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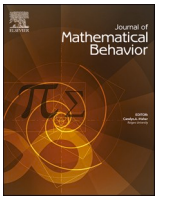

Considering, piloting, scaling and sustaining a research-based precalculus curriculum and professional development innovation

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ABSTRACT

We report a case study of scaling a *research-based curriculum and professional development innovation*. We describe the *Pathways Precalculus Curriculum and Professional Development* (PPCPD) project and provide an overview of its development and components. In doing so, we detail how research informed its development and refinement, illustrate why we claim the PPCPD innovation is research-based, and document ways in which it is educative for both instructors and students. We describe results from a case study in which the PPCPD scaled to 13 sites that piloted the innovation, 12 of which locally scaled the innovation and attempted to sustain its use. We report findings from survey and interview data that reveal key variables that led to the sites' sustaining or not sustaining the PPCPD innovation. We further highlight the importance of conceptualizing curricular scaling as an opportunity for continuous learning among the project leaders, local leaders, and precalculus instructors during all phases (considering, piloting, locally scaling, and sustaining) of the PPCPD.

1. Introduction

Many reports document U.S. university students' low persistence in precalculus courses and the barriers this creates to completing their degree programs (e.g., Bressoud, 2015; Hsu et al., 2008; Leyva et al., 2020). The National Council of Teachers of Mathematics (1980) report, *An Agenda for Action*, called for secondary and university/college mathematics curricula and instruction to support students' deeper engagement with mathematical ideas and habits of reasoning. Since then, many professional organizations have outlined similar goals (e.g., American Mathematical Association of Two-Year Colleges, 2006; Mathematical Association of America, 2018; National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). Unfortunately, few exemplars have shown precisely how U.S. mathematics precalculus instruction might systematically achieve these goals. The cultural emphasis on procedural learning is widespread and pervasive in U.S. classrooms (e.g., Boston, 2012; Litke, 2020; Stigler & Hiebert, 2009), thus perpetuating images of teaching and curricular supports that advance this focus. Although some projects (such as

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NSF-funded curricula) developed and implemented a more conceptually focused approach, most did not or could not provide the sustained support necessary for teachers to shift away from cultural images of procedural mathematics teaching (e.g., Dancy & Henderson, 2008).

In this study, we[1] report a case study of scaling a *research-based curriculum and professional development innovation*. We describe the *Pathways Precalculus Curriculum and Professional Development* (PPCPD) project, designed to support university precalculus students in engaging more deeply with mathematics ideas known to be critical for success in calculus; we also provide an overview of its development and components. In doing so, we describe how research informed its development and refinement and illustrate why we claim the PPCPD innovation is educative for both instructors and students. Once the innovation had proven effective at the developers' site (university), it was scaled to 13 other universities over 15 years.

The following research questions guided this study:

- What processes and attributes of the PPCPD innovation are associated with 12 universities considering, piloting, locally scaling, and sustaining it?

- What attributes of the PPCPD led to discontinuing or scaling back the innovation to fewer sections?

This study is unique in its focus on understanding the mechanisms that led to locally scaling (or not scaling) and sustaining (or not sustaining) a research-based precalculus innovation requiring substantial new learning for instructors. The case study demonstrated that during the PPCPD innovation's local scaling and sustaining phases, stakeholders needed to confront the misalignment between precalculus instructors' conceptions of precalculus mathematics and the PPCPD's research-based learning goals. The study's results further revealed that scaling and sustaining the innovation was associated with (i) an initiator at a local site learning about and valuing the projects' learning goals, (ii) departmental administrators valuing or not obstructing consideration of the PPCPD, (iii) initial professional development led by a Pathways developer that engages local leaders and instructors in understanding and valuing the project's learning goals, (iv) a local Pathways coordinator providing yearly two- or three-day professional development training for new Pathways instructors and weekly ongoing support for new and continuing Pathways instructors, (v) a site leader comparing student learning and performance in their current program with the Pathways project, and (vi) project developers and the site initiator working together to adapt the innovation to the site's local needs. Scaling back (two sites) was associated with discontinuing the weekly professional development, increasing student complaints, and subsequent withdrawal of administrative support for continuing the PPCPD. Piloting the PPCPD without scaling (one site) was associated with a change in the precalculus coordinator. We further highlight the importance of conceptualizing curricular scaling as an opportunity for continuous learning among the project leaders, local leaders, and precalculus instructors during all phases (considering, piloting, locally scaling, and sustaining the PPCPD).

In what follows, we provide an overview of the literature on scaling educational innovations and describe research that informed the PPCPD's development. Our reporting describes features of the research-based innovation that made it challenging to scale. We also discuss the innovation's educative features for supporting instructors' use of Pathways materials. We then present a general framework to illustrate common processes and variables across sites during each phase of considering, piloting, locally scaling, and sustaining the research-based curriculum and professional development innovation. We share results from survey and interview data gathered from the sites' project coordinators, select department chairs, and instructors, while highlighting barriers that we encountered and subsequent site and project adaptations. We conclude with our analysis of a purposive sample of four sites that represent important themes in the process of scaling (or not scaling) and sustaining (or not sustaining) the PPCPD innovation.

2. Background

2.1. Scaling curricular innovations

Scaling educational innovations is "a complex endeavor" that "often fails or is only successful to a limited extent" (Krainer et al., 2019, p. 395). Mathematics education researchers have identified barriers to scaling beyond a single teacher (local scaling) or single school (broader curricular scaling). These barriers include the need for sustained professional development to support teachers' relearning and the cost in both money and time to invest in activities such as ongoing professional learning communities (Krainer et al., 2019; Roesken-Winter et al., 2015). For educational innovations to succeed in scaling, they need (i) sustained professional development and teacher mentoring, (ii) adopters to perceive the innovation as personally beneficial, and (iii) options for potential adopters to pilot the innovation while evaluating and adapting the model for the local environment (Cai et al., 2020; Krainer et al., 2019). Consistent with these findings, Cai et al. (2020) call for mathematics educators to shift away from a *scaling as replication* perspective. Instead of seeking to replicate an innovation in its original form, they call for approaching scaling as an *ongoing process of adapting and learning* within new educational settings. Studying necessary adaptations for implementing innovations in different contexts can reveal new knowledge for improving those innovations more broadly. This perspective is consistent with implementation integrity—that of

[1] In this paper, "we" references the work of this author team. Two of the paper's authors are also Pathways developers, but the development team includes non-authors as well. Therefore, we will use the third person when referencing Pathways developers' perspectives and work. When speaking about Pathways developers' observations, experiences, and opinions, the members of this author team who are also developers consulted with other members of the development team to ensure the statements are accurate and representative of the entire Pathways development team.

