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The Precalculus Concept Assessment: A Tool for Assessing Students' Reasoning Abilities and Understandings

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This article describes the development of the Precalculus Concept Assessment (PCA) instrument, a 25-item multiple-choice exam. The reasoning abilities and understandings central to precalculus and foundational for beginning calculus were identified and characterized in a series of research studies and are articulated in the PCA Taxonomy. These include a strong understanding of ideas of *rate of change* and *function*, a process view of function, and the ability to use covariational reasoning to examine and represent how two covarying quantities change together. This taxonomy guided the PCA development and now provides the theoretical basis for interpreting and reporting PCA results. A critical element of PCA's design was to identify the constructs essential for learning calculus and to employ methods to assure that PCA items are effective in assessing these constructs. We illustrate the role that cognitive research played during both the design and validation phases of the PCA instrument. We also describe our Four-Phase Instrument Development Framework that articulates the methods used to create and validate PCA. This framework should also be useful for others interested in developing similar instruments in other content areas. The uses of PCA are described and include (a) assessing student learning in college algebra and precalculus, (b) comparing the effectiveness of various curricular treatments, and (c) determining student readiness for calculus.

Precalculus courses are taught in high schools, community colleges, and universities across the United States. The name suggests that the course content is focused on knowledge that prepares students for calculus; however, fewer than half of the students who successfully complete precalculus go on to enroll in calculus. One study from Arizona State University revealed that only 23% of students who earned a C or better in precalculus at the university level subsequently completed Calculus I, and fewer than 5% of these students successfully completed Calculus II (Thompson et al., 2007b). This pattern of losing students in calculus has been cited as a major reason why students choose to leave Science, Technology, Engineering, and Mathematics (STEM) majors, resulting in a high cost to our nation's intellectual power and economic well-being. Our nation's high attrition rate in STEM majors has become so severe that leaders in government and industry are calling for colleges and universities to understand and address the problem (Business Higher Education Forum, 2007).

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While the number of U.S. college students persisting in their study of calculus is known to be in sharp decline, the play of factors contributing to this decline is not fully understood. However, research does suggest that a student's decision to leave a STEM major typically is not the result of one event but arises from a multitude of factors including conceptual difficulties, poor teaching by faculty, course pace, course load, inadequate high school preparation, inadequate academic advising, financial problems, and language barriers (Seymour, 1992a, 1992b, 2001; Seymour & Hewitt, 1997). While not all of these factors are under the control of schools and universities, issues related to teaching, academic advising, course pace, high school preparation, and conceptual learning of students might be impacted by educators and educational institutions.

Another commonly cited reason for high attrition from precalculus to calculus is that standards for what should be taught in courses like precalculus mathematics are unclear and inconsistent (Arismendi-Pardi, 1996; Schmidt, Houang, & Cogan, 2002). Moreover, there is now substantial research on what is involved in learning key ideas of algebra through beginning calculus (e.g., Carlson & Ramussen, 2008). However, a cursory examination of the commonly used curricula suggests that this research knowledge has had little influence on precalculus level curricula. A review of the literature has also revealed that there are currently no research-based instruments available for assessing concepts of precalculus or beginning calculus. In the absence of research-based curricula or instruments, teachers tend to rely on their own opinions about what students need to learn as they plan instruction.

Adding to the complexity of the interplay of attrition factors, there is currently no widespread agreement among curriculum developers or instructors about the conceptual foundations that are needed for calculus. Even a casual review of precalculus level textbooks makes apparent this lack of consensus. To further complicate matters, results from the Third International Mathematics and Science Study indicate that instructional mathematics practices in American schools tend to focus on fragmented procedures and topics rather than concepts that are central to continued learning of mathematics (National Center for Education Statistics, 1995; Schmidt et al., 2002). This disjointed instruction might be leaving many intellectually capable U.S. students without the foundational knowledge, confidence, and interest they need to attempt more advanced courses in mathematics.

The Precalculus Concept Assessment (PCA) that we have developed should be useful in helping curriculum developers and instructors gain greater clarity about the conceptual foundations needed for calculus learning. We also hope that our approach to developing the PCA instrument will serve as a model for others who take on the challenge of developing multiple-choice, conceptually focused instruments to assess students' knowledge in specific content areas. It is our goal that this article will provoke reflection on the role of cognitive research in informing instrument development and will contribute to higher standards for the processes of instrument development.

The following section provides a review of research that has provided insights into the foundational understandings and reasoning abilities that students need for success in a first course in calculus (Carlson, 1995, 1997, 1998; Monk, 1992; Kaput, 1992; Dubinsky & Harel, 1992; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Smith, & Persson, 2003; Carlson, Larsen, & Lesh, 2003; Carlson & Oehrtman, 2004; Oehrtman, 2008a, 2008b; Oehrtman, Carlson, & Thompson, 2008; Thompson et al., 2007). This review is followed by a description of the Precalculus Concept Assessment Taxonomy that our team developed to synthesize the understandings and abilities identified by that research. We then describe the Precalculus Concept Assessment (PCA) itself, a 25-item multiple-choice instrument that we have developed to assess the reasoning abilities and understandings enumerated in the PCA Taxonomy. A detailed description of the approach we took in developing and validating PCA items is provided. We then illustrate the process we used to validate answer choices for each PCA item. The article concludes by describing the meaning of a PCA score and the knowledge that can be generated by administering the PCA to students.

Our goal was to develop an instrument to assess essential knowledge that mathematics education research has revealed to be foundational for students' learning and understanding of central ideas of beginning calculus. A critical first step in the development process was to review the literature related to learning key ideas of precalculus and calculus and to conduct a sequence of qualitative studies to identify tasks that were most effective in assessing this knowledge.

ESSENTIAL KNOWLEDGE FOR LEARNING CALCULUS: AN OVERVIEW OF THE RESEARCH

As students progress through secondary and undergraduate mathematics courses, instructors frequently require them to manipulate algebraic equations and practice techniques for computing answers to specific types of problems. This strong emphasis on procedures without a concomitant focus on developing coherent understandings has not been effective in developing conceptions that support meaningful interpretation and use of concepts when solving novel problems (Hiebert & Carpenter, 1992; Thompson, 1994a; Thompson, Philipp, Thompson, & Boyd, 1994; White & Mitchelmore, 1996). Research into student learning of precalculus and introductory calculus has also uncovered information about the complexities of learning and understanding the concept of function, the central conceptual strand of the mathematics curriculum, from algebra though calculus.

It has been well documented that many precalculus level students' reasoning is dominated by a static image of arithmetic computations used to evaluate a function at a single numerical value (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1995, 1997, 1998). One result of this constrained view, also labeled an *action view of function*, is that students tend to view functions only in terms of symbolic manipulations and procedural techniques disassociated from an underlying interpretation of function as a more general mapping of a set of input values to a set of output values (Breidenbach et al., 1992; Dubinsky & Harel, 1992; Carlson, 1998). Once students can begin to imagine a continuum of input values in the domain of the function producing a continuum of output values, they are said to have developed a *process view of function* (Breidenbach et al., 1992). Students who are unable to conceptualize a function as a process have difficulty composing and inverting functions, and are thus unable to use functions effectively in solving word problems (Breidenbach et al., 1992; Dubinsky & Harel, 1992; Carlson, 1998).

The ability to interpret the meaning of a function modeling a dynamic situation also requires attention to how the output values of a function are changing while imagining changes in a function's input values. This ability has been referred to as *covariational reasoning* (Thompson, 1994b; Carlson, 1998) and has been documented to be essential for representing and interpreting the changing nature of quantities in a wide array of function situations (Carlson et al., 2002; Thompson, 1994c) and for understanding major concepts in beginning calculus (Carlson et al., 2003; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Kaput, 1992; Rasmussen, 2000; Thompson, 1994b; Zandieh, 2000). A more detailed examination of what is involved in having, and not having, a process view of function reveals the critical role that this function view plays in covariational reasoning.

A Process View and Interpreting the Meaning of a Graph

Students with an action view of function often think of the graph of a function as being only a curve (or fixed object) in the plane; they do not view the graph as defining a general mapping of a set of input values from the independent axis to a set of output values on the dependent axis. They then interpret the location of points, the vertical line test, and the "up and over" evaluation of the slope of functions as geometric properties of the graph and not as properties of a more general mapping or relationship between two varying quantities. Another manifestation of an action view is that students conflate the geometric shape of a graph with the physical characteristics of the real-world scenario that the graph describes. For example, they see a distance-time graph of a cyclist riding on a hilly terrain as representing the path of the cyclist instead of the covariation of time and distance of the cyclist as he or she moves away from some location (Monk, 1992; Carlson, 1998).

Students' weak understandings of functions have also been related to their inability to interpret and express contextual relationships (i.e., functions defined by words or an actual context) using algebraic symbols. When asked to express s as a function of t, many high performing precalculus students did not know that their objective was to construct a formula in the form of $s = <$ some expression containing a $t >$ (Carlson, 1998). Such weak understandings and highly procedural orientations appear to contribute to students' inability to use symbols meaningfully to construct an algebraic formula to characterize how two quantities covary in a physical context (Carlson, 1998).

