insights into the students' images of the problem contexts and how these images influenced their solutions (e.g., formulas, graphs, and explanations).

4.1. The box problem

> Starting with an 11 in. × 13 in. sheet of paper, a box is formed by cutting equal-sized squares from each corner of the paper and folding the sides up.
> Write a formula that predicts the volume of the box from the length of the side of the cutout.

The box problem asked the students to create a formula that related the length of the side of a square cutout from four corners of a piece of paper to the volume of the box created by folding up the sides of the paper. All students initially approached solving the problem by drawing a diagram or using the given piece of paper. As an example, Travis oriented to the problem by drawing a picture of a piece of paper (Fig. 1) and denoting the square cutouts on this drawing. As he drew and labeled his diagram, he discussed the problem's context (Table 1).

Travis first labeled the dimensions of the paper and denoted the cutout by writing x next to the length of the square cutout (lines 1–3). After identifying what appeared to be his understanding of the problem's goal, "a formula that predicts the volume of the box" (lines 3–4), Travis stated the general formula for the volume of a box (lines 4–5). When attempting to describe this formula in terms of the box, he accurately stated how the height of the box and the cutout length are related. However, he appeared to conceptualize the fixed length and width of the paper as the length and width of the box (lines 5–8). Travis did not correctly refer (through motions or speech) to the length and width of the box, nor did he clearly

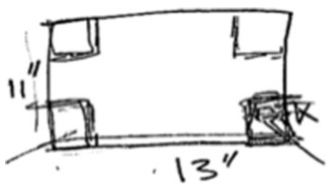

Fig. 1. Travis's drawing of the box.

Table 1
Travis's first description of the box.

1	Travis:	Uh, I'm drawing the corners that are going to be cutout so they are equal
2		sides. So, I will label one of the sides of the cutouts as x, and, uh, it's 11 by
3		13, so I'll label that (*labeling the length and width of the paper*). Um (*pause*) a
4		formula that predicts the volume of the box. The volume of the box is length
5		times width times height (*writing*). Um (*pause*) so we would take the length,
6		which is 13, times 11, which is the width, and then the height is x, whatever
7		the side of the length of the cutout is (*writing* $V = 13 \cdot 11 \cdot x$). And. uh, that
8		will be the volume.

identify the process for making a box (e.g., removing the cutouts and folding up the sides of the box). His response suggests that he imagined a fixed length and width of the box; these dimensions did not depend on or vary with the length of the side of the square cutout. His image of the box was further revealed as the interviewer asked him to explain his solution (Table 2).

During this interaction, Travis continued to identify the height of the box as having a varying measure that was equal to the length of the side of the cutout (lines 2–3 and 9–12). He also imagined folding up the sides of the paper to form the height of the box, revealing that he held an image of a box in mind (lines 9–12). Yet, Travis did not appear to conceive of the length and width of the box as distinct from the dimensions of the original sheet of paper (lines 5–6 and 21–23), nor did Travis mention a relationship between the length and width of the box and the length of the cutout. This further supports our claim that Travis's image of the situation consisted of a box with dimensions not dependent on the length of the cutout, even though he imagined the height of the box varying directly with the length of the side of the cutout. It is noteworthy that although Travis's formula ($V = 13 \cdot 11 \cdot x$) was incorrect from an observer's perspective, his formula *was compatible* with his image of the problem's context.

Another student, Matt, responded to the box problem by initially describing how the paper (with initial dimensions of 8.5 in. × 11 in.) relates to the box. Matt first used a sheet of paper to discuss the constructed box, rather than representing the situation with a drawing of the paper and square cutouts (Table 3).

Matt described a shallow box with a wide base, appearing to imagine a box that results from cutting small squares from the paper's corners (lines 1–3). Matt then imagined how increasing the length of the square cutout causes the box to become narrower due to the base of the box becoming "less as wide," while the depth of the box increased (lines 4–5). Matt attempted to relate the varying dimensions of the box to the volume of the box by explaining that the volume increases and then decreases as the box became "so thin, but so deep" (lines 9–13). Matt's description reveals that his image of the problem's context included an understanding that variations in the cutout length caused variations in the three dimensions

Table 2
Travis continues his description of the box.