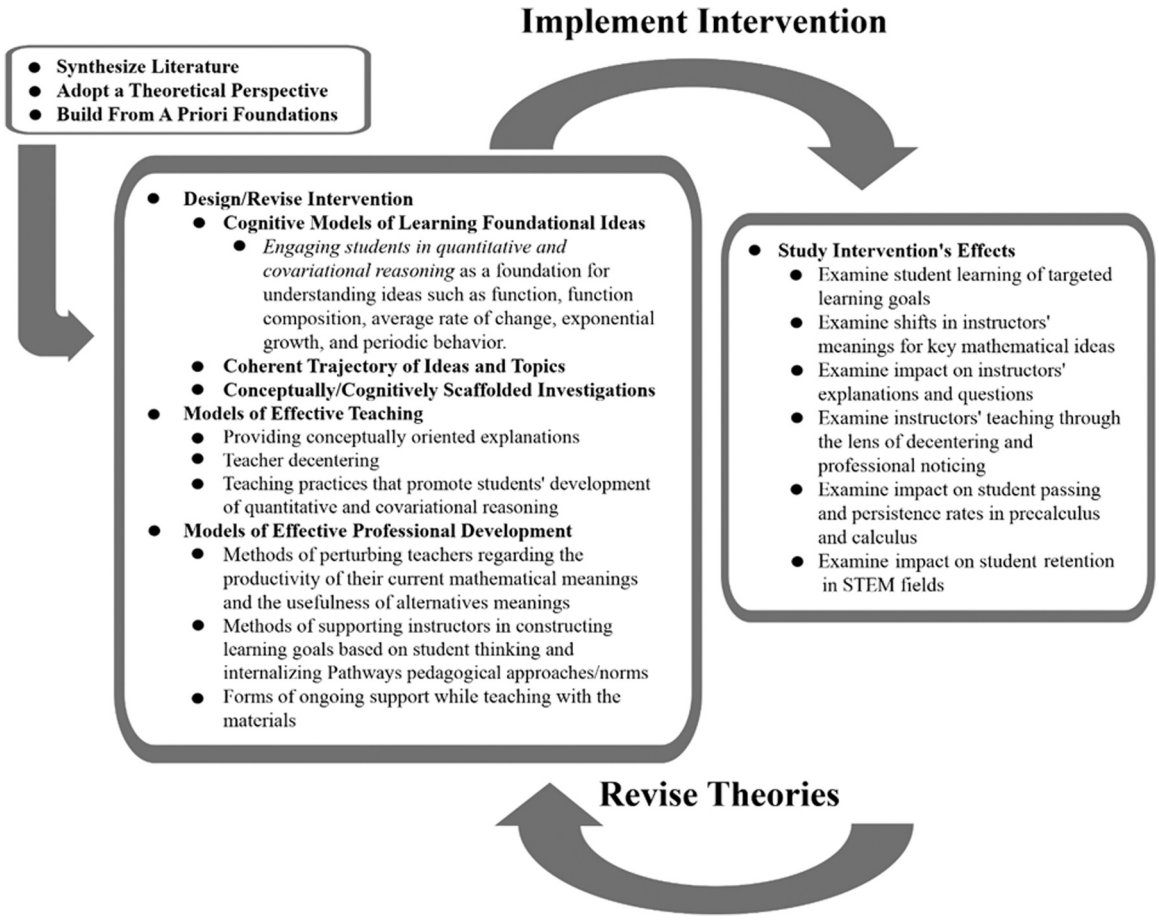

Fig. 1. The PPCPD ongoing development, research, and design cycle.

adapting the innovation to better support student learning in a local context (Cannata et al., 2021), rather than implementation fidelity—expecting that an innovation be implemented identically to its original design.

2.2. Mathematics curriculum development, scaling efforts, and professional development

Several innovative mathematics curricula have been developed and scaled over recent decades. A notable example is the NSF-funded calculus reform project that resulted in seven reform calculus textbooks designed to support students in developing a conceptual understanding of ideas through increased use of technology (Hallett, 2006; Hurley et al., 1999; Tucker & Leitzel, 1995). According to Tucker and Leitzel (1995), professional development workshops that engaged faculty in completing the more conceptually oriented problems accompanied early adoptions of the reform textbooks. At the height of the calculus reform movement, approximately one-third of all students taking calculus in the U.S. were using a reform curriculum. However, within a few years, the use of calculus reform textbooks had declined significantly after backlash from more traditionally oriented faculty (e.g., Windham, 2008). There is also no documentation of the workshops being sustained beyond the initial funding period.

Attempts to shift mathematics curricula and instruction toward more meaningful learning have been initiated at the K-12 level over the past 60 years. Most notable are two of 13 NSF-supported projects developed in the 1990 s: i) the middle school Connected Mathematics Project (CMP) and ii) the secondary (9−12) Core-Plus Mathematics Program (CPMP). Both curricula were successfully adopted, scaled, and sustained in many U.S. schools. Studies of student learning in these curriculum projects reported positive growth in students' reasoning and problem-solving abilities (e.g., Senk & Thompson, 2003). Despite this reporting, NSF-funded innovative curricula never held more than 20% of the K-12 mathematics market share as of 2006 (Education Market Research, 2006). Classroom observations of these projects' scaling efforts revealed that most teachers needed support to build richer connections of the mathematics ideas required to provide conceptual responses to student questions (e.g., Tarr et al., 2006). This finding points to problems with the U.S. system/approach to preparing and supporting mathematics teachers and is not the fault of individual teachers (Ma, 2010).

Studies have identified a teacher's mathematical knowledge for teaching as a primary influence on their pedagogical practices (e.g., Ball et al., 2001, 2008; Depaepe et al., 2013; Grossman et al., 2009; Hill et al., 2008; Moore et al., 2011; Silverman & Thompson, 2008: Shulman, 1986, 1987) and learning goals for students (e.g., Marfai, 2016; Tallman, 2021). Other studies characterized the relationship

between an instructor's mathematical meanings for teaching a specific idea, their image of how students learn that idea, and the instructor's explanations and questions (Baş-Ader & Carlson, 2022; Tallman, 2021). Researchers also documented that a teacher's mathematical meaning for teaching an idea can impact a teacher's ability to make sense of and act on a student's thinking (Carlson et al., 2024; Rocha, 2023). Repeated efforts to make sense of students' thinking as they learn an idea (decentering) can similarly impact the teacher's image of how students learn that idea. Carlson et al., 2024 and Rocha (2023) also claim that this evolving symbiotic relationship between an instructor's mathematical meaning for an idea and the instructor's decentering ability relative to that idea can advance through sustained efforts to support the instructor in reflecting on their students' learning in relation to their learning goals and instructional choices (for more details see Carlson et al., 2024). Thus, focused professional development is key for advancing a teacher's instructional practices, particularly when the professional development targets significant shifts in teachers' conceptions of a course's ideas to include research-based learning goals and a greater focus on understanding and advancing students' thinking (e.g., Baş-Ader & Carlson, 2022; Musgrave & Carlson, 2017).