A Process View and Computational Reasoning

A student with an action view of function tends to rely solely on computational reasoning. For the real valued function, f, defined by $f(x) = 2x^2 + 1$, students with an action view are confined to seeing the defining formula as a computational procedure for finding a single answer for a specific value of x. They view the formula as a set of instructions: square the value for x, multiply this number by two, and then add one to get the answer. In contrast, a student with a process view sees $2x^2 + 1$ as a means of mapping any input value of the function represented by x to an output value represented by $f(x)$. When asked to evaluate $f(3)$ the student thinks about the evaluation as the enactment of a process that takes 3 as an input and produces an output value that is referenced as $f(3)$. When a student no longer has to imagine each individual operation for an algebraically defined function, they are able to imagine a continuum of input values in the domain of the function producing a continuum of output values. In one study (Carlson, 1998) 43% of college algebra students attempted to evaluate $f(x + a)$, given that f is defined by the formula $f(x) = 3x^2 + 2x - 4$, by adding a onto the end of the expression for f rather than inputting $x + a$ into the formula that defines f. When probed to explain their thinking, the students typically explained that when you add a to one side of the equation you must also add a to the other side. They were clearly not thinking of the formula as a means of mapping inputs to outputs but as two expressions separated by an equal sign.

A student with a process view of function is also able to understand what it means to solve an equation of the type $f(x) = c$ given f and c. As one example, when asked to solve the equation $f(x) = 5$, given that f is defined by the formula $f(x) = 2x^2 + 1$, the student would understand that he or she was being asked to find the input value(s) that corresponds to the output value 5. The student would also understand that the expression $2x^2 + 1$ provides a general means of determining the output value for a given value of x, resulting in the student setting this expression equal to 5, then performing the algebraic steps to isolate x. Students with an action view of function divided the expressions on both sides of the equal sign by f (Carlson, 1995). They were also unaware that the parentheses serve as a marker for the input and that f represents the name of the function. These students clearly did not view the defining formula for f as a means of converting input to output. Consequently they were unable to view solving an equation as a reversal of this process, that of taking the output value and finding the corresponding input value(s).

The Process View of Function and Covariational Reasoning

As students begin to explore dynamic function relationships, such as how speed varies with time or how the height of water in a bottle varies with volume, they must consider *how* one variable changes while imagining successive amounts of change in the other. Recall that we refer to such mental actions as *covariational reasoning* (Carlson et al., 2002). We view covariational reasoning as a refinement and extension of a process view of function and elaborate here why we claim that a process view of function is essential for imagining and describing *how* two quantities covary.

When interpreting what a graph conveys about the covarying nature of two quantities students must engage in covariational reasoning. As a simple example, a student who has a strong process view of function might see the algebraic formula $A(s) = s^2$ for $s \geq 0$ as a means of determining the area of a square for a set of possible side lengths. The student would view the formula as an entity that accepts any side length s as input to produce a unique value for the area. He or she would have no difficulty determining the side of the square for given values of the area (reversing the process). The student might also begin to notice that, as the value of the length (represented by s) increases, the value of the area (represented by $A(s)$) also increases. By exploring images of actual squares, numerical patterns, and/or by constructing a graph of this function, the student might also observe that as one successively increases the value of s by some fixed amount, the amount by which $A(s)$ increases gets larger and larger. From the graph, the student might also observe that as the value of s increases so does the slope of the graph. When asked to explain why the graph gets steeper, the student would be able to unpack the notion of slope (steepness) by describing the relative changes of input (side length) values and output (area) values, while stepping through values of s. Such reasoning patterns are exactly the kind of thinking that students need to employ to determine growth patterns and growth rates of various function types (Thompson, 1994a, 1994b).

Our past work to characterize the thinking involved in reasoning flexibly about dynamically changing events has led to our decomposing *covariational reasoning* into five distinct mental actions (Carlson et al., 2002). These five categories of mental actions (Table 1) describe the reasoning abilities involved in the meaningful representation and interpretation of dynamic function situations. Students initially are able to recognize that two quantities in a problem context

TABLE 1
Mental Actions of the Covariation Framework (from Carlson et al., 2002)

Mental Action	*Description of Mental Action*	*Behaviors (Graphical Representation)*
Mental Action 1 (MA1)	Coordinating the dependence of one variable on another variable	• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing a monotonic straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate of change of the function with uniform increments of change in the input variable	• Constructing secant lines for contiguous intervals in the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the changes in the instantaneous rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

change together in a dependency relationship (MA1). A student who engages in mental action 2 is able to imagine the direction of change of one quantity while imagining a change in the other quantity—as the temperature increases the amount of water in a reservoir decreases. As students' covariational reasoning abilities advance they are able to coordinate the *amount of change* of one quantity while imagining changes in the other quantity (MA3)—as each cup of water is added to a conical vase, the height of the water in the vase goes up more and more. Students who engage in mental action 4 are able to attend to how the *rate of change of one quantity with respect to the other* changes while imagining changes in one quantity—as each cup of water is added to a conical vase, the *rate of change of the height with respect to volume* increases until the vase is full. Mental action 5 involves imagining how one quantity changes while imaging *continuous changes* in a second quantity—as the water is poured into the vase, the height of the water in the vase increases at an increasing rate. It is noteworthy that mental actions 1 and 2 of the Covariation Framework must precede mental actions 3, 4, and 5, while mental actions 3, 4, and 5 are not necessarily hierarchical. A student might be able to engage in mental action 5 without being able to coordinate how the change in one quantity affects the change in another (Carlson et al., 2002). A student might also be able to speak about the changing nature of the *rate of change* of two quantities (MA4) without understanding or speaking about how the *amount* of one quantity changes while imagining successive changes in the other quantity (MA3).

In our work, we have developed beginning calculus modules that include tasks and prompts to promote these mental actions in students. After three iterations of refining these modules (based on our analysis of data of students' reasoning when working through them), we observed dramatic gains in beginning calculus students' *covariational reasoning* abilities over the course of one semester (Carlson et al., 2003). Other studies have also revealed that covariational reasoning is a foundational reasoning ability needed for understanding concepts of limit (Monk, 1987; Oehrtman, 2008b), derivative (Zandieh, 2000), related rate problems (Engelke, 2007), accumulation (Carlson et al., 2003; Thompson & Silverman, 2008), and the Fundamental Theorem of Calculus (Thompson, 1994b).

These studies led us to articulate the foundational reasoning abilities and understandings needed for learning key ideas of calculus that have formed the PCA Taxonomy. The PCA Taxonomy guided the selection of items for the original draft of the PCA, while the process of validating the PCA led to further refinement of the PCA Taxonomy. The qualitative studies that were conducted during the process of validating the PCA also provided useful insights about the specific understandings that were assessed by PCA items that had not been previously uncovered in other qualitative studies.

The primary motive for our development of the PCA Taxonomy and the PCA was to provide useful knowledge about the central ideas of precalculus that are needed for learning calculus and a valid tool that teachers and curriculum developers could use to assess student learning of these ideas. We wanted to help curriculum developers and teachers by characterizing the primary reasoning abilities and understandings that should be a focus of precalculus level mathematics. We also wanted to have a valid tool that we, and others, could use to assess the effectiveness of curriculum and instruction in preparing students to be successful in calculus.

A TAXONOMY OF FOUNDATIONAL KNOWLEDGE FOR BEGINNING CALCULUS

The PCA Taxonomy (Table 2) consists of three categories of *Reasoning Abilities* and three categories of *Understandings* that have emerged in our studies (e.g., Carlson, 1995, 1997, 1998; Carlson et al., 2002; Oehrtman et al., 2008) to be essential for understanding and using central concepts of precalculus, and understanding key concepts of beginning calculus. The PCA Taxonomy was not intended to be a comprehensive inventory of all content in precalculus courses; however, it does serve to focus the PCA on crucial reasoning abilities and understandings that all students should have prior to enrolling in a calculus course. It should be noted that there are significant and complex interactions among the subcategories so that no one subcategory can be completely isolated.

Reasoning abilities included in the Taxonomy are a *process view of function*, *covariational reasoning*, and *computational abilities*. A student who has a *process view* of function conceives of a function as an entity that accepts a continuum of input values to produce a continuum of output values. A student who engages in *covariational reasoning* in the context of a function is able to analyze multiple aspects of how the input and output values of a function change together. *Computational abilities* include facility with manipulations and procedures that are needed when evaluating algebraic representations of functions and solving equations. The latter category does not include the entire spectrum of algebraic abilities that students need to learn in a precalculus course. Rather, it focuses on those skills involved in evaluating a function, solving an equation, composing a function, and inverting linear and exponential functions.

The various understandings in the PCA Taxonomy articulate specific function concepts needed to construct and use functions meaningfully and are arranged into three categories. The first includes understandings of the meaning and usefulness of function evaluation, rate-of-change,

TABLE 2
The PCA Taxonomy

Reasoning Abilities	
R1	*Process view of function* (items 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 17, 20, 22, 23) View a function as a generalized process that accepts input and produces output. Appropriate coordination of multiple function processes
R2	*Covariational reasoning* (items 15, 18, 19, 24, 25) Coordinate two varying quantities that change in tandem while attending to how the quantities change in relation to each other
R3	*Computational abilities* (items 1, 3, 4, 10, 11, 14, 16, 17, 21) Identify and apply appropriate algebraic manipulations and procedures to support creating and reasoning about function models
Understandings	
Understand meaning of function concepts	
ME	Function evaluation (items 1, 5, 6, 11, 12, 16, 20)
MR	Rate of change (items 8, 10, 11, 15, 19, 22)
MC	Function composition (items 4, 5, 12, 16, 17, 20, 23)
MI	Function inverse (items 2, 4, 9, 10, 13, 14, 23)
Understand growth rate of function types	
GL	Linear (items 3, 10, 22)
GE	Exponential (item 7)
GR	Rational (items 18, 25)
GN	General non-linear (items 15, 19, 24)
Understand function representations (interpret, use, construct, connect)	
RG	Graphical (items 2, 5, 6, 8, 9, 10, 15, 19, 24)
RA	Algebraic (items 1, 4, 7, 10, 11, 14, 16, 17, 18, 21, 22, 23, 25)
RN	Numerical (items 3, 12, 13)
RC	Contextual (items 3, 4, 7, 8, 10, 11, 15, 17, 18, 20, 22)

function composition, and function inverse. The second category includes an understanding of the growth patterns of various function types: linear, exponential, rational, and general nonlinear functions. The third category involves understanding the meanings that are conveyed by various function representations (graphical, algebraic, numerical, and contextual) and the connections that exist among these representations.