1	Int.:	OK, so that's the volume of what then?
2	Travis:	The box from the sheet that you originally had taken the cutout from the corner
3		and folded up.
4	Int.:	OK, so what's the 13, the 11, and the *x* represent again then?
5	Travis:	The 13 is the length of the paper, 11 inches is the width of the paper, *x* is height
6		of the cutout from a corner.
7	Int.:	OK, so that's the height, so why does that end up being the height of the box?
8		Why is the cutout. . .
9	Travis:	Um, because when you take this away from the paper and you fold this up
10		(*pointing to the cutout on his diagram*), say we took that out. This length would
11		be the same as this length (*referring to the cutout and height*), the height. When
12		you would fold it up it would be the same as the length of the cutout.
13	Int.:	OK.
14	Travis:	So if you fold it up, then it goes up, which would be the height.
15	Int.:	OK. So what are the, what are the various quantities of this situation that we are
16		working with?
17	Travis:	Um, quantity of, number of inches for the length, number of inches for the
18		width (*tracing the full dimensions of the paper*), number of inches for the
19		height.
20	Int.:	OK. So what do the 13 and 11 represent then again?
21	Travis:	Number of inches of length of the side of the box, 11 is the number of inches of
22		the width of the side of the box, and *x* is the number of inches in the height of
23		the box.

Table 3
Matt's description of the box.

1	Matt:	Like this is what happens, basically, when you first, OK, this is the paper (*pointing*
2		*to the paper*), um, the volume first is a wide box, very shallow (*moving hands*
3		*apart to show a wide box that is shallow*), when you first start cutting. And then
4		eventually you start cutting it starts getting, the box starts getting less as wide
5		(*moving hands horizontally together*) but deeper (*moving hands vertically apart*).
6	Int.:	OK, so what do you mean by deeper?
7	Matt:	The box becomes deeper, the depth, so it's height times width times depth, I think.
8	Int.:	OK.
9	Matt:	Depth, yeah. So eventually, since it's going, the depth is going to be increasing, the
10		box is going to be getting not as wide as it was, and eventually it's going to end up
11		(*moving hands horizontally together*) where the volume increased so much that it
12		starts decreasing because it's getting so thin, but so deep (*moving hand vertically*
13		*apart*), but it's still decreasing.

Table 4
Matt investigates a cutout length.

1	Matt:	If you had like, if you say the cutout was 1 inch, so that would mean, that would
2		mean, 7 times 10 times 1. Because you know that if 1 inch has been cut off, or if
3		0.5 inches have been cut off, because 0.5, 0.5 (*pointing to the two corners of the*
4		*paper*), this would be 0.5 (*referring to the length of the cutout*).
5	Int.:	OK, so now why would that be 0.5 then?
6	Matt:	Because you're cutting off (*pointing to the side of the paper)*, if the width, if the
7		width is 8, and, if we know that before the cutout, oh my, the width is 8.5.

Table 5
Matt continues to investigate a cutout length.

1	Matt:	So the original width of the paper is 8.5 inches (*pointing to the width of the paper*).
2		If you cut off 0.5 from each side (*pointing to the two corners*) the cutout is 0.5,
3		you're going to come down to 7.5 inches for width.
4	Int.:	So why is it 7.5 again?
5	Matt:	Because you're cutting out 0.5 here (*pointing along the side of the paper in one*
6		*corner*) and 0.5 here (*pointing along the side of the paper in the other corner*).
7		That way it adds up to one inch off the 8.5, which is the width (*tracing along the*
8		*entire width of the piece of paper*).

of the constructed box. Specifically, Matt imagined that increasing the length of the cutout caused decreases in the base dimensions of the box and corresponding increases in the height of the box.[4]

Next, Matt determined a formula for the volume of the box in terms of the length of the cutout. Matt first stated, "The length is going to be this (*pointing to the length of the paper*), the 11 in., the width is going to be 8 in. (*pointing to the width of the paper*), and the depth, or height, is going to be *x*." Even though Matt's initial formula was the same as Travis's formula, the underlying imagery that led to this formula was not the same. Matt's formula did not reflect the quantitative structure he had constructed while orienting himself to the problem. As Matt tried to reconcile his formula with his image of the problem's context, Matt paused for several seconds. He then realized that there was a conflict and he proceeded to consider a specific cutout length (Table 4).