2.3. Research-based curricula

Many curriculum projects claim to be based in research, but, as Clements (2007) noted, their use of the term "research-based" often does not reflect scientific research. The term frequently describes nonscientific "market research" that provides no information about the curriculum's impact on student learning relative to research-informed learning goals, nor does it explain how the textbook authors leveraged research on understanding and learning a course's ideas to inform its learning goals and design. In contrast, Clements defined criteria for a curriculum to be research-based that include (i) the curriculum development process is informed by prior research and practice, (ii) targeted evidence of the curriculum's effectiveness is collected along several dimensions, (iii) the curriculum's usability, including how instructors and students interpret elements of the curriculum's features, is evaluated, (iv) modifications to the curriculum are based on data collected, and (v) results and novel insights are disseminated to the education community. The relative lack of ongoing reporting of peer-reviewed qualitative research and validated quantitative data related to most mathematics curricula "calls into question much of what is currently used in classrooms" (Clements, 2007, p. 60).

2.4. Educative curriculum materials

Curricular innovations, especially research-based ones like the PPCPD built on current understandings about how students learn mathematical ideas, may target learning goals unfamiliar to instructors. Even with professional development, instructors may need additional help implementing the innovation in ways consistent with its design. *Educative curriculum materials* (Beyer et al., 2009; Davis & Krajcik, 2005; Taylor et al., 2015) describes materials that include features intentionally embedded in their design to advance teachers' pedagogical content knowledge and subject matter knowledge necessary to implement the curriculum successfully. These features[2] include (i) guidance on anticipating how students might think about activities within the curriculum, the source of these meanings, and how to respond, (ii) support for teachers in improving their understanding of subject matter content with a focus on connecting representations of ideas and fostering the discipline's unique practices, (iii) explicit support to help instructors see connections across units in the course so that the instructors can effectively plan and present daily lessons consistent with the course's "big ideas," (iv) developers' rationale to instructors about the reasons for developers' design choices "to integrate their knowledge base and make connections between theory and practice" (Davis & Krajcik, 2005, p. 5), and v) support for teachers' autonomy in the teaching process because teachers are knowledgeable about local conditions and must, by necessity, modify any curriculum materials to maximize students' learning potential.

In the following section, we discuss the PPCPD's initial and ongoing development, including how the developers addressed lessons learned from other NSF-supported curricula's scaling efforts, how the PPCPD is research-based, and why its design is consistent with the idea of producing educative curriculum materials to support instructor learning.

2.5. The PPCPD's research-based design and refinement cycle

The PPCPD—now in its ninth edition (Carlson et al., 2022)—has been repeatedly refined since its first edition in 2006. The 2006 PPCPD innovation consisted of a set of materials for student use, including an online textbook with embedded, conceptually focused videos and interactive applets. Each section of the online textbook has an accompanying cognitively scaffolded investigation that students complete during class. The investigations and written homework for each investigation are in a printed workbook that students bring to class. Developers designed the companion teacher resources and professional development (described in more detail in a later section) to be educative for instructors preparing to use the materials with their students.

A radical constructivist theory of knowledge development guided the development of the Pathways materials and professional development. A basic tenet of this theory is that every person must construct knowledge through their experiences (Steffe & Thompson, 2000). These experiences can be fostered through an individual's actions and reflections when working a problem, listening to an explanation, and through social interactions. Pathways developers' commitment to supporting students in constructing strong

[2] We synthesized these features from the following sources: Ball and Cohen (1996), Barab and Luehmann (2003), Ben-Peretz (1990), Brown and Campione (1996), Clandinin and Connelly (1991), Collopy (2003), Davis and Krajcik (2005), Heaton (2000), Petish (2004), Remillard (2000), Schneider and Krajcik (2002), Shkedi (1998), and Wang and Paine (2003).

meanings of precalculus ideas led to them designing interventions they conjectured would achieve this goal. Similarly, their design of instructor materials and professional development targeted instructors' construction of strong meanings for precalculus ideas, how they are connected, and how students learn them.

The developers began the materials design by synthesizing literature on how students understand key mathematical ideas (e.g., average rate of change, function composition, function inverse, exponential growth, and trigonometric functions) necessary to succeed in calculus (e.g., Carlson, 1997, 1998; Carlson et al., 2001, 2002, 2003, 2010; Infante, 2007; Jacobs, 2002; Moore, 2010; Smith, 2008; Smith & Thompson, 2007; Ström, 2008; Thompson, 1994). This literature provided the theoretical basis for their initial hypotheses and learning trajectories for precalculus ideas. These *hypothetical learning trajectories* (Simon, 1995) for each module (unit) and investigation informed the question scaffolding, videos, and interactive applet design. Pathways developers were learners themselves—they viewed their innovations as their best conjectures (at the time of development/adaptation) for supporting improvements in precalculus teaching and learning.

Pathways researchers also conducted classroom observations (e.g., O'Bryan & Carlson, 2016; Teuscher et al., 2011, 2016) to gain insights into an instructor's effectiveness in supporting student thinking and learning. This data also informed refinements of the instructor resources. When scaling Pathways to other sites, researchers continued to collect data to assess: i) the curriculum's impact on student learning (e.g., Hower et al., 2023; McNicholl et al., 2021), and ii) professional development's impact on instructors' meanings (e.g., Musgrave & Carlson, 2017; Ström, 2008) and their teaching practices (e.g., Marfai, 2016; Rocha, 2022, 2023; Rocha & Carlson, 2020; Tallman, 2015; Teuscher et al., 2016; Underwood & Carlson, 2012). As the PPCPD advanced, studies produced further refinements of the theoretical descriptions of how students learn specific precalculus ideas (e.g., Bowling, 2014; Kuper, 2018; Moore, 2010; O'Bryan, 2018). This cycle repeats indefinitely (Fig. 1).

The Pathways researchers also used the 25-item, multiple-choice Precalculus Concept Assessment (PCA; Carlson et al., 2010) to assess student learning of precalculus ideas foundational for learning calculus. Researchers validated this assessment to evaluate students' understandings (Carlson et al., 2010). The quantitative data from PCA documented Pathways students' learning and provided insights for further refinement and expansion of both the PPCPD materials and professional development. Sites considering the Pathways materials frequently use the PCA instrument as one data source for gauging the effectiveness of their current approach. These sites reported post-mean scores of less than 8 (out of 25) in their site's precalculus course. The site initiator frequently expressed that they and their departmental administrators were surprised by their low PCA mean score. These course coordinators, at sites where PCA was administered before their official pilot, conveyed that their precalculus students' performance on PCA influenced their decision to consider the Pathways materials and professional development (e.g., McNicholl et al., 2021). Since the PCA was validated to assess student understanding of key ideas of the function concept, and student scores on the PCA correlated with success in calculus,[3] developers continued to administer the PCA as a pre-post-assessment. This data, reported in the results section, provided one measure of Pathways students' learning.