Specific PCA items that provide information about students' abilities for each reasoning ability and understanding are listed with the Taxonomy in Table 2. Since responding to each item typically involves the use of multiple reasoning abilities and understandings, each PCA item appears in multiple categories of the PCA Taxonomy.

THE DEVELOPMENT OF THE PRECALCULUS CONCEPT ASSESSMENT (PCA)

Background

The development of the PCA was inspired and informed by work in physics education research. This work has resulted in instruments such as the Mechanics Diagnostic Test (Halloun & Hestenes, 1985a, 1985b), the Mechanics Baseline Test (Hestenes & Wells, 1992), and the Force Concept

Inventory (Hestenes, Wells, & Swackhammer, 1992).[1] The processes employed to develop these instruments began with the creation of initial taxonomies to characterize broad categories of student thinking related to important concepts of force and mechanics central to most introductory physics courses. These taxonomies guided the development of open-ended questions to assess their constituent concepts. Question wording was refined based on student interviews with these items and distractors were gleaned from the most common student response categories. Further interviews were conducted with students to verify that questions were interpreted correctly and that distractor choices corresponded to common student understandings and misconceptions. The taxonomies and items went through cycles of refinement until items were validated to be consistently interpreted and to assess the knowledge in the Taxonomy to which it was associated.

Lissitz and Samuelsen (2007) have argued that the most critical issue in instrument development is the process used to establish the content validity of the tool. They claim that at the heart of test validation is the content elements (e.g., questions and distractors) of the test (including the test development procedures), the relationship between the elements, and the student cognitive behaviors relative to the elements of the test. They further advocate that a test's reliability should not depend on external measures for its meaning and that test validation processes should not begin by relating the test to other measures. They argue that test validation should initially involve an iterative approach that begins by determining the constructs worthy of assessment; this should be followed by iterations of a process to assure that the tool measures these *internal* constructs. They also claim that such an iterative development process has the potential to contribute theory about the cognitive processes and other attributes measured by the tool. In the case of the PCA, our validation process has contributed to the refinement of our theory about what it means to have an action and process view of function, what is involved in covariational reasoning (Table 1), and how these reasoning abilities interact with several important understandings outlined in the PCA Taxonomy (Table 2).

The Four Phases of the PCA Development

We used a four-phase approach to develop and validate the PCA, similar to those used to develop instruments in physics education and consistent with what has been advocated by Lissitz and Samuelsen (2007). During the first two Phases of the PCA development, and prior to developing our first version of the PCA, we engaged in a series of studies to understand the reasoning abilities and understandings that are foundational for constructing the central ideas of precalculus and calculus (Carlson, 1995, 1997, 1998; Carlson et al., 2002; Carlson et al., 2003; Oehrtman, 2004, 2008a, 2008b; Engelke, 2007; Oehrtman, Carlson, & Thompson, 2008). These studies led to the development of the first draft of the PCA Taxonomy. We constructed a set of open-ended questions to gain insights into students' reasoning abilities and understandings articulated in the PCA Taxonomy. After administering these questions in an open-ended format we interviewed students to determine the reasoning abilities and understandings required to provide a correct response. We also began the process of refining the wording of the questions and identifying potential distractors for these PCA items. In Phase III of the PCA development we validated the

[1]The development of PCA preceded the development of the Calculus Concept Inventory (CCI) by more than 5 years. The CCI had no bearing on the development of the PCA.

multiple-choice items and engaged in ongoing refinement of the PCA Taxonomy. This Phase involved eight cycles in which we (a) administered the 25-item multiple-choice version of PCA, (b) conducted follow-up interviews, (c) analyzed student written work and interview data, and (d) revised the PCA Taxonomy and 25-item version of the PCA. The outcome of these iterations was a stable version of the PCA, with all items validated to assess the reasoning abilities and understandings described in the categories of the PCA Taxonomy. This Phase also produced the five answer choices for each PCA item. By the end of Phase III we felt assured that the PCA measured the constructs worthy of assessment and that the PCA Taxonomy articulated the meaning of these constructs.

In Phase IV we examined the data from administering the validated 25-item version (eighth iteration) of the PCA to various populations of students. Our goal during this phase was to establish the meaning of PCA scores and to correlate these scores with an external measure. A more detailed description of the processes we employed during these four Phases of the PCA development follows.

Phase I. We began our work by investigating precalculus, second semester calculus, and beginning graduate students' understanding of the function concept (Carlson, 1995). An assessment tool that contained 34 open-ended items was developed for the purpose of gaining insights about the reasoning abilities and understandings that are central to the concept of function in precalculus and calculus. The knowledge assessed by this collection of items was articulated in a "Function Framework" (Carlson, 1995, p. 15–25), a precursor to the current PCA Taxonomy. Drawing on function studies and Carlson's experience with precalculus level mathematics, the initial framework included 11 broad abilities critical for students' success in beginning calculus. Twenty-one of the 34 open-ended items in this tool had been used effectively in previous studies to reveal unique insights about the conceptual obstacles students encounter, understandings they lacked, or the reasoning abilities they needed when learning or using the function concept to solve novel problems (Dubinsky & Harel, 1992; Kaput, 1992; Leinhardt, Zaslavsky, & Stein, 1990; Monk, 1992; Sierpinska, 1992; Vinner & Dreyfus, 1989).

The process of further developing the first collection of items then began. In cases where several items assessed the same attribute, Carlson reviewed interview transcripts reported in published articles and selected the item that, in her judgment, was more effective. In particular, she wanted items that assessed understandings and reasoning abilities that were essential for learning calculus, and items that students were not likely to answer correctly unless they possessed the understanding(s) the item intended to assess. At the end of this process, there were understandings and abilities in her original 11-item framework for which she had no items. The remaining 13 items were developed by Carlson, with regular review and feedback from one mathematician and one mathematics educator, to assess understandings that had been documented in the literature to be critical for success in beginning calculus (e.g., covariational reasoning, function composition), but for which there were no precalculus level items in the literature. As one example, the bottle problem was developed by Carlson to assess precalculus level students' ability to attend to the covariation of two quantities. Once these new items were developed, four mathematicians were asked to review them for mathematical correctness and their relative importance in precalculus and beginning calculus. Only those items that were deemed by all four mathematicians to assess function abilities needed for calculus were retained.

Prior to conducting Carlson's (1995) initial study, all function items in this open-ended assessment were piloted in clinical interviews to assure that the wording of each item was clear and that students' interpretations of each item were consistent with the author's intent. Whenever the wording was unclear or the intent was misinterpreted, the item was modified to address the ambiguity or misinterpretation that was revealed, and clinical interviews were conducted again with at least 10 precalculus level students (Carlson, 1995). This assessment tool was a precursor to the initial version of the PCA; fourteen of the 34 items served as the basis for the development of multiple-choice items for the first version of the PCA. The 14 items were selected because: (a) interviews revealed that only students who possessed the understanding or ability being assessed were able to provide a correct answer, (b) the items could be adapted to a multiple-choice format, and (c) the item was appropriate for precalculus level students. The process we employed, in general, to convert open-ended items to multiple-choice PCA items is illustrated later in this article.

Phase II. Since our goal was to design a test that would assess students' understandings and reasoning abilities needed for learning central ideas of beginning calculus, we engaged in a series of preliminary studies to investigate the process of acquiring these reasoning abilities and central ideas, including concepts of function (Carlson, 1997, 1998; Engelke, Oehrtman, & Carlson, 2005), limit (Carlson et al., 2002; Oehrtman, Carlson, & Thompson, 2008), rate-of change (Carlson et al., 2002; Engelke, 2007), and accumulation (Carlson & Bloom, 2005). In designing these studies we created new tasks and used many of our original 34 PCA items to probe specific reasoning abilities and cognitive processes exhibited by students. The data we collected provided categories of responses that we used in developing PCA item answer choices. The product of this work also led to the development of our first version of the PCA Taxonomy and associated PCA.

Phase III. Our next step was to start the process of establishing internal content validity. We began by administering the first multiple-choice version of the PCA to 146 students at the completion of a precalculus course. We then cycled through eight iterations of (a) administering the revised version of the PCA, (b) analyzing the quantitative data generated, (c) conducting follow-up interviews, (d) analyzing student written work and interview data, and (e) revising the PCA Taxonomy and 25-item version of the PCA. After completing eight iterations of this cycle we emerged with a stable 25-item version of the PCA; one in which all PCA distractors for each PCA item were selected by at least 5% of 162 precalculus level students.

In each administration of the PCA during Phase III, we directed students to show all work to justify the answer choice selected. They were also informed that they would receive a quiz grade for completing the assessment, two points for each item with one point awarded for justifying the answer choice and one point for selecting the correct answer. When analyzing the quantitative data, we identified answer choices that were selected by fewer than 5% of the students for possible replacement. Alternate distractors were developed by analyzing written responses to identify answers that emerged in student work that we had not previously encountered. We then targeted these students for clinical interviews. During the clinical interviews students were asked to verbalize their thinking as they completed those items. We also continued to cluster the common reasoning patterns for each answer choice for each item. As a result of analyzing the data from these interviews, we made minor changes in the wording of some items, and we replaced 16 (of the 125) distractors, the ones chosen by fewer than 5% of students (this process is illustrated and described in more detail in the following section).