Matt's process of determining values for the width and length of the box for a specific cutout length led him to refine his understanding of how the box's length and width are related to the cutout length. After initially confusing the length and width of the box as being the same as the length and width of the original sheet of paper (previous to Table 4), he selected a specific cutout length of 1 inch and described the new values for the length and width of the box that resulted from removing only one cutout length from the length and width of the paper, which he revealed when stating that he would multiply 10 times 7 times 1 to determine the volume of the box (lines 1–2). When examining the paper, he realized that he would need to cut one-half of an inch from each edge of the paper to remove one inch from a particular dimension (length or width) (lines 3–4). His quantitative structure of the problem's context now included the correct image of the length and width of the box as being determined by subtracting two cutout lengths from the fixed length and width of the paper. During the process of refining his image of how the quantities are related, Matt also caught his mistake of using 8 inches as the original width of the paper, rather than 8.5 in. (lines 6–7). Matt then spontaneously restated how the width and length of the box are related to the dimensions of the paper and the length of the side of the square cutout, as if to continue to solidify his understanding of these relationships (Table 5).

After recognizing that he made an incorrect statement about the width of the paper (lines 6–7 in Table 4), Matt refined his image of the quantities composing the box and explained how a cutout of 0.5 in. related to the width of the box and the width of the paper (lines 1–3, 5–8). It was only when he considered a specific value (Tables 4 and 5) for the cutout that he was able to refine his image of the problem's context by imagining the length and width of the base of the box as being determined by subtracting *two* cutout lengths from the fixed dimensions of the paper. Matt's refinement of his formula illustrates how a student's image of a problem's context can adapt as he attempts to reconcile differences in his imagery and his products. Prior to writing a formula Matt did not appear to imagine *two* cutout lengths being removed from the sheet of paper, although he did imagine that an increase in the cutout length causes the two base dimensions of the box to decrease in length. It was only when he attempted to determine the box's dimensions when considering a specific cutout length that he began to conceptualize the length and width of the box as two cutout lengths less than the paper's length and width.

Following Matt's description of the dimensions of the box resulting from a cutout of 0.5 inches, he created a formula for the volume of the box based on a cutout of length *x*. Matt first wrote the formula $V = (8.5 - x)(11 - x)x$. He then returned to the sheet of the paper and explained, "The width of the paper in inches minus *x*, which is the height, which would be also the cutout (*identifying the cutout on the sheet of paper*) of each. . .sorry, 2*x*, because it's two sides. . .it's two cutouts because it's (*identifying cutout from each corner*) from each side." By returning to the context of the paper and imagining the building of

[4] We note that Matt did not articulate exactly how the base dimensions changed (decreased by 2 cutout lengths) as the cutout length increased. But, unlike Travis, Matt did imagine the base dimensions changing.

the box, he was able to correct the error in his formula. The quantitative relationship he constructed between the dimensions of the box, the paper, and the cutout, supported him in correcting this formula by recognizing that the length and width of the box resulted from subtracting two cutout lengths from the length and width of the sheet of paper.

Overall, Matt's actions reveal him making multiple incorrect descriptions, but continually correcting these descriptions as he persisted in comparing his formulas and calculations to his emergent image of the box's quantities and how they change together. It is important to note that Matt refined his image of the problem's context (e.g., considering a specific cutout in relation to the dimensions of the box and paper) while reconsidering the process of constructing a box and attempting to determine specific values for a box's quantities. In contrast, Travis did not refine his initial image of the quantities and their relationships (e.g., the box width is the same as the paper width); yet we observe that Travis's solution did align with his image of the box's quantities and how they are related.

The finding that Matt and Travis did not initially construct correct formulas to model the relationships between the quantities in the box problem was a common phenomenon among the nine students who participated in this study. The students who did not determine a correct formula typically constructed an image of the problem's context that was inconsistent with the intention of the problem statement. For instance, four of the students (including Travis) did not construct a quantitative structure that included their conceptualizing a dynamic relationship between the length and width of the box, the cutout length, and the dimensions of the sheet of paper. These students conceived of the base of the box having fixed measurements, as opposed to varying with respect to the cutout length. Such images led to the students creating incorrect formulas, although their formulas were consistent with their expressed images of the problem's context (e.g., their formulas reflected a box with base dimensions of a constant measure).