Because we are reporting a case study, the purpose of which is to illuminate the rationale for and implementation of a set of decisions, why they were taken, and how they were implemented (Schramm, 1971; Yin, 2009), the results should not be assumed to be generalizable; nor is it possible to replicate this study. We now turn to elaborating the process of developing and refining the PPCPD. We also explain why we claim the PPCPD is both *research-based* and *educative*.

2.6. Designing the PPCPD's professional development model and instructor supports

Pathways developers designed the PPCPD to support instructor learning through both the curriculum and professional development provided by the Pathways professional development team and local leaders. This model of professional development aligns with a combination of what Roesken-Winter et al. (2015) describe as the personnel strategy (focused on supporting instructors in enhancing their professional knowledge and practices) and the material strategy (designing materials to be used by the local facilities to support ongoing instructor growth). Initial two- or three-day professional development workshops at a site introduce new Pathways instructors to Pathways learning goals and model strategies for engaging students in constructing productive mathematical meanings. Workshop leaders engage participants in discussing general and specific learning goals for Pathways precalculus students, including developing students' confidence and competence in using their mathematical thinking to make sense of problems and ideas independently. Workshop leaders also discussed attributes of effective instructors, including providing conceptually focused explanations, posing questions to advance students' thinking, and orchestrating learning experiences that support students in constructing strong meanings for the course's ideas. After these initial discussions, workshop leaders engaged participants in crafting solutions to capstone questions in the Pathways investigations that require conceptualizing and representing quantities in applied contexts and considering how the quantities' values are related and vary together. Exploring and explaining these covarying relationships became the primary way of analyzing new contexts and developing functions to represent dynamically changing quantities in applied contexts.

As workshop participants complete Pathways tasks, leaders ask them to reflect on and share the thinking they used when constructing their responses. Other participants are expected to pose questions when an explanation is unclear or fails to meet the standards of *speaking with meaning* (Clark et al., 2008). These activities support instructors in shifting their practices to focus more on understanding and developing their students' thinking. If needed, the workshop leader might also intervene by posing questions to illuminate productive and unproductive ways of thinking when responding to tasks. After participants share their thinking, the

[3] 77% of 248 Pathways precalculus students who continued on to calculus, enrolling in six different sections, who scored 13 or higher on the 25-item test passed the first-semester calculus course with a grade of C (Carlson et al., 2010).

workshop leader sometimes models a conceptually focused explanation that illuminates the following: (i) the clarity that comes with making a drawing of the quantitative relationships described in the problem statement; (ii) being specific in defining variables[4] for the varying quantities to be related; and (iii) consistently referencing quantities when developing and discussing algebraic and graphical function models. The materials' initial design aligns with the features of educative curriculum materials. The instructor support materials and lesson scaffolding encourage teachers to reflect on their own and their students' mathematical meanings as they use the materials to plan and deliver instruction. Modeling how to use the materials productively in workshop settings is key to initiating instructors' interactions with the curriculum and instructor resources in ways that support their continued learning as they begin to teach with the materials.

2.7. Pathways research-based learning goals

As mentioned earlier, research related to implementing the Pathways curriculum shows that instructors need support in shifting to value and embrace research-based learning goals that differ from traditional approaches to teaching mathematical ideas (O'Bryan & Carlson, 2016; Rocha, 2023; Underwood & Carlson, 2012). As a point of emphasis, it is understandable that precalculus instructors will need support in preparing to use the Pathways materials. This is because the over-arching learning goals for students' reasoning and targeted understandings for each precalculus idea have only been described in research papers and are not included in other precalculus curriculum materials.

We also aspire to support precalculus students (and instructors) in viewing precalculus as a coherent body of ideas that all students can conceptualize. As students move through the course, they are supported (through focused question scaffolding) in habitually conceptualizing quantities in a situation, considering how they are related (quantitative reasoning), and considering patterns in how two quantities' values change together (covariational reasoning) as a foundation for constructing algebraic and graphical function representations for each function type (linear, exponential, polynomial, rational, periodic, etc.). For a more detailed description of Pathways instructional conventions for supporting students' spontaneous engagement in quantitative and covariational reasoning, see Carlson et al. (2022).

As one example, we illustrate in the context of the idea of exponential growth how the PPCPD approach differs from traditional formula-first approaches which have been documented to be ineffective for both students (e.g., Carlson et al., 2010) and instructors (e.g., Ström, 2008). It has been documented that precalculus students upon completion of a traditional precalculus course have difficulty reasoning about the algebraic form of a doubling function (Carlson et al., 2010) and commonly provide $f(x) = x^2$ as an example of an exponential function. Traditional precalculus textbooks typically define exponential functions through their algebraic form while providing minimal (if any) meaningful discussion or examples exploring how the values of two quantities portrayed as related exponentially vary together. In contrast, Pathways materials include question sequences that ask students to explore doubling, tripling, or increasing by 25% some initial value over equal time periods since an initial population begins growing. As students consider how the number of decades since a population began growing and the size of a population (that is tripling each decade) are changing together, they reason about two key ideas developed in parallel. First, the lessons support students in seeing that the coordinated changes in quantities' values occur regardless of where the measurement begins within the function's domain. That is, the population triples when the time elapsed changes from 0 decades to 1 decade since the growth began, 3 decades to 4 decades since the growth began, or 2.145 to 3.145 decades since the growth began. Second, students begin to represent two triplings as 3^2 (two triplings being equivalent to one nine-tupling), five triplings as 3^5 (five triplings being equivalent to one 243-tupling), and t triplings as a 3^t-tupling (Kuper & Carlson, 2020). When given an initial population (such as 56,000 people), the question scaffolding supports most students in spontaneously representing the result of t triplings of 56,000 with the expression $(56{,}000)(3^t)$. The lessons continue by supporting students in considering non-integer interval sizes (such as what happens every year or month if the population triples every decade). Our lesson scaffolding supports students in conceptualizing a definition for exponential functions grounded in coordinated changes they can identify regardless of the representation used for modeling the relationship.[5]

2.8. Preparing to scale the PPCPD to other universities

As the Pathways materials (e.g., Ström, 2008; Moore, 2010) and professional development (Carlson et al., 2007) were locally scaled at the development site, developers repeatedly refined the PPCPD by engaging in multiple passes through the Pathways research-design cycle (Fig. 1). Our criteria for scaling the Pathways courses to other sites were that all Pathways sections at the development site achieved statistically significant gains in student learning on the PCA (Carlson et al., 2010) and that post-PCA mean scores at the development site were significantly better than the post-PCA mean scores of sections of precalculus at multiple universities using another approach (Carlson et al., 2010). The post-PCA mean score for the Pathways sections at the development site ranged from 12.2 to 15.9, compared to post-mean scores of 7.8 to 8.5 for non-Pathways sections at the development site and other universities.

[4] Research studies (Moore & Carlson, 2012; Carlson, O'Bryan, & Rocha, 2022) call for variable definitions to specify what is being measured while including the starting point, direction, and measurement units.