Phase IV. Subsequent to establishing internal content validity of the PCA, we turned our attention to examining the meaning of a PCA score. During this phase we administered the PCA as a pre-post-assessment to precalculus level students and inservice secondary teachers. We examined pre-post-PCA mean scores and PCA subscores to determine if the PCA is sensitive to instructional interventions.[2] Following instructional treatments, we have observed pre–post-shifts in PCA mean scores and in PCA subscores in particular PCA Taxonomy categories (e.g., understanding the meaning of function composition). After establishing the content validity of the PCA in Phase III, and following the recommendation of Lissitz and Samuelson (2007), we examined external measures to further establish PCA's validity as a tool for determining students' preparedness for beginning calculus. We did this by correlating students' post-course PCA scores with their course grades. Results of these analyses are described in more detail later in this article.

The following section illustrates and elaborates on our method of refining PCA questions and validating PCA answer choices for the 25 items that appear on the final version of the PCA. We note that the following description does not follow the four-phase development process in temporal order. Instead, we provide a detailed description of how we implemented the above Phases for select PCA items. The items discussed in the following section were selected because they include different areas of content focus relative to the PCA sub-dimensions and are representative of diverse issues we encountered during the validation process (e.g., conversion of an open-ended word problem to two different PCA items and multiple iterations to settle on an item's answer choices).

ILLUSTRATING THE PROCESS OF PCA ITEM REFINEMENT

In this section we illustrate and discuss the process of converting two research tasks into PCA multiple-choice items. Following these descriptions we conclude by illustrating the process we employed to validate PCA item distractors. As will be revealed in these discussions, converting an interview-based item that was used productively in research to a multiple-choice item involved a lengthy process. The work on each item involved: (a) identifying the most common student responses, (b) developing answer choices for each response type, (c) re-administering the item to at least 50 students, (d) identifying incorrect answer choices that were selected less than 5% of the time, (e) administering the item in an open format and/or conducting clinical interviews to identify new answer choices, (f) revising the answer choice, and (g) repeating steps *c* through *f* until all answer choices had been selected by at least 5% of students who completed the newly revised multiple-choice item. In this section we also illustrate how interview data from complex open-ended items led to our identifying multiple important reasoning abilities or understandings, each of which could be converted to a multiple-choice item to assess a narrower range of abilities (as defined in the PCA Taxonomy) than the original open-ended item. We conclude the section with an illustration of how the interview data informed the PCA Taxonomy classification for PCA items.

[2]The subscores are generated from scoring clusters of items that are associated with each category of the PCA Taxonomy.

Refining a Covariation Item

The initial version of the Bottle Problem (Figure 1) was designed to assess students' ability to construct a graph of a dynamic situation in which two quantities covary, specifically focusing on the ability to engage in MA1-MA3 level covariational reasoning.

The most common response provided by precalculus students was drawing a straight line that rises to the right, i.e., an increasing linear function (Carlson, 1995, 1998). When asked to explain the rationale for this response one student said, "As the water comes in, the height goes up." Such students appeared to attend only to the direction in which the height changed while imagining increases in the amount of water (MA2 of the covariation framework). Most second semester calculus students in the study and some college algebra students constructed a concave up graph. Analysis of the interview data revealed that these students were imagining the water getting higher and higher in the bottle. To them, "higher and higher" in reference to water in the bottle meant "steeper and steeper" on the graph. They were not covarying the amount of volume of water with the height of the water in the bottle. Other college algebra students constructed a strictly concave down graph and provided a justification indicating that they were trying to construct a shape that matched the side of the bottle (Carlson, 1998). As was elaborated in our literature review, this type of error in which the shape of the graph is made to match some physical attribute of the motion or event is widely reported in the literature (e.g., Monk, 1987, 1992).

Students who gave the correct response (a concave down graph, followed by an inflection point where the graph becomes concave up, followed by a linear portion in which the slope of the line is the same as the slope of the tangent line at the point just before it becomes linear) were able to explain how changes in the amount of water were related to changes in the height of the water at various locations in the bottle. For example, one student explained, "The planes get bigger as you go up, and then they get smaller again. So, the water poured at the same amount would take more to fill up a circular plane with a bigger radius, so the height would slow down" (MA3). The vast majority of students, however, produced a line with an incorrect slope when representing the cylindrical part of the bottle. Since we wanted a multiple-choice item that assessed one primary covariational reasoning mental action (in this case, the ability to coordinate changes in two quantities as they covary (MA3)), we removed the neck of the bottle when administering the item in multiple-choice format.

Five-point scoring rubrics emerged from identifying the varied responses produced by students. The initial refinement of the rubrics resulted from reviews by two mathematicians and two mathematics educators reading and providing wording improvements and suggestions for other answer choices. Following this refinement, three mathematicians and three mathematics educators each scored 15 exams using the rubrics and noted expressions that were unclear in the text of the

Imagine this bottle filling with water. Sketch a graph of the water's height in the bottle as a function of the amount of water in the bottle. Explain the thinking you used to construct your graph.

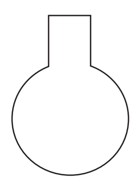

FIGURE 1 The bottle problem.

TABLE 3
Bottle Problem Rubric

Award one point for each of the following graphical features:
a) The graph of the function is always increasing
b) The initial part of the graph is concave down
c) The graph changes from concave down to concave up
d) The graph becomes linear
e) The slope of the linear portion of the graph is the same as the slope of the tangent line at the last point of the curve before becoming linear

rubrics. The rubrics were again refined and this process was repeated. The refined rubrics were then validated by having another team of experts (three mathematicians and three mathematics educators) score 10 exams. Comparison of these scores revealed a high level of consistency across reviewers in the scoring of open-ended items using the refined rubrics (Carlson, 1995, p. 32).

For the Bottle Problem rubric (Table 3) we drew from the analysis of student interviews. Option "a" awards one point when students recognize that the graph should rise to the right. Interviews revealed that these students possessed MA2 level thinking; they reported that they were imagining water pouring in and the height going up. Students who constructed an increasing straight line thus received one point for their construction. Data from the initial open-ended responses and interviews revealed that many students drew the first part of the graph concave down (a correct construction for the lower part of the bottle). However, they failed to attend to how the covariation of height and volume changed at the widest point of the bottle and thus did not include the concave up portion following the initial concave down portion. Students who produced this construction were awarded two points (one because the graph was increasing and one because the initial part of the graph was concave down). Students who received three points (one for a, one for b, and one for c) created a concave down, then concave up graph, but failed to recognize that the graph should become linear for the cylindrical part of the bottle. (Again, this was a common response in the interviews when students responded to the initial open-ended items.) Students who received four points created a linear graph for the cylindrical part, but with an incorrect slope (i.e., they received points for a, b, c, and d). Students who received all five points also made sure that the slope of the linear portion of the graph was the same as the slope of the tangent line of the graph just before it becomes linear. Use of this rubric required hand scoring with the grader determining which of the above features were present in the student's graph.

Converting the Bottle Problem to Multiple Choice Format

The conversion of the Bottle Problem to a multiple-choice format involved modifying the question to eliminate the cylindrical portion of the bottle. This modification was necessary to assure that students who were able to reason covariationally about the spherical portion of the bottle did not miss the problem by choosing an answer that was incorrect for the cylindrical part but still correct for the spherical portion of the bottle. Had we retained the cylindrical part of the bottle in the question, we would not have been able to claim that all students who chose an incorrect response did not engage in covariational reasoning. This was one of the few PCA items in which the

TABLE 4
PCA Answers and Associated Reasoning Patterns

Answer Choice	*Common Student Reasoning*
a	The water gets higher and higher on the bottle
b	Focuses on the left side of the bottle and matches its shape to the concave down graph (iconic interpretation)
c	Covaries the amount of water with change in height (correct answer, MA3) or change in rate (correct answer, MA5).
d	As more water is added to the bottle the height of the water goes up (MA2)
e	The water rises slowly at first, then faster, then slower again

validation procedures produced no new answer choices. The five graphs (of height as a function of volume) for the spherical portion of the bottle emerged during clinical interviews in past studies (Carlson, 1998; Carlson et al., 2002). The five answer choices (Figure 2) were included on the first version of the PCA and remained stable throughout Phase III of the PCA validation process.

Analysis of clinical interview data with 47 students showed that all students who chose the correct answer were able to engage in covariational reasoning. Most ($n = 38$) coordinated the amount of change of one variable with the amount of change of the other variable. For example, they described how the height of water in the bottle changed while imagining changes in the amount of water (MA3). The other nine students described how the rate at which the bottle was filling changed while imagining more water pouring into the bottle (MA5). Both groups illustrated correct reasoning patterns that led to their selecting the correct response. The students who selected each of the incorrect responses did not engage in covariational reasoning. A common response of students who chose answer *a* was to say that the upward shaped graph represented more and more height. Students who chose answer *b* commonly interpreted the graph as a representation of the shape of the left side of the bottle (Table 4). Further analysis of follow-up interviews revealed that 23 of the 38 students who chose and justified the correct answer by covarying the amount of water with the change in height were unable to explain what the graph's concavity conveyed

Assume that water is poured into a spherical bottle at a constant rate. Which of the following graphs best represents the height of water in the bottle as a function of the amount of water in the bottle?