All students frequently referred to the length of the paper, the width of the paper, the length of the cutout, the length of the box, and the width of the box as "the length." Although this could easily be interpreted as a poor choice of wording on a student's part, the students often subsequently interchanged various dimensions as they continued to solve the problem, implying they had not conceived of the various lengths as distinct quantities. As an example, Travis did not conceive of the base dimensions of the box as distinct from the dimensions of the paper, and he was unable to determine a correct formula. This ambiguity could result from the use of the same word for both the attribute being measured (e.g., "length" as a linear extent) and as an object with that attribute (e.g., "length" of the box as the name of one side, as opposed to the height or width). We also frequently observed students referring to a quantity using only one of (i) the attribute being measured (e.g., length), (ii) the object to which the attribute belongs, (e.g., a side of the cutout), or (iii) the unit of measurement (e.g., inches). With only one of the three aspects of the quantity specified, the referent is typically ambiguous, and may easily lead to conflation, or blurring, of the various quantities in the context.

In light of the students' difficulties in responding to the box problem, and in order to gain further information about their images of the problem's context, the interviewer asked each student (after he or she settled on a solution) to discuss how the various sides of the box were formed from the original paper. The interviewer's prompting resulted in each student reconsidering the construction of the box and refining her or his image of the problem's context. As an example of a student refining his image of the box problem and then correcting his solution after interviewer prompting, consider Travis's actions immediately after the interaction presented in Table 2. When describing the box in relation to the original sheet of paper, Travis paused for several seconds and then claimed, "[I] did it wrong. . .cutting out a piece [would] reduce the length and the width," which implied that he had altered his image of the problem's context. Without further prompting from the interviewer, Travis characterized the relationship between the length of the cutout and the dimensions of the box (Table 6).

As a result of Travis imagining the "actual length" and width of the base of the box as varying with the length of the side of the cutout, Travis correctly constructed formulas for the length and width of the box (lines 1–3 and 6–10). When describing why $13 - 2x$ and $11 - 2x$ represented the length and width of the box, respectively, he described the quantitative relationships reflected by each term and operation. He described that subtracting "by 2 x's" reflected taking two cutouts away (lines 7–9). Also, Travis understood x as representing the number of inches of the length of the side of a cutout (lines

Table 6
Travis's modified image of the box.

1	Travis:	So, the actual, the actual length would be 13 minus 2 x, because there's two
2		cutouts on the side, so you have to multiply the x by 2. So that would be
3		length, width would be 11 inches minus 2 x, and the height would be x again.
4		(*pause*)
5	Int.:	Ok, so why the 13 minus 2 x, and 11 minus 2 x.
6	Travis:	Um (*pause*) because on one side of the paper, when you would take the
7		cutouts, you would take two cutouts and one cutout is x, is the length of one
8		side of the cutout (*pointing to the cutout in the diagram*). So, since you're
9		taking two away, you'd have to subtract it by 2 x's, by two cutouts. And the
10		same thing with the width, two times also.
11	Int.:	Ok, so what would the resulting volume be then, of the box?
12	Travis:	Um, it would be the length (*pause*) times the width times the height.
13	Int.:	Ok, so what does x again represent here?
14	Travis:	That's, uh, the length, or the number of inches in the side of a cutout of the
15		paper.

14–15). Thus, it appears that Travis had developed a quantitative structure involving a relationship between the dimensions of the box, the dimensions of the paper, and the length of the side of a cutout.

Similar to Travis, when each student modified their image of the problem's context so that their image was compatible with the problem's intentions, they had little difficulty determining the correct formula. These students justified their formulas by referencing the quantities of the situation and the relationships between these quantities. In comparing Matt's and Travis's actions, Matt's initial dynamic image of the situation better supported his ability to determine a correct solution (and construct correct quantitative relationships) with minimal interviewer intervention. Travis, on the other hand, failed to provide a correct solution until the interviewer's questioning prompted him to refine his mostly static image of the situation. It was not until Travis imagined the entire process of making the box (removing the cutouts and then folding up the sides of the box, where the sides of the box covaried with the cutout length) that his quantitative structure aligned with the box as intended by the problem statement.