[5] The PCA includes an item to assess students' ability to represent exponential growth algebraically. Only 38% of 601 calculus I students (at the beginning of their semester) (Carlson et al., 2015) selected the correct answer, as compared to over 80% of students (at the end of a Pathways course) selecting the correct answer.

Because the instructors at the development site included graduate student instructors, full-time instructors, adjunct faculty, and tenure track faculty, the developers needed to train new instructors at the development site each semester. Through repeatedly introducing a new team of instructors to the Pathways materials at the development site, developers created and refined strategies for introducing new Pathways instructors to the ways of thinking entailed in learning, understanding, and teaching precalculus ideas meaningfully to students. These experiences gave the project leaders confidence that other sites could successfully use the materials, provided they received adequate support in understanding Pathways learning goals and preparing to lead discussions to support student learning of the ideas.

The PPCPD model scaled to other sites consists of educative materials and professional development that support teachers using (and learning from) the student materials and instructor resources. The Pathways materials that were scaled to the first five Pathways sites included (i) an online textbook with embedded videos, (ii) student investigations (aligned with each textbook section) with cognitive scaffolding to support students in constructing strong meanings for precalculus ideas, (iii) teacher resources that include written learning goals for each investigation and conceptually focused solutions that explain the thinking associated with solution steps, (iv) PowerPoint slides that include linked applets, all investigation questions and the associated conceptually oriented solutions and explanations, (v) written homework questions and exam question banks, and (vi) ongoing professional development training for all users. During the past seven years, developers added online homework sets and interactive online lessons designed to help students construct productive meanings for the same ideas targeted in the in-class investigations. The insights into effective ways to advance students' thinking and understandings have also informed updates to our instructor materials (e.g., updates to the notes in the instructor investigations and the creation of animated tools to engage students in exploring mathematical ideas). Our observations of instructors using Pathways materials continues to reveal both complexities and insights about approaches that are more (and less) effective for advancing instructors' conceptions and instructional practices. The PPCPD developers have made progress in making Pathways research-based materials educative and coherent for students and instructors. A study comparing the level of coherence of 10 precalculus textbooks (O'Meara & Vaidya, 2021) reported that Pathways Precalculus student materials were the most coherent among this collection.

3. Methods

In this section, we describe the methods for our case study of scaling the Pathways materials beyond the development site. We chose a case study methodology because our goal is to understand how and why universities considered, piloted, locally scaled (or did not scale), and sustained (or did not sustain) the PPCPD. Of the 13 sites that piloted, one did not scale the innovation beyond the initial pilot, 12 chose to pilot and subsequently scale the innovation locally, and 11 of the 12 adopters have sustained the innovation as of this writing. According to Schramm (1971) and quoted by Yin (2009, p.17), a case study's purpose is to illuminate a set of decisions, why they were taken, and how they were implemented. Consistent with the case study methodology, we describe the decisions that we and local stakeholders (e.g., site initiators[6] and department chairs) made during the piloting, local scaling, and sustaining phases of the Pathways innovation, including why and how adaptations were implemented. We also report survey data from the local coordinator of the 12 implementation sites that chose to adopt and locally scale the PPCPD. We did not invite the site initiator that did not scale the PPCPD to complete the survey because the survey questions primarily focused on decisions and processes beyond the initial pilot. However, we report data collected from this site initiator via a different email questionnaire that inquired about their pilot and the rationale for that site deciding not to scale or continue using the Pathways materials. Another data source was the author team, which provided historical and personal accounts of their experiences and descriptions of the innovation, how it was developed, and how it was adapted. Two of the authors participated in the research and development, and one of these authors led the initial design and research teams. Two other authors led the piloting and local scaling of the PPCPD at their respective universities, and one author served as the project evaluator.

3.1. Data collection

We began data collection for this case study by administering a 60-item written survey (Appendix A) to the Pathways Precalculus coordinator at each university. The coordinators described the processes and approaches they had personally experienced in paragraph-form responses. Clusters of questions probed for descriptions of how and why their department considered, piloted, locally scaled, and sustained the PPCPD. Other clusters of questions probed for descriptions of the coordinator's role in (and rationale for) considering and piloting the PPCPD, the nature of their weekly professional development meetings that included a request to provide a sample agenda of their weekly professional development sessions, their descriptions of barriers that emerged and how they responded, and specific actions they took to support their instructional team in advancing their students' thinking and understandings. Each coordinator was paid $200 for their time to complete the survey. While coding and compiling the coding results into one table, we identified new questions about the rationale for decisions and their implementation. This led to our designing interview protocols for conducting clinical interviews with four department chairs (Appendix B), six course coordinators (Appendix C), and five instructors (Appendix D). We designed the interview protocols to gain more specific information about a site's implementation and approach to navigating obstacles while piloting, locally scaling, and sustaining the PPCPD. Four of this paper's authors conducted these interviews.

[6] We refer to the person initiating exploring the PPCPD for potential use at a site as the site initiator.