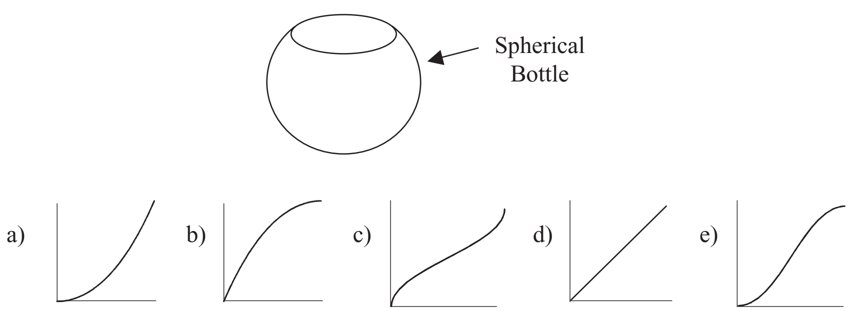

FIGURE 2 PCA Item 15: A covariation task.

about the *rate* at which the height changed with respect to the volume (MA5). This resulted in our classifying the bottle problem as assessing MA3 level covariational reasoning. Since our previous research had also revealed that second semester calculus students were unable to explain what a concave up or concave down graph conveys about the covarying phenomena (Carlson, 1998), we decided to develop a different item to assess only this ability. That item probed students' ability to covary the input quantity (the water) with the *rate of change* information that is conveyed by a graph. We provided students with five answer choices that characterized how the *rate* is changing for the concave down portion of the graph. The answer choices were (a) increasing at an increasing rate, (b) increasing at a decreasing rate, (c) increasing at a constant rate, (d) decreasing at an increasing rate, and (e) decreasing at a decreasing rate. As with all other items, the validation procedures described in Phase III were employed.

Converting a Composition Task and Linking PCA Items to the PCA Taxonomy

Carlson's (1995) early work on students' understanding of the function concept (Phase I of PCA development) included a contextual word problem (Diameter-Area Problem) that prompted students to define a formula and construct a graph to relate two varying quantities by initially composing two formulas (area vs. radius and radius vs. diameter). This ability has been shown to be essential for solving applied problems in calculus (Carlson, 1998; Engelke, 2007). As students read the words in the original Diameter-Area Problem they needed to set a goal to find a formula that accepts values for one quantity and produces values for another quantity—R1 of the PCA Taxonomy. They then had to (a) construct two algebraic formulas to relate the two varying quantities (RA), and (b) recognize the need to apply function composition (MC), and (c) use the idea of function inverse (MI) to relate the two quantities (i.e., diameter and area) as requested in the original problem.

> *Original Diameter-Area Problem*
> Express the diameter of a circle as a function of its area and sketch its graph.

In the Carlson (1995) study, only 2 of 30 high performing precalculus students (who received a course grade of A) were able to write the algebraic formula for this item, and none were able to sketch the graph. It was also the case that none of 16 high performing second semester calculus students were able to produce the graph, while 7 (of these 16) were able to write the algebraic formula. This resulted in our modifying this item to eliminate the directive to "sketch the graph" since this question assesses two major abilities, and even high performing second semester calculus student were unable to respond to this part of the question.

Follow-up clinical interviews with students from the three groups who constructed the correct formula revealed that these students illustrated the circle expanding with a picture that had concentric circles and varying lengths of the radius that were labeled with an r. After constructing this dynamic illustration of the event, they typically wrote the formulas $A = \pi r^2$and $d = 2r$. In doing so, they verbalized that they needed to combine these formulas to express diameter as a function of area. Their written work and explanations revealed that they viewed $A = \pi r^2$ as a means of finding a value of the area for any value of the radius. They also appeared to recognize that the formula $d = 2r$ provides a means for determining the length of the diameter for various lengths of radius. As these students kept in mind their goal to find the diameter of the circle as a function of the area, they often verbalized that finding this function would require the construction of a formula with d equal to some expression that contained the variable A. One precalculus student explained:

> Let's see, I need to write d equal to something, that something must have an A in it so I can put in area and get out diameter. I know that I can find area if I know the radius, and I know that I can find radius if I know diameter ... I just divide the diameter by 2. I know how to write a formula to get area when I know diameter, but ... what I really need is d equal to something in terms of A, so now I can solve this formula for diameter to get a formula that takes in area and puts out diameter.

As can be seen from this excerpt, this student began by articulating that his goal was to express diameter in terms of area. With this goal in mind he appeared to have little difficulty knowing to compose the two formulas to determine diameter as a function of area. As the need arose he also computed the inverse function by defining radius as a function of diameter. In addition, he was able to access and apply the algebraic procedures needed to determine his answer. His explanations of these actions suggest that he held a process view of function. In the second line of the excerpt he spoke of his goal to create a formula that accepts area as input and produces diameter as output. In the last line of the excerpt he stated that he had produced a formula that "takes A and produces a value for d." Analysis of interview data with the seven high-performing calculus students who also provided a correct answer to this question revealed that they each exhibited a process view of function.

The Emergence of Two PCA Function Composition Items

Analysis of interview data from 12 precalculus students who provided incorrect responses to the Diameter-Area function composition item pointed to a bifurcation of student responses into two general categories. The first group consisted of students ($n = 7$) who did not recognize the need to compose two formulas as part of the process of defining the circumference of a circle as a function of its area. They successfully wrote formulas for area (in terms of radius) and diameter (in terms of radius), but they failed to establish a goal to write diameter as a function of area, resulting in their subsequent algebraic steps being unproductive. Students in the second group ($n = 5$) also wrote correct formulas for area and diameter ($A = \pi r^2$ and $d = 2r$) but, unlike those in the first group, they managed to express a goal to write diameter in terms of area. However, they were unable to make progress. Four of the five students stated that they did not know what to do next. They failed to recognize the need to invert the diameter formula, $d = 2r$, as a necessary next step in their expressed goal to construct the formula for diameter as a function of area. These distinctly separate response categories uncovered during Phase I led us to develop two different PCA composition items in Phase II to assess the two primary understandings of function composition required by the original Diameter-Area Problem.

One of these new items (PCA item 17) assesses only the ability to recognize the need to compose two functions to obtain a new function. The second new item (PCA item 4) requires that students invert one function prior to composing two functions to obtain the required composite function.

Developing and Validating Two Function Composition Items

Discussions of the reasoning patterns and understandings needed to solve a particular problem are provided below to illustrate how the interview data generated during Phases I, II, and III of the development process informed the PCA Taxonomy classification for each PCA item. The composition word problem (item 17 discussed above) was designed to assess students' ability to use function composition to write one quantity in terms of another. The problem asks students to express the area of a circle in terms of the time (in seconds) for the situation where the circle is created by a ball hitting the water and traveling outward at a speed of 5 cm per second. Our intent in designing this item was to remove the complexity of inverting a function (or reversing the function process) in the Diameter-Area problem.

Interviews conducted during the validation process revealed that, as with the Diameter-Area composition task, answering this item (Figure 3) entails thinking of a function as a process that accepts input to produce output (R1). In the process of relating area and time students also recognized that $(5t)^2$ is equal to $25t^2$, requiring use of algebraic procedures to obtain the final form of the solution (R3). Although it might be the case that not viewing $(5t)^2$ as $25t^2$ suggests that a student does not have a process view of function, our interview data did not provide compelling evidence for that conclusion.

After iterating the cycles (described in Phase III) to refine the answer choices, we administered the PCA to 672 precalculus students. Follow-up interviews with 12 students from the 114 (17%) who had selected the correct answer for item 17 revealed that that each of these students constructed a dynamic image of a circle rippling outward. Their utterances during the interviews support that these students were imagining time as an input and radius as an output. They then re-conceptualized the radius in terms of the time and constructed the formula to determine the area of a circle for varying amounts of time (R1). Many students who failed to produce the correct answer were also unable during follow-up interviews to coordinate the output of one process with the input of another. These students were only able to seek patterns and provide justifications such as, "There's a 5 they gave me for some reason, and I'm going to want to use it. So, I'm going to guess d. D's the only one with a 5 in it."

A ball is thrown into a lake, creating a circular ripple that travels outward at a speed of 5 cm per second. Express the area, A, of the circle in terms of the time, t, (in seconds) that have passed since the ball hits the lake.

a) $A(t) = 25\pi t$

b) $A(t) = \pi t^2$

c) $A(t) = 25\pi t^2$

d) $A(t) = 5\pi t^2$

e) $A(t) = 10\pi t$

FIGURE 3 PCA Item 17: A composition contextual problem.

TABLE 5
Answer Choices and Rationale for PCA Item 17 (672 Precalculus Students)

Answer Choice	*Percent*	*Approach*
a	21	Failed to speak about a function as a process and failed to recognize the need to square *t*
b	7	Expressed that the answer matched the form of the area function
c	17	Correct answer (common reasoning described above)
d	34	Failed to speak about a function as a process and did not recognize the need to square 5 when squaring the expression for the radius.
e	20	Failed to speak about a function as a process; multiplied 5 by 2 and provided no logical rationale for this decision
x	1	No Answer

Phase III validation of item 17 produced percentages of students (from administering PCA to 672 precalculus students) who selected each answer choice. The common reasoning that led to each answer choice emerged from analyzing data from clinical interviews with students (Table 5).