The students' actions on the box problem support the emphasis Smith and Thompson (2008) place on the mental structures students create when solving problems and reasoning about relationships between quantities. The students involved in the present study determined volume formulas that were compatible with their constructed quantitative structures, regardless if the formulas were correct or incorrect from the interviewer's perspective. In the cases that the students determined incorrect formulas, they had little difficulty explaining these formulas, and thus their mistakes appear to have been situated within their image of the problem's context (e.g., not appropriately conceiving of the length and width of the box in relation to the cutout length). When they subsequently modified their images of the problem's context, the students altered their volume formulas to reflect these modifications and described the formulas in terms of their reconstructed images of the quantitative relationships. These actions reveal that the students did not lack the reasoning abilities needed to create the proper formulas; rather, the accuracy of their formulas was dependent on their mentally constructed images of the box's quantities.

4.2. The Ferris wheel problem

> Consider a Ferris wheel with a radius of 36 feet that takes 1.2 min to complete a full rotation. April boards the Ferris wheel and begins a continuous ride on the Ferris wheel. If the platform to board the Ferris wheel is 8 feet off of the ground at the bottom of the Ferris wheel, sketch a graph that relates the total distance traveled by April and her vertical distance from the ground.

The Ferris wheel problem provided insights into how a student's image of a problem's context is critical to her or his ability to determine a graph that correctly represents covarying quantities' values. When first attempting the Ferris wheel problem, none of the students drew a diagram of the situation before drawing an accurate graph, although their initial graphs did depict April's distance from the ground as increasing and then decreasing as the distance she traveled around the circumference of the Ferris wheel increased. However, the concavity of each student's graph was incorrect (e.g., Fig. 2). When asked to justify the concavity of his or her graph, each student verbalized that as the total distance traveled by April increases, April's vertical distance from the ground increases and then decreases. But when initially probed to explain the shape of their graphs, the students were unable to speak accurately about how the changes in April's distance from the ground were changing. They also did not consider the rate of change of April's distance from the ground with respect to her distance traveled.

In an attempt to further identify how the students' images of the problem's context related to their graph, the interviewer asked each student to explain why he or she drew a "curved" graph. As an example of a student's approach to this problem, Charles first drew a graph that was concave down relative to the first three-quarters of April's revolution, concave up for the last quarter of a revolution, and concave up for an additional quarter of a revolution (Fig. 2). Charles described that his graph was concave down over the first quarter of a revolution because, "the change in [height from the ground] is decreasing for every equal amount of [arc length] (*denoting changes of arc length and vertical distance on his graph for the first quarter*

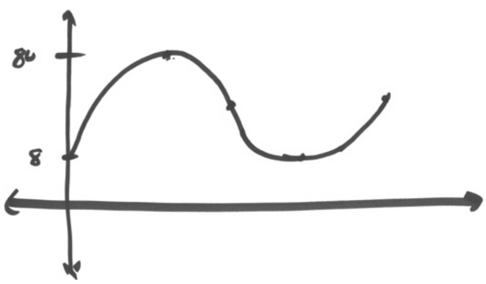

Fig. 2. Charles's initial graph.

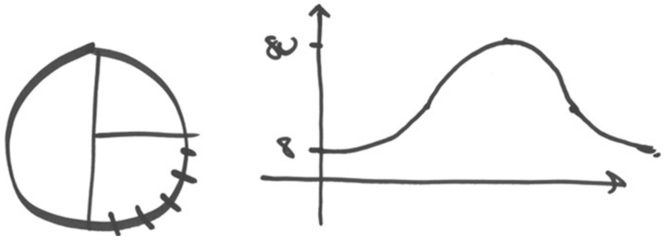

Fig. 3. Charles's corrected graph and diagram of situation.

of a revolution)." Charles's explanation was correct relative to the concavity of *his graph*; yet, the graph itself was incorrect relative to the situation (from the interviewer's perspective) over the first quarter of a revolution (e.g., the correct graph is concave up for the first quarter of a revolution).

In response to Charles using his graph, as opposed to the problem's context, to describe the relationship between the quantities, the interviewer asked Charles to explain the shape of his graph by using a diagram of the Ferris wheel (Table 7).

In this excerpt, Charles reiterated that April's distance from the ground increased as her total distance traveled increased (lines 1–3). In an attempt to justify the concavity of his graph, Charles identified successive increases in the value of the input quantity by marking off equal sized arcs on the path formed by April's path around the Ferris wheel (lines 9–12 and 14–16). As a result of marking off equal intervals of arc length on his diagram (Fig. 3) and noticing that the changes in height were increasing rather than decreasing (lines 25–27), Charles questioned the accuracy of his original graph (lines 20–23). After considering how the amounts of change of the two quantities' values changed together, Charles proceeded to modify his graph to be an accurate reflection of the relationship between April's vertical distance from the ground and the distance she traveled from her starting position on the Ferris wheel (Fig. 3).