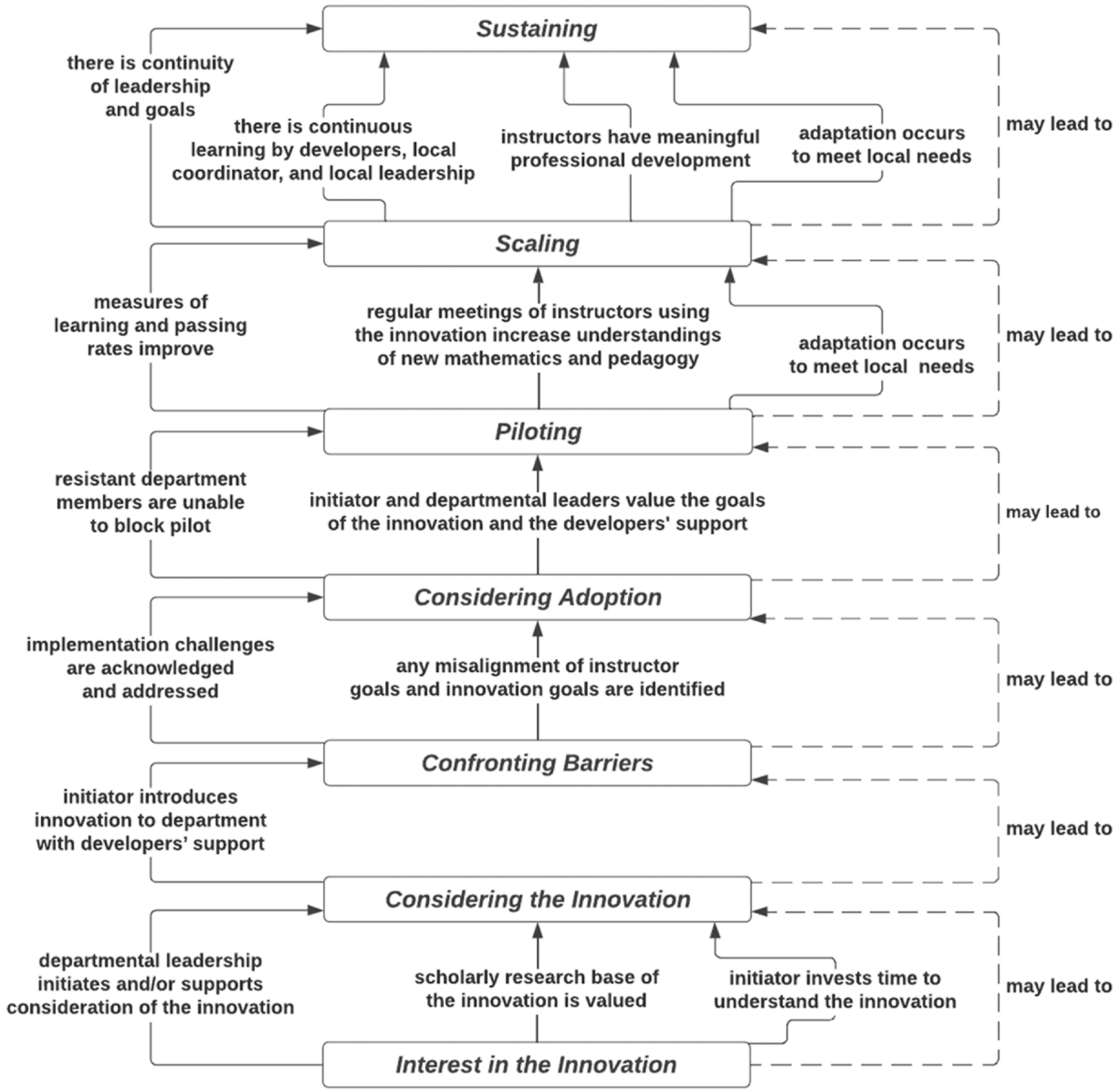

Fig. 2. Considering, Piloting, Scaling, and Sustaining an Instructional Innovation (The CPSSII Framework).

We selected the clinical interview subjects based on their diverse roles and approaches to scaling and sustaining (or not sustaining) the PPCPD. Specifically, we interviewed department chairs, course coordinators, instructors, and graduate teaching assistants at four sites. The four sites were selected because of the diversity in how they scaled and/or sustained the PPCPD, with (i) Site A sustaining for 10 years (as of this writing); (ii) Site B being sustained for 5 years, then discontinued; (iii) Site C reducing the number of sections and sustaining; (iv) Site D reducing the number of sections and adapting their local implementation, followed by their scaling the PPCPD back up to all sections.

We also note that data had been collected in prior years at each case site (according to each site's criteria) to measure the PPCPD's effectiveness. Since the criteria for effectiveness was not uniform across sites, these sites used a variety of quantitative assessments (e.g., PCA, departmental precalculus final exams), longitudinal data (such as student passing and persistence rates in precalculus and calculus), observations of Pathways classes, and student and instructor feedback. In compiling our results, we often did not have access to this data. However, many survey respondents and interviewees referenced these data sets in their responses. Together with the project leaders' personal experiences from site visits and the local coordinators' responses to the survey, these data sets became part of the data corpus for this case study. Collectively, this data was used to illuminate the rationale for the stakeholders' decisions and to inform ongoing adaptations to the PPCPD materials and professional development.

3.2. Data analysis

We used grounded theory tools of open coding and axial coding in our initial data analysis (Corbin & Strauss, 2015) and used pattern coding (Miles et al., 2020) in our second coding cycle. Consistent with pattern coding methodology, we grouped our labels and notes into a smaller group of explanatory themes and categories to advance our understanding of the phenomenon of adopting and scaling the PPCPD (our case); more specifically, the essential conditions and processes that led (or did not lead) to locally scaling and sustaining the PPCPD. We coded the coordinator survey responses to identify specific decisions and to understand why they were made and how sites operationalized the decisions. We continued by identifying similarity and diversity in responses as we attended to the causal conditions for a site considering the PPCPD, piloting and local scaling strategies, obstacles to adopting/scaling, and the consequences of adopting/scaling. As already noted, additional questions arose that prompted sampling of participants for clinical interviews. This process is consistent with Yin's (2009, p. 70) observation that a good case study investigator typically uncovers new leads when collecting and/or analyzing their data, resulting in the investigator pursuing additional lines of inquiry and seeking new data sources. We then performed pattern coding for the entire data corpus to identify themes, explanations, and relationships that aided in forming categories with explanatory power (Miles et al., 2020). Loosely holding our units of meaning early in the coding was critical to this process, as was a willingness to lay aside the more tenuous themes until categories emerged with high levels of empirical grounding (Miles et al., 2020). For example, our initial coding of the survey data identified professional development as a key variable for locally scaling Pathways materials to a larger group of instructors at a particular site. However, this initial coding did not specify the nature or duration of the professional development at a particular site. Our subsequent interviews and comparison of sites' implementation strategies further revealed that an important feature of the professional development was its primary focus on advancing instructors' conceptions of the ideas the curriculum targets and how those ideas connect across lessons and units. Our data collection and coding also revealed that sustaining the professional development training is important beyond the first few years after a site adopts the PPCPD. Through this process, we identified critical categories that led to answering our research questions. Critical categories for this analysis formed around key themes that either fit the data at all sites or were not contradicted by the data at any site and confirmed by the site initiator and/or a member of the Pathways implementation team. In the next section, we present the framework that resulted from this analysis.

4. Results

Recall the guiding questions for our inquiry include: 1) What processes and attributes of the PPCPD innovation are associated with it being considered, piloted, locally scaled, and sustained at each site?; and 2) What attributes of the PPCPD led to discontinuing or scaling back the innovation at a site? Open and axial coding results for each survey and clinical interview produced key themes for the data we compiled into one combined coding. Our pattern coding produced a grouping of the themes clustered as transitions between the general categories of our framework (Fig. 2) that characterizes the process of a site having an initial interest in the PPCPD to their sustaining the innovation for a minimum of 5 years.

The process typically began with a faculty member at a site expressing initial interest in learning about the PPCPD (see bottom box in Fig. 2). If the departmental leadership supported Pathways learning goals, these faculty subsequently invested time to understand and consider the innovation while identifying barriers (e.g., faculty resistance to professional development). If the site devised approaches for confronting these barriers (e.g., paying faculty to attend professional development), the site advanced to consider adopting the innovation and subsequently moved to pilot the innovation while collecting comparative data. It is noteworthy that departmental faculty who did not support Pathways learning goals were not able to block the site initiator and other faculty from moving forward with the pilot. If the pilot data revealed greater learning gains and retention of Pathways students compared to a site's current approach, the site moved forward to scale Pathways to more precalculus sections. The sites that retained a local coordinator who valued Pathways learning goals and continued leading on-site professional development for Pathways instructors have all sustained Pathways, some for as long as 14 years.