As can be seen by reviewing the PCA Taxonomy, the processes employed in Phases I–III revealed that item 17 provides information about students' performance on the following reasoning abilities and understandings: process view of function (R1); ability to identify and apply appropriate algebraic manipulations and procedures (R3); understand the meaning of function composition (MC); understand, construct and use an algebraically defined function (RA); and interpret contextual function information (RC).

A Composition-Inverse Item

A second function composition item (Figure 4) was developed to assess students' ability to recognize the need to invert one formula, prior to composing two formulas. Since we recognized that this ability was lacking in some students when responding to the original Diameter-Area Problem, we wanted to devise an item to isolate and assess this ability. We created a second contextual composition PCA item that required inverting one function (rather than two as in the Diameter-Area Problem described earlier) as they worked toward the goal of relating two varying quantities.

Which of the following formulas defines the area, A, of a square in terms of its perimeter, p?

a) $A = \dfrac{p^2}{16}$

b) $A = s^2$

c) $A = \dfrac{p^2}{4}$

d) $A = 16x^2$

e) $p = 4\sqrt{A}$

FIGURE 4 PCA Item 4: A composition and inverse contextual problem.

Item 4 asks students to represent the area, A, of a square in terms of its perimeter, p. Completing this problem required students to initially recall the formulas for area and perimeter of a square, $A = s^2$ and $p = 4s$. Constructing a formula to define the area, A, in terms of the perimeter, p, requires inverting the formula $p = 4s$ to obtain the new formula, $s = \frac{p}{4}$. The formula $A = s^2$ is then composed with $s = \frac{p}{4}$ to obtain the formula, $A = (\frac{p}{4})^2 = \frac{p^2}{16}$. The answer choices for item 4 emerged after four cycles of (a) administering the PCA, (b) analyzing the quantitative data generated, (c) conducting follow-up interviews, (d) analyzing student written work and interview data, and (e) revising the PCA Taxonomy and 25-item version of the PCA (Phase III). The answer choices for this item emerged from analyzing student work during the first two administrations of PCA. Follow-up interviews in subsequent administrations of PCA revealed the common student reasoning that was associated with each answer choice. A more detailed description of how we selected particular answer choices for PCA items is provided below.

As with item 17, a student who provides a correct answer to item 4 is able to view a function as a process that accepts input and produces output (R1), apply appropriate algebraic manipulations and procedures to support reasoning about the creation of a new function model (R3), understand the meaning of function composition (MC), understand how to interpret a word problem context and represent this contextual information (RC) using an algebraic formula (RA). But unlike item 17, this item also assesses students' ability to understand and use function inverse when solving a contextual problem (MI).

Phase III of the validation process for item 4 produced percentages of 672 precalculus students who selected each answer choice. The common reasoning that led to each answer choice resulted from analyzing data from interviews with students (Table 6). For this item the answers were finalized after four iterations of administering the item, analyzing written responses, revising the distractor and re-administering the item (Phase II of the validation process).

The methods for determining and validating the multiple-choice options for select PCA items are described and illustrated in the next section.

THE REFINEMENT OF PCA ITEM ANSWER CHOICES

During Phase II of the PCA development the original answer choices for each item emerged from interviews with students who completed the open-ended version of the task. In cases where

TABLE 6
Answer Choices and Rationale for PCA Item 4 (672 Precalculus Students)

Answer Choice	*Percent*	*Approach*
a	25	Correct answer
b	20	Finds area as a function of its side
c	34	Does not square denominator
d	3	Plugs in 4s and squares to get area
e	18	Finds perimeter as a function of area or just recognizes the relationship by substituting numbers
x	1	No answer

five answer choices did not emerge from this analysis, the original developer (Carlson) devised answers that she conjectured were likely candidates for student incorrect responses. In many cases we included "none of the above" as the fifth answer choice. This decision enabled us to identify new answer choices by examining written solutions of students who selected "none of the above." The refinement of PCA answer choices during Phase III involved eight iterations of administering PCA in multiple-choice format to groups of at least 50 students (actual numbers varied for each item since not all items were validated at the same time).

When analyzing the quantitative data after each administration, distractors selected by fewer than 5% of the students were removed and replaced prior to the subsequent administrations. The replacement distractor was obtained from reviewing the written work of students who had selected "none of the above." During interviews with these students we prompted them to explain the reasoning that supported their answers and written work, and as a result we sometimes noticed that not all students had based their selection of a particular distractor on reasoning that we had previously anticipated. This led to further modification of our answer choices until each distractor was satisfied by two criteria. First, it had to be selected by at least 5% (of over 200 students) and second, a common rationale for providing that answer had to emerge in the student interviews. In some cases this required eight cycles of (a) changing the distractor, (b) administering the item to at least 50 students, (c) analyzing the quantitative data, (d) conducting focused clinical interviews with 10 students, and (e) refining the answer choices and re-administering the multiple-choice format of the exam. This was a tedious process that played out over a period of four years as we cycled repeatedly through the five steps described above, until every distractor for all PCA items had been selected by at least 5% (of over 200 students) who completed that multiple-choice item. This process is illustrated for two PCA items.

Refining the Answer Choices for a Graphical Composition Task

A function composition task (item 5) was originally administered in an open-ended format to 47 precalculus and 32 beginning calculus students. This item provides the graphs of the functions, f and g, and asks students to determine the value of $g(f(2))$.

During Phases I and II of the study, analysis of interview data revealed that students who provided a correct response to this question: (a) possessed a process view of function (PCA Taxonomy, R1), (b) understood what it means to evaluate a function (ME), and (c) understood how to compose two functions (MC) given in a graphical representational context (RG). When the item was administered in an open-ended format, the most common answers provided by students were -2, 1, 0, and 2, resulting in our including these answers as options on the initial multiple-choice version of this item (Figure 5). The initial fifth answer choice was "none of the above."

Follow-up interviews revealed that five students who chose answer *e* were expecting an answer of 4. The students reasoned that they had found $g(f(2))$ by locating the point $(2, f(2))$ on the graph of f. They then drew a vertical line that intersected the graph of g at $g(2)$ (i.e., the point $(2, 4)$) and noted that 4 was the value of $g(2)$. In the words of one student, "So, f of 2 is negative 2, that we know, so what's g of negative 2? It would be 4" (the student put his finger on the point $(2, -2)$ on the graph of *f;* then followed this point straight up to the point $(2, 4)$ on the graph of g). This student followed by saying, "But 4 is not an option so it must be answer *e*."

TABLE 7
Answer Choices and Rationale for PCA Item 5 (672 Precalculus Students)

Answer Choice	*Percent*	*Approach*
a) –2	10	Determined the value of $f(2)$ only
b) 1	43	Started with an input of 2 and used the graph of f to determine that the output of f is –2; then used –2 as the input to g and determined that the output of g at –2 (y-value) is 1
c) 4	28	Determined that $f(2) = -2$, but used $x = 2$ as the input for g, thus finding $g(2)$
d) 2	6	Determined the value of $f(g(2))$
e) not defined	13	Determined that $f(2) = -2$; then located $x = -2$ and determined that $g(-2)$ is undefined

This data led to our replacing distractor *c,* an answer of 0 (it was selected by only 2% of 122 students), with the answer 4. Changing the answer choice *c* from 0 to 4 resulted in 28% of 672 precalculus students selecting this answer (see Table 7) on subsequent administrations of PCA. Follow-up interviews conducted in subsequent iterations of PCA also revealed that some students believed that since f is not defined for the input value 2, $g(f(2))$ would not be defined. We then replaced the answer choice "none of the above" with "not defined." Follow-up interviews did not reveal another answer choice. Our processes of verifying the common reasoning associated with each answer choice revealed that all 13 students who selected the correct response demonstrated a process view of function when providing their rationale for their answer choices. In contrast, none of the students who chose an incorrect response demonstrated a process view of function when discussing the thinking that had supported his or her response. The finding that a process view of function is needed for using function composition is consistent with findings that were reported in earlier studies (e.g., Carlson, 1995, 1997, 1998). Data from the administration of this item in a multiple-choice format to 672 students (Table 7) indicated the five answer choices all represented common responses of precalculus level students.

Refining the Answer Choices for an Average Rate of Change Item

The average rate of change problem (item 11) was originally administered in an open-ended format to 47 precalculus and 32 beginning calculus students. During Phases I and II of the study,

Use the graphs of f and g to evaluate $g(f(2))$

a) –2

b) 1

c) 0

d) 2

e) none of the above

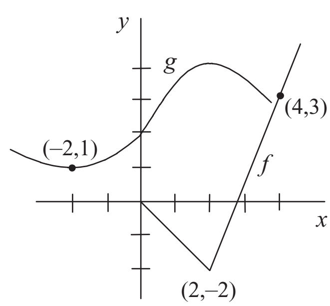

FIGURE 5 Initial PCA Item 5: Graphical composition problem.

analysis of interview data revealed that students who provided a correct response to this question possessed a process view of function (PCA Taxonomy, R1). They were also able to identify and apply appropriate algebraic procedures and calculations (R3) to construct an algebraic formula (RA) of the average rate of change of a contextual situation (RC)—the distance that a car had traveled. When administering the item in an open-ended format, the most common answers provided by students were 6 ft/sec, 10 ft/sec and 11 ft/sec. These three answers were included as answer choices for the initial version of PCA.

> *PCA Item 11: Average Rate of Change Problem*
> The distance s (in feet) of a car moving in a straight line is given by $s = t^2 + t$,where t is measured in seconds. Find the average rate of change (velocity) for the time period from $t = 1$ to $t = 4$.