In both cases, Charles's description of his graph was correct. In the first instance, his graph shaped his description of amounts of change between the two quantities; in the second instance, he refined his image of the problem's context by using a diagram to consider amounts of change between the two quantities. The resulting quantitative structure became the basis for correcting the concavity of his graph. Charles was only able to produce a correct graph after he refined his image of the problem's context by exploring the relationships between the various quantities. Similarly, each student in the study provided a correct graph only after they had considered how one of the quantities changed while considering equal increments of change of the other quantity. Consistent with the students' actions on the box problem, the students only formed a correct solution (in this case a graph) after refining their image of a problem's context through constructing a robust and correct quantitative structure.

Table 7
Charles investigates April's ride.

1	Charles:	So, as the total distance is increasing (*tracing the arc-length*), we
2		notice that the height is increasing but for every successive change of
3		total distance (*making marks at equal changes of arc-length*), let's
4		say right here it's eight, well if I drew a bigger one, I'd be able to
5		show it more precise.
6	Int.:	Here, go ahead and I've got some extra pieces of paper. Go ahead and
7		if you want to draw it on there somewhere (*handing him a sheet of*
8		*paper*) a little bigger.
9	Charles:	(*Drawing a larger circle and drawing a vertical and horizontal*
10		*diameters through the circle*) So, as, I guess we can assume this is
11		ninety degrees (*referring to the compass*) we can make an angle
12		(*attempts to use protractor on compass*). . .
13	Int.:	So what are you trying to do right now?
14	Charles:	Well I see, this thing moves (*referring to the compass*), I'm trying to
15		show that, um, I'm trying to make, well I could use the protractor,
16		I'm just trying to change, show successive change in input.
17	Int.:	Could you use the [waxed string] to do that?
18	Charles:	Well, actually, yes I can.
19	Int.:	So, you're trying to show successive changes in what?
20	Charles:	In input, which would be the total distance (*marks a distance on a*
21		*waxed string mumbling to himself, then marking off successive arc-*
22		*lengths on the circle*). Ok, so, for every change in total distance, right
23		here (*referring to arc-length*), he, well, hmmm.
24	Int.:	What makes you go hmmm?
25	Charles:	Because it seems as total distance increases (*referring to the arc-*
26		*length*), the actual change in the height is increasing instead of
27		decreasing.

The students' solutions to the Ferris wheel problem further emphasize the importance of students imagining the situation described by the problem and continuing to refine their image of the problem's context. The Ferris wheel problem also highlights the power of students exploring the context of the problem when attempting to construct a graph that represents how the quantities in a problem's context change in tandem over a continuum of input values.

5. Discussion

When initially responding to the box problem, students often did not conceive of the length and width of the box as distinct from the length and width of the paper. However, these students did conceive of the length and width of the box as measurable attributes, although they did not conceive of their values in terms of a relationship between the varying length of the side of the cutout and the dimensions of the original sheet of paper. If a student did not first imagine an accurate relationship between the dimensions of the base of the box, the length of the cutout, and the dimensions of the paper, the student commonly constructed a formula (e.g., $V = 13 \cdot 11 \cdot x$) that *correctly* reflected their static image of the box's base, but incorrectly characterized the quantitative relationships intended by the problem statement. It was only after the students imagined the process of making the box and considered how the relevant quantities of the situation changed in tandem that they created a correct volume formula. For instance, Matt initially formed an image of the situation that included all dimensions varying as the length of the cutout increased. He then refined this image by considering the dimensions of a box for a specific cutout value. In instances when students considered a value of one quantity, and attempted to compute the values the other relevant quantities in the problem context, he or she adapted (if necessary) the structure of her or his conceptualization to align with the quantitative relationships as intended[5] by the problem statements.