Arrows connecting general categories in the CPSSII framework illustrate key themes characterizing common reasons leading to the innovation being advanced from one category to another (such as "acknowledgement of implementation challenges" as one element in the process of transitioning from "confronting barriers" to "considering adoption" of the innovation). By following the links in the CPSSII framework, we see that a site's initial interest in the innovation might lead to the site considering the innovation, with reasons for considering the innovation including (i) the materials are research-based, (ii) one or more local leaders took time to learn about the innovation's resources (e.g., read and work through the investigations), and (iii) the innovation met a local need and/or was acceptable to the department chair. The CPSSII framework emerged from our data and provides a general framing of the phases and rationale for a Pathways site moving from their site initiator expressing interest in the Pathways research-based mathematics instructional innovation to a site scaling and sustaining the innovation. We also offer the CPSSII Framework as a more general guide to inform others working to scale their research-based educational innovation to new sites and locally scale their innovation to all sections at a particular site.

The following sections provide a narrative account of the detailed contexts from which we gathered our data. This narrative also includes the critical categories that comprise the components of and linkages within the CPSSII framework. After combining the labels generated by the axial coding and establishing the key themes, then clustering these key themes into general categories, we reviewed each survey to determine the number of sites that presented the characterization of that theme. This data often provides useful insight into the variation in how sites moved from having an initial interest to sustaining the PPCPD (e.g., some sites scaled to all sections after an initial pilot and others scaled incrementally).

Table 1
From Initial Interest in the Innovation to Considering the Innovation.

Key Themes in the Transition from "Interest in the Innovation" to "Considering the Innovation" within the CPSSII Framework	Total
Departmental leadership initiates and/or supports consideration of the innovation	
Departmental leadership open to considering another approach/new materials	12/12
Current curriculum/approach (e.g., Emporium, ALEKS, traditional textbook) not adequately preparing students for calculus	7/12
Looking for an approach to improve student passing rates	11/12
Representative Interview Quote: *The undergraduate committee was pretty much convinced that this is what we needed to try. They were also convinced that instructors needed an attitude to go into this kind of a course, because it's not quite what we had before—it is not quite what you expect. It was not just insisting on doing the high school material once again… It was different material and a somewhat different approach.* –Department Chair	
Scholarly research base of the innovation is valued	
Valued materials that were informed by research	12/12
Representative Interview Quote: *The administration was concerned about high [failing] rates. So we got an* ad hoc *committee together and we settled on Pathways after a semester's long discussion [of the literature on precalculus teaching and learning].* –Course Coordinator	
Site initiator invests time to understand the innovation	
Heard about the materials by attending a research talk or conference presentation or reading published research	11/12
Read select investigations and completed select homework	12/12
Visited home institution to observe Pathways classes	4/12
Met with Pathways leaders in person or via Skype/Zoom to learn more about the materials, how they were developed, and how they are used	12/12
Representative Interview Quote: *What I really like about the Pathways materials is this covariation notion. To understand really what's happening with functions is to understand how one variable is changing with respect to the other. I think that is fundamental, and [the Pathways team] has proven that and shown that.* –Course Coordinator	

The following sections describe the conditions that led to local sites transitioning from one phase (category) to the next. We included the key themes (and axial codes related to each theme) associated with the sites' transitioning to a new category in a table in each of the following narratives. The first section presents the conditions associated with the sites transitioning from having an initial interest in the PPCPD to considering the innovation. The table presents the number of sites (out of the 12 that locally scaled the innovation) that presented each theme (e.g., the department chair was open to considering new materials) listed in the table.

4.1. From initial interest in the innovation to considering the innovation

In all instances, the department chair was at least open to considering a curriculum change. Faculty expressing interest in the PPCPD did so as a potential solution to a perceived or acknowledged problem at their site, such as discontentment with their current precalculus curriculum and approach (e.g., Emporium, ALEKS, or a procedurally-oriented textbook), low passing and/or persistence rates in precalculus level courses, student performance on assessments valued by faculty, and/or a belief that their current approach was not preparing students for calculus. Sometimes, department or school administration explicitly directed faculty to seek solutions to these issues. Survey responses (see Table 1) revealed that most site initiators whose sites eventually adopted the PPCPD (11 out of 12) first learned about the PPCPD by attending a presentation at research conferences or reading a research paper by a member of the lead author's research team. These faculty members' interest was also piqued when they learned that the curriculum's development was grounded in research on understanding and learning precalculus ideas most important for calculus, and that other Pathways users had reported high student passing rates (typically 70% or higher).

4.2. From considering the innovation to confronting barriers

After becoming aware of the Pathways materials, faculty contacted Pathways developers to learn more about the innovation. If the faculty's interest persisted, developers gave them access to the student materials and instructor resources to explore the cognitive scaffolding, interact with online applets, view course videos, and consider whether the materials addressed their perceived needs. Most sites also considered local content coverage requirements. All 12 site initiators surveyed reported working through investigations in the Pathways student workbook. Most (eight of the 12) described the conceptual scaffolding of the investigations as an attribute they believed would be useful for their instructors and students. After reviewing the materials, all site initiators contacted a Pathways developer to arrange a phone or video conversation to learn more about the Pathways printed and online resources for students and instructors, how research guided the content design decisions, and how the materials' cognitive scaffolding impacted student learning. In all cases, the site initiator inquired about passing rates, content coverage, and evidence of greater student learning. See Table 2.

For example, one site initiator said:

> I wanted to see evidence that students were going to be better prepared for calculus than in the traditional pre-calculus curriculum—that students would have a conceptual understanding of things like function composition, co-variational reasoning, and, uh you know, unit circle and triangle trigonometry and how they are connected.

Meetings with site initiators focused on contrasting Pathways learning goals for understanding key ideas (e.g., average rate of change, exponential growth) with those of other popular precalculus materials and sharing pre-post learning gains on validated instruments. Pathways developers discussed cross-cutting reasoning abilities and demonstrated how specific foundational ideas develop across lessons and units. Developers also commonly discussed processes for implementing the curriculum, research that informed the curriculum design, the nature of professional development offered to instructors, and instructor and student resources.

4.3. From confronting perceived barriers to the decision to pilot

As we have mentioned, both the style of the scaffolded student investigations and the student learning goals in the PPCPD differ from most texts and common approaches used by many instructors. Even though the conceptual scaffolding had been refined and documented to advance students' meanings, it was natural that some faculty and/or departmental leaders would not immediately recognize the value in asking questions targeted to develop more robust ways of thinking and understandings of precalculus ideas than what are commonly targeted (e.g., two quantities in a constant rate of change relationship vary together so that the ratio of corresponding changes in the two quantities' values remains constant). While the site initiator and other lead faculty had become more convinced that the PPCPD provided greater learning opportunities for both students and instructors, it was natural that some faculty did not value Pathways learning goals or were concerned that the shift to using the materials would be challenging for some of their precalculus instructors. Our coding of the coordinator surveys revealed key themes (Table 3) associated with the transition from confronting perceived barriers to the decision to pilot the PPCPD.