All students who provided the correct response, 6 ft/sec, provided a reasonable justification for their answers. They indicated that they had divided the *change in distance* by the *change in time*. In the words of one student, "I computed $s(4)$-$s(1)$**,** giving 18 feet for the change in distance. I then computed the *change in time* by subtracting 1 from 4, so since the average is *change in distance* over *change in time* ... the answer has to be 6." For the response 10 ft/sec, most students indicated they were thinking of the arithmetic mean instead of the average rate of change. A student replied to a follow up probe to explain his answer by saying, "I added them up and divided by 4 ... it is like finding my average test score. I add up the test scores and divide by the number of tests." Analysis of responses from students who answered 11 ft/sec revealed that they thought computing the average involved adding two numbers, $s(1)$ and $s(4)$, and dividing by 2. Since the question asked for the average change in distance for the time interval from 1 second to 4 seconds, these students computed the values $s(4)$ and $s(1)$, then added these values and divided by 2. These three items were placed as answer choices on the first multiple-choice version of PCA. We also included the answer 18 because we conjectured that some students would only compute the change in distance, $s(4) - s(1)$. We included "none of the above" as our fifth answer choice in hopes of finding other possible distractors by examining the work of students who selected this option.

Analysis of the quantitative data after administering the first version of PCA to 98 precalculus and 52 college algebra students revealed that less than 3% of these students selected the answer $s(4) - s(1)$. Analysis of written work of students who chose "none of the above" revealed that some students had computed $s(4)$; then they divided by 4 to obtain an answer of 5 *ft/sec.* Consequently we included this answer choice on the second version of PCA in place of the answer $s(4) - s(1)$. We also retained the answer choice, "none of the above." Analysis of the quantitative data from administering the second version of PCA revealed that 9% (of the 84 students) selected the response, $\frac{s(4)}{4}$. Student work of those who chose "none of the above" did not reveal other common responses on this administration of PCA.

No new answers emerged in the third administration of this item to 53 precalculus students. Therefore we retained the same answer choices when we administered the fourth version of PCA to 86 college algebra and 123 precalculus students. Analysis of student work on this item revealed that students who chose "none of the above" commonly provided the calculation, $\frac{s(4)-s(1)}{2}$. We then conducted follow-up interviews with 12 students who had provided this response. Analysis of the interview data revealed that most were viewing $s(4)$-$s(1)$ as the *change in distance*, but they also thought that computing the average involved dividing by 2. When prompted, one student

TABLE 8
Answer Choices and Rationale for PCA Item 11 (1,501 Students)

Answer Choice	*Percent*	*Approach*
a) 5 ft/sec	13	Computes: $\frac{s(4)}{4}$
b) 6 ft/sec	15	Correct: $\frac{s(4)-s(1)}{4-1}$
c) 9 ft/sec	14	Computes: $\frac{s(4)-s(1)}{2}$
d) 10 ft/sec	38	Computes: $\frac{s(4)+s(3)+s(2)+s(1)}{2}$
e) 11 ft/sec	19	Computes: $\frac{s(4)+s(1)}{2}$

explained, "I need to divide by 2 to average two numbers ... I computed $s(4) - s(1)$ to find the change in distance ... then, to get the average, I divided by 2." This answer choice was included on this item along with the other four answer choices. Analysis of the quantitative data from 123 precalculus students revealed that 12% of these students selected this new answer choice. The item was subsequently administered to a total of 1501 precalculus, college algebra, and calculus students. The percentage of these students who selected the final five answer choices, along with the common computation that led to each response, is provided in Table 8. In follow-up interviews, some calculus students arrived at the correct answer incorrectly, by computing the arithmetic mean of derivatives evaluated at either two or four points: $\frac{s'(1)+s'(4)}{2}$ or $\frac{s'(1)+s'(2)+s'(3)+s'(4)}{4}$. Since s is a quadratic function, these computations produce the correct answer, but the students did not express an understanding of the meaning of average rate and expected their procedure to work for any other function type. No precalculus or college algebra students provided an incorrect justification for the correct answer. Since the assessment is focused on precalculus, we did not revise the item to eliminate this possibility (for example, by choosing a cubic function) to avoid conflation of computational and conceptual errors.

As has been illustrated by the above two examples, the process of obtaining five answer choices that are selected by at least 5% of students requires detailed analysis of student reasoning. Determining the common thinking associated with each answer choice can also become a lengthy process that involves interviewing a large number of students.

VALIDITY AND RELIABILITY OF THE PCA

As shown in the PCA Taxonomy, the PCA assesses reasoning abilities and conceptual understandings that interact in complex ways. Each PCA item provides information about students' reasoning abilities and understandings from multiple categories in the PCA Taxonomy, and no two items assess the same combination of categories. For example, items 4, 5, and 17 require that the student exhibit a process view of function (R1) and an understanding of the meaning of function composition (MC), but they vary in their use of function inverse (only item 4 requires an understanding of function inverse) (MI). They also vary relative to the representational context in which students are asked to interpret, use, or construct a function (RG, RA, RC). As such, it is not reasonable to expect that a student who selects the correct answer on item 17 would necessarily select the correct answer on items 4 and 5. Consequently, establishing uni-dimensional scales for measuring a single variable or the same combination of variables would require that PCA contain a cluster of items that assess exactly the same categories of the Taxonomy combined in exactly the same way. Thus establishing meaningful correlations among items would require a prohibitively long instrument. Fortunately, our qualitative studies provided confidence that each item did indeed assess the PCA Taxonomy sub-dimensions associated with that item. They also led us to the conclusion that it is not feasible to reduce the factors assessed by such an instrument to a manageable number without it becoming too narrow to adequately assess the intended reasoning abilities and understandings that are foundational for understanding central ideas of beginning calculus. We chose to develop a representative, streamlined instrument with little redundancy among the ways in which theoretical categories (i.e., the sub-dimensions of the PCA Taxonomy) are combined in the items.

As a result, no two PCA items assess exactly the same combination of understandings and reasoning abilities articulated in the PCA Taxonomy. The PCA assesses the composite effect of the identified reasoning abilities and understandings on students' abilities. This approach is consistent with viewing the complex interactions among categories in the PCA Taxonomy as producing an emergent effect (Cohen, Cohen, Teresi, Marchi, & Velez, 1990) related to important precalculus reasoning abilities, rather than trying to establish independent uni-dimensional measures of underlying latent variables. We therefore anticipate low inter-item correlation, which is typical for emergent variables and is also born out in our data.

Despite the treatment of the PCA score as an emergent variable, we found that 592 of the 600 pair-wise correlation coefficients were positive. Thus the 25-item test is able to achieve a Cronbach alpha of 0.73 (see Table 9) indicating some degree of overall coherence. Since the theoretical and empirical structure is multidimensional, we caution against interpreting this alpha value as a standard measure of reliability and attributing any observed effects to particular dimensions (DeVellis, 1991; Netemeyer, Bearden, & Sharma, 2003). We argue instead that the value of an individual PCA score is based on the validity of students' explanations for answer choices in the clinical interviews and through correlation with other measures such as the calculus course grades described in a later section. Thus, we have confidence interpreting an individual student's score as a broad indicator of reasoning abilities and understandings relative to the PCA Taxonomy.

As shown in Table 9, the alpha coefficients for each of the categories in the Taxonomy range from 0.19 to 0.62, reflecting the interaction effects between categories in the PCA Taxonomy and the limited number of items assessing each category to achieve a streamlined instrument. This has resulted in our restricting our interpretations of these subscores to assessment of groups of students. For example, while it would not be appropriate to draw inferences about the abilities of an individual student relative to PCA subscores, class-level data might provide sufficient aggregation to assess PCA Taxonomy categories for which curriculum and instruction were more or less effective. Since we refined PCA items through multiple iterations until student explanations stabilized, most who respond with the correct distractor should be reasoning with the standard pattern identified in our research. Our confidence in inferring the underlying reasoning pattern for correct answers is strong since over 300 interviews with students during the instrument refinement process produced no evidence of a student selecting a correct answer and not demonstrating the reasoning abilities that we claim the item assesses. However, it is possible that when selecting an incorrect response, some students might have relied on idiosyncratic reasoning patterns that differ

TABLE 9
Post-Course Subscores and Reliability Coefficients from 902 Precalculus Students

Category	% Correct	Alpha
PCA Total	41	0.73
Reasoning abilities		
Process	35	0.62
Covariation	50	0.46
Computational	42	0.41
Understand meaning of function concepts		
Evaluate	52	0.52
Rate	26	0.38
Composition	41	0.47
Inverse	31	0.43
Understand growth rate of function types		
Linear	33	0.27
Exponential	38	N/A
Rational	56	0.19
Nonlinear	46	0.43
Understand function representations		
Context	31	0.52
Graphical	41	0.57
Algebraic	40	0.47
Numerical	41	0.36

from the ones that we identified in our studies. Regardless of this possibility, distractor choices are highly reliable when applied to a group of students, such as those in a class, where overall trends are very likely to indicate the standard underlying reasoning patterns that our qualitative data revealed to be common for each answer choice.

PCA ASSESSMENT RESULTS

The PCA has been administered to thousands of students at over 45 post-secondary institutions since completing our validation of the final version of PCA. In our own work we have administered it to college algebra and precalculus students to determine the degree to which the instrument revealed useful insights about the student reasoning abilities and understandings described in the PCA Taxonomy. We have also used it to evaluate the impact of instructional interventions on student learning and to assess student readiness for calculus. The results from administering the PCA for these purposes are provided in the following sections of this article.