Similar to the students' solution attempts to the box problem, the students had difficulty determining a correct graph during the Ferris wheel problem. When first responding to the Ferris wheel problem, the students created graphs by imagining the value of one quantity increasing or decreasing with respect to another quantity's value increasing. The students' non-linear graphs indicated changing rates of change between the two quantities, but the interviewer's questioning revealed that their image of the problem's context did not include a robust quantitative structure including how the changes in the quantities change together, or how the rate of change of the first quantity with respect to the second quantity changes for variations in the first quantity. As a result, the students constructed incorrect graphs and then used these graphs (as opposed to a diagram of the situation) to describe rates of change and amounts of change between the two quantities' values. The students only created a correct graph when they modified their image of the problem's context by using a diagram of the situation to consider how the values of the two quantities changed together. After using the diagram to construct a more developed image of the problem's context, the students determined a correct graph that was supported by their more robust quantitative structure.

The students' actions during the interviews revealed that their solutions aligned with their emerging images of a problem's context. Students' incorrect solutions stemmed from constructing images of the problem contexts that were underdeveloped or inconsistent with the intention of the problems. As the interviewer continued to ask the students to explain their thinking, they frequently adjusted their images of the problem contexts. When a student refined her image of a problem's context, the student also altered her solution to reflect these refinements. Thus, in the case of both incorrect and correct solutions to a problem, the students' formulas and graphs reflected their image of a problem's context and students only changed their solutions after modifying their image of a problem's context.

6. Conclusion and implications

The present study illustrates that a student's mental image of a problem's context can either hinder or support the student's ability to determine a correct formula or graph. In the case that students do not construct a quantitative structure compatible with the problem context as intended by the problem statement, they are not equipped to determine formulas and graphs that correctly represent how the quantities are related and change together. The influence of a student's image of a problem's context on her solution offers one explanation for Carlson's (1998) observation that students had difficulty producing graphs that conveyed a relationship between two covarying values. The weaknesses in these students' abilities to reason covariationally, when attempting to construct a formula or graph in an applied problem context, may have been related to the students holding different (and sometimes underdeveloped) images of the relevant quantitative relationships in the problem.

As a student progresses in solving a problem that requires relating quantities that vary in tandem, her understanding of the quantities of a problem and how these quantities change together emerges as the student imagines the problem's context. The students in the present study were able to advance their conceptualization of a problem's quantities by imaging the described situation (including the use of diagrams), mentally playing out variations in quantities, and considering specific values to explore relationships between quantities (e.g., see Matt's actions on the box problem). As the students engaged in these actions they refined their images of the problem contexts in ways that impacted their formulas and graphs. These

[5] We use *intended* in place of *described* to highlight that a problem statement does not objectively define the problem's context. Instead, a student's cognitive activity produces an image of a problem's context.

findings support Smith and Thompson's (2008) suggestion for an increased emphasis on quantitative reasoning as a means of orienting to a novel problem context and supporting students in creating meaningful formulas and graphs. This finding may also shed light on the mental processes that have been characterized in problem solving as sense making (Schoenfeld, 1992, 2007) or orienting (Carlson & Bloom, 2005; Pólya, 1957).

The present study provides information about students' reasoning when making sense of novel applied problems and should therefore be useful to curriculum designers and instructors. It is common practice to ask mathematics students to solve problems based on contextual situations. It is easy to imagine a teacher using formulas or graphs (or asking students to produce formulas or graphs) before students have constructed a well-developed image of a problem's context. In such a situation, the student has little to no conceptual foundation for interpreting the teacher's products. Furthermore, the student is deprived of the opportunity to construct her own quantitative structure as a basis for determining meaningful formulas and graphs.

Our findings suggest that curriculum and instruction should attend to the emergent nature of students' images of problem contexts by frequently prompting them to reason about the quantities in a problem's context and how they change together. As an example, in the box problem, students may benefit from engaging in and discussing the process of making boxes from a sheet of paper *before* performing numerical calculations and creating a formula to model relationships between the quantities' values. Students should be given the opportunity to discuss the constraint of having square cutouts of equal dimensions on all four corners and how these cutouts impact the dimensions of the box.

Instruction should also support students in reflecting on a problem's context as an approach for checking a solution and correcting incorrect solutions. When a student determines an incorrect formula or graph, the student's solution might (correctly) reflect an impoverished quantitative structure. This suggests that the student would benefit by reconsidering the problem's context and the relationships represented by their formula and graph. The emergent nature of students' quantitative reasoning provides insights into the necessary foundations for productive metacognitive problem solving behaviors, such as monitoring (Schoenfeld, 1992) and control (Carlson & Bloom, 2005). By maintaining a focus on problem contexts, students might be supported in constructing images of problem contexts that support their monitoring the correctness and effectiveness of their solution attempts.

Based on our findings, we call for other studies to investigate the process by which students' images of problem contexts are refined to align with the intent of a problem. In order to better support students in engaging in actions that lead to their developing robust quantitative structures, it is important that we understand the productive and unproductive mental actions involved in constructing an image of a problem's context. The present study reveals the importance of students imagining a situation and engaging in activities to conceptualize, vary, and covary the values of related quantities as they attempt to solve word problems. Most importantly, our study emphasizes that these activities play a critical role during students' attempts to determine meaningful and correct formulas and graphs. If mathematics educators expect students to consider mathematics as a tool to understand and model real world situations, then it is important that students are provided opportunities to make sense of the words in a novel problem so that they conceive of situations in ways that support their viewing formulas and graphs as representing how the quantities of a situation vary in tandem.

Acknowledgements

Research reported in this paper was supported by National Science Foundation Grants No. EHR-0412537. Any conclusions or recommendations stated here are those of the authors and do not necessarily reflect official positions of NSF. We thank Michael Oehrtman for reviewing and contributing to an earlier version of the article and thank Pat Thompson for his numerous conversations about quantitative reasoning. Also, thank you to SIGMAA on RUME for providing us the opportunity to present and prepare an earlier version of the article at the 2009 annual conference.

References

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Shoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education, III. Issues in Mathematics Education* (pp. 115–162).

Carlson, M., Jacobs, S., Coe, E., Larsen, S. & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Carlson, M. P. & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics*, *58*(1), 45–75.

Castillo-Garsow, C. C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth*. Unpublished Ph.D. dissertation.

Clark, P. G., Moore, K. C. & Carlson, M. P. (2008). Documenting the emergence of speaking with meaning as a sociomathematical norm in professional learning community discourse. *The Journal of Mathematical Behavior*, *27*(24), 297–310.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–589). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Dubinsky, E. & Harel, G. (1992). The nature of the process conception of function. In G. Harel, & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA notes* (pp. 85–106). Washington, D.C.: Mathematical Association of America.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction*, *25*(4), 439–478.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction*, *28*(4), 383–432.

Harel, G. & Dubinsky, E. (1992). *The concept of function: Aspects of epistemology and pedagogy, MAA notes*. Washington, D.C.: Mathematical Association of America.

Kaput, J. (1992). Patterns in students' formalization of quantitative patterns. In G. Harel, & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and pedagogy, MAA Notes* (pp. 290–318). Washington, D.C.: Mathematical Association of America.

Monk, S. & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. *CBMS Issues in Mathematics Education*, *4*, 139–168.

Moore, K. C. (2010). *The role of quantitative reasoning in precalculus students learning central concepts of trigonometry*. Unpublished Ph.D. dissertation.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Oehrtman, M., Carlson, M. & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson, & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27–42). Washington, D.C.: Mathematical Association of America.

Pólya, G. (1957). *How to solve it, A new aspect of mathematical method* (2nd ed.). Princeton, NJ: Princeton University Press.

Saldanha, L. & Thompson, W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkings, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th annual meeting of the north American chapter of the international group for the psychology of mathematics education* (pp. 298–303). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York: Macmillan.

Schoenfeld, A. H. (2007). Problem solving in the United States, 1970–2008: Research and theory, practice and politics. *ZDM Mathematics Education*, *2007*(39), 537–551.

Smith, J., III & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). New York, NY: Lawrence Erlbaum Associates.

Strauss, A. L. & Corbin, J. M. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks: Sage Publications.

Thompson, P. W. (1989). *A cognitive model of quantity-based algebraic reasoning*. Paper presented at the annual meeting of the American Educational Research Association.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, *25*(3), 165–208.

Thompson, P. W. (1994a). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*(2–3), 229–274.

Thompson, P. W. (1994b). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education: Issues in mathematics education* (pp. 21–44). Providence, RI: American Mathematical Society.

Thompson, P. W. (2000). Radical constructivism: Reflections and directions. In L. P. Steffe, & P. W. Thompson (Eds.), *Radical constructivism in action: Building on the pioneering work of Ernst von Glaserfeld* (pp. 412–448). London: Falmer Press.

Thompson, W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOMê*. Laramie, WY: University of Wyoming.