All coordinators completing the survey expressed that some of their precalculus faculty held learning goals for precalculus students (such as the belief that precalculus should primarily focus on developing students' algebraic skills) that differed from those of Pathways developers. Some faculty and/or departmental leaders also expressed concern that not all site instructors could successfully use the materials. The reasons they cited include (i) that teaching with the materials requires that instructors reconceptualize precalculus ideas, (ii) that instructors' initial implementation of the curriculum would require extra planning time, (iii) that instructors may be uncomfortable adopting new pedagogical strategies to shift to a more conceptually scaffolded, student-centered teaching approach, and/or (iv) that some instructors might believe that deficiencies in procedural skills explain students' struggles in calculus, rather than students having unproductive meanings for key mathematical ideas. For example, one site initiator said:

> I was concerned at the beginning about the skill level and whether some instructors would think the curriculum needed more skill practice. […] When the instructors that teach calculus say, "they don't understand algebra", nobody says they don't understand the ideas needed to learn calculus. Maybe the faculty just don't know what understandings students need. All they see is that students can't expand a binomial or don't know exponents.

Finally, because the Pathways curriculum focuses more tightly on a collection of cross-cutting reasoning abilities and mathematical ideas important for calculus, some faculty pointed out that the list of topics they typically cover in a precalculus course did not perfectly align with the table of contents in the Pathways workbook.

A factor in a site's departmental leadership and faculty deciding to pilot the innovation typically began with the site initiator and Pathways developers identifying the perceived barriers and developing a plan for addressing them. The process of deciding on a plan for addressing barriers varied across sites. For example, some site initiators visited the PPCPD development site to observe Pathways classes and attend the site's weekly professional development seminar. They also spoke with Pathways instructors about their experiences in using the materials for the first time. This visit to the development site gave the site initiator ideas for addressing perceived barriers at their local site. Other site initiators invited Pathways developers to lead a departmental colloquium and, in some cases (five of 12), the Pathways developers also led a 4–8 h workshop to introduce the Pathways learning goals to resistant faculty.[7] During the departmental colloquium the Pathways developer(s) discussed the research informing Pathways material development and engaged participants in reviewing the student and instructor resources (student investigations, applets, videos, and conceptually oriented online lessons and homework).

During the introductory workshop, faculty completed and discussed student-level tasks with a focus on experiencing the ways of thinking and understandings promoted by the conceptual scaffolding. This workshop was instrumental for some faculty at each site to embrace the PPCPD learning goals and instructional supports more broadly. The sites' Pathways coordinators and Pathways workshop leaders commonly conveyed that this workshop positively impacted resistant instructors to be receptive to piloting the materials with their students. Some resistant faculty also expressed that working through select tasks and reviewing the instructor resources resulted in their gaining an appreciation for Pathways learning goals and the materials' conceptual scaffolding. In many cases, the workshop participants also expressed appreciation for the affordances of their students engaging with similar tasks and constructing the research-based understandings targeted by the Pathways materials. This suggests that faculty views about precalculus learning goals are

[7] In more recent years, it has become common for the site initiator to arrange a remote meeting with Pathways developers and other precalculus faculty to provide an overview of the Pathways learning goals, resources, and data and to engage the participants with select tasks from the materials.

Table 2
From Considering the Innovation to Confronting Barriers.

Key Theme in the Transition from "Considering the Innovation" to "Confronting Barriers" within the CSSII Framework	Total
Site initiator introduces the innovation to their department with developers' support	
The site initiator invited a Pathways leader to give a departmental colloquium. The colloquium was attended by interested faculty and administrators. The colloquium speaker: (i) introduced the Pathways learning goals; (ii) described research that had informed the design principles; (iii) shared research findings highlighting the targeted ways of thinking and understandings, (iv) provided an overview and demonstrated use of Pathways resources for instructors and students, while highlighting unique attributes of Pathways for advancing student thinking, (v) shared comparative data of student learning and retention, and (vi) engaged with faculty about their goals and the local context for teaching precalculus.	12/12
A Pathways developer provided a departmental professional development workshop for interested faculty. The workshop participants were engaged in working through key tasks in the curriculum, with the workshop leader highlighting the utility of the meanings targeted in the investigations and how they connect to other ideas in the course.	5/12
Interested faculty were given access to Pathways materials via an online portal.	10/12[a]
Representative Interview Quote: *When we first started, [the developers] came out and gave the workshop and then we would have weekly seminars of 90 min. So, when we scaled up… we invited [all instructors] to a workshop… And they have access to all of the instructional support that comes with the materials, you know advice about how to teach the lessons and the big ideas for instructors.* –Course Coordinator	

[a] An online portal with easy access to digital versions of the materials was not available at the time that the first two sites considered the materials.

Table 3
From Confronting Perceived Barriers to the Decision to Pilot.

Key Themes in the Transition from "Confronting Perceived Barriers" to "The Decision to Pilot" within the CPSSII Framework	Total
Misalignment of instructor goals and innovation goals are identified	
Some course instructors' goals for student learning did not align with Pathways goals. (i.e., "Precalculus mathematics should focus on algebraic skills needed for calculus.")	12/12
Representative Interview Quote: *I was concerned at the beginning … about the skill. You know, … you can't please some people in our department. And when the people who teach calculus say "they don't understand algebra," Nobody says, "they don't understand the ideas behind [the skill]." Nobody says that. They all say, "they can't expand a binomial right" or "they don't know exponents." So, you know, that was a concern at the beginning… [But] it shouldn't be that [students] either can do it without understanding OR [that they] understand it. And I mean, you should be able to … do both.* - Course Coordinator	
Acknowledgement of implementation challenges	
The research-based materials include tasks that require faculty to advance their understanding of precalculus ideas and/or adapt their pedagogical strategies.	12/12
Implementing the materials for the first time requires additional planning to understand Pathways targeted learning goals for upcoming lessons and to prepare to discuss solutions to Pathways investigations and homework.	12/12
Departmental leaders were concerned that not all faculty could be successful using the materials.	8/12
Representative Interview Quote: *There were so many things that I just had not thought about, you know. When looking at exponential functions or trigonometric functions, I realized when I was teaching out of pathways that there was just an awful lot that I had never really thought about. I just accepted procedure-based approaches and just used them, and had never really given a lot of thought about, "okay I can prove that these things work, but why should they work? What's the motivation behind the way we define the sine function? Or what's the motivation behind the differences between exponential growth and linear growth?" And it's just, it was a real revolution to me, and kind of humbling too. It's like, oh my gosh, I've been teaching this