Assessing Precalculus Students' Learning

Mean scores for students who completed the PCA at the end of both college algebra and precalculus provide benchmark data for anyone interested in using the PCA. Because of the qualitative research that has guided the development of PCA items and the validation of PCA answer choices, analysis of student response patterns on both individual and clusters of PCA items are useful for gauging student learning. We report on select insights about student learning that emerged from examining PCA scores and response patterns on PCA items.

The final 25-item version of the PCA has been administered to students upon completion of college algebra and precalculus courses taught in small classes with different teachers at a large public university and nearby community college. The mean score for 550 college algebra students was 6.8 with standard deviation of 3.2. For 902 pre-calculus students, the mean score was 10.2 with standard deviation of 4.1. These scores are disconcerting given our data on students' success in calculus discussed below. Further analysis of clusters of items indicate that even at the end of precalculus, most students are procedurally focused, have not developed a process view of functions, and lack covariational reasoning abilities (see Table 9).

Students' strong procedural focus is illustrated by examining three questions on the PCA that ask students to evaluate a function composition at a given value, where the function information is presented using different representations for each item: algebraic, tables of values, and graphs. When 902 precalculus students were asked to compute a function composition for a particular input value given an algebraic formula, 94% responded correctly (Engelke, Oerhtman, & Carlson, 2005). However, only 50% of the students answered correctly when the functions were presented graphically (item 5), and 47% answered correctly when the functions were given in a table of values. This result and corresponding interview data suggests that these students were proficient in carrying out the procedures for composing two functions defined in an algebraic representational context, while they were less proficient in composing two functions in a graphical or numerical representational context.

Two additional questions imbed composition in the context of other problems (items 17 and 4 above). Item 4 asks students to construct an algebraic formula of a function that defines the area of a square in terms of its perimeter. Item 17 asks students to find the area enclosed by a circular ripple in a pond as a function of time, given that the radius increases at a constant rate. While 24% of the students could find the area of a square as a function of its perimeter, only 16% were able to determine the area of a circle as a function of time. Students' lower success on composition problems when presented in non-algebraic representations or embedded in context indicates that these problems reveal weaknesses in students' process view of function. We claim that a process view of function is necessary for providing a correct solution to this cluster of PCA items, although students might miss these problems for other reasons as well (e.g., computational or numerical weaknesses). Interview data (see validation discussions above) for each of these items supports this interpretation.

Assessing College Algebra Students' Learning and Reform Program Evaluation

An alternate 25-item version of PCA[3] was used to assess the effectiveness for a College Algebra Redesign (CAR) project at our institution that involved use of curriculum that had a strong

[3]The alternate 25-item version contained 15 items from the version of PCA that is the focus of this article. Student scores on these 15 items were correlated highly with students' overall PCA score. We added 10 new items that were validated using the same procedures as described in this article. Four items assessed students' covariational reasoning abilities; three items assessed students' ability to represent rate of change information; and three items assessed students'

TABLE 10
Comparison of PCA Scores and Subscores for Students in Traditional College Algebra Classes and Students in the College Algebra Redesign (CAR)

	Traditional (n = *36)*		*CAR* (n = *143)*	
	Pretest	*Posttest*	*Pretest*	*Posttest*
Mean	7.9	8.7	7.4	11
SD	2.4	3.7	3.1	3.4
Process	26%	34%	24%	39%
Covariation	27%	33%	31%	52%
Computational	36%	40%	35%	51%
Evaluate	42%	51%	35%	56%
Rate	24%	29%	30%	39%
Composition	33%	44%	26%	63%
Inverse	27%	29%	24%	41%
Linear	32%	36%	39%	48%
Exponential	22%	25%	20%	37%
Rational	42%	22%	34%	38%
Nonlinear	26%	38%	30%	49%
Context	27%	31%	29%	38%
Graphical	27%	35%	27%	45%
Algebraic	35%	36%	33%	48%
Numerical	26%	26%	28%	37%

focus on promoting covariational reasoning ability in students. The average gain for students in the traditional sections was 0.8 points with very small gains on most of the sub-scores (see Table 10). The average gain for CAR students was 3.6 with notably stronger gains than the students in traditional sections in most of the categories, including those items involving standard computational skills. Examination of this data suggests that the PCA is able to detect shifts in student understanding relative to overall PCA score. We can also observe differences in student pre- post- shifts for the various understandings and reasoning abilities characterized by the PCA Taxonomy categories.

Assessing Readiness for Calculus

While the PCA was not designed for use as a placement exam, it performs reasonably well in this regard. We administered the PCA as a pretest to 248 students entering first-semester introductory calculus courses with six different instructors at our institution and compared final course grades with PCA scores. The most discriminating cut-off score was 50%. Specifically, 77% of the students scoring 13 or higher on the 25-item test passed the first-semester calculus course with a grade of C or better while 60% of the students scoring 12 or lower failed with a D or F or withdrew from the course (see Table 11). The correlation coefficient between the initial

understanding of the growth rate of exponential functions. The alternate instrument was intended to better reflect specific program objectives.

TABLE 11
Performance of First-Semester Calculus Students by PCA

	PCA Score > 12		*PCA Score < 12*	
	n	%	n	%
Pass (A, B, or C)	98	77	49	40
Fail (D, F, or W)	29	23	72	60
Total	127		121	

PCA score and final course grade was 0.47. In comparison, an Education Testing Services study reported data from five colleges and universities showing Math SAT has a correlation of about 0.19 with calculus grades while locally produced placement tests and the MAA placement test have a correlation with calculus grades ranging from 0.34 to 0.6 with a weighted average of 0.37 (Bridgeman & Wendler, 1989).

DISCUSSION

Creating the PCA entailed a four-phase development and validation process that began by reviewing the available literature on knowing and learning precalculus and beginning calculus. We then conducted a series of focused studies to characterize specific reasoning abilities and understandings that are essential for learning beginning calculus (Phase I). These studies resulted in the preliminary PCA Taxonomy and an instrument with open-ended questions to assess students' knowledge of the constructs articulated in the Taxonomy. In Phase II we conducted clinical interviews to validate the questions and identify the answer choices for the initial multiple-choice version. Our cognitive studies continued in Phase III as we validated all distractors for each PCA item. After administering the multiple-choice exam to students, we conducted interviews to characterize student reasoning that led to their selecting a specific distractor. In Phase III we also continued the process of ensuring that each question assessed the constructs that we intended (as described by the subdimensions in the PCA Taxonomy). This validation process required eight iterations of administering the exam and conducting follow-up interviews with students. In Phase IV we used quantitative data from administering the final version of PCA to establish the meaning of a PCA score.

We agree with Lissitz and Samuelsen (2007) that the process used in establishing content validity of an assessment tool is likely the most critical issue in instrument development. To have used some other instrument as a primary source for establishing the validity of PCA, rather than conducting our own empirical studies, would have assured our overlooking important constructs that are foundational prerequisites for learning calculus and are now included in the PCA. Clinical interviews with students during the PCA validation also made it possible to identify common responses for each PCA item and to know what reasoning patterns and understandings led students to select particular answer choices.

For those interested in developing a concept assessment instrument, we caution that the process of determining both the questions and the five answer choices can be laborious. However, in reflecting on our development process, we do not recommend skipping any steps since each step of the four Phases that we have described informed the next. We recommend establishing an organized database early in the process to efficiently track changes in the questions and answer choices and to document students' reasoning behind the selection of each distractor. In retrospect, had our database been more organized in the beginning, we might have had improved efficiency in validating PCA answer choices.

The PCA has already been demonstrated to be useful for assessing pre-post learning and for comparing various approaches to teaching precalculus and college algebra. We have established PCA baseline scores for both college algebra and precalculus that can be employed by others intending to use the PCA to assess student learning. Analysis of PCA sub-scores for all students in a course might also provide helpful insights into student proficiency in specific categories or subcategories in the PCA Taxonomy, thus providing useful information for curricular and/or instructional adjustments. Another potential use of the PCA is as a calculus readiness exam. Recommended cut points for PCA scores (see Table 11) can be advisory to students about their preparedness for calculus. In addition, the PCA provides both a theoretical foundation for precalculus level curricula and courses and a tool for assessing the effectiveness of a particular instructional approach. As a result, we are optimistic that the PCA Taxonomy and Instrument will aid curriculum designers and instructors in focusing instruction on essential knowledge for learning calculus. We are optimistic that, in these roles, the PCA can contribute to increased matriculation of students into calculus and improved student success in calculus.

Many high schools and community colleges across the United States have large numbers of students who leave their institution bound for a particular local university. Mathematics program evaluators at these feeder institutions want to know whether their students, having completed courses through the precalculus level, possess the background to compete favorably (or at least perform successfully) when they enter beginning calculus at the university. PCA results provide objective data that can be used to inform feeder institutions about the ability of their precalculus programs to provide adequate preparedness for calculus. (A local community college that feeds to our institution used the PCA as one component in their overall program evaluation.) In cases where PCA scores for an institution suggest the need for improvement, the subscores can indicate possible directions for curricular and instructional interventions.

In closing, we emphasize that the construct of instrument validity must be rethought to focus on cognitive processes that are essential for students' continued learning. We urge others who are involved in developing content-focused, multiple-choice exams to initially review the cognitive research in the content area in which the assessment is being developed as a first step in determining the content focus of the instrument. They should then view the development of the instrument as a continuation of a research program in that area. In mathematics and science education there is now extensive research documenting the processes involved in learning and understanding central ideas in many content domains. As such, it seems logical that a careful review of this body of research should be the starting point for instrument design and that qualitative studies as described in our four-phase development framework be continued as a natural process of producing a valid instrument.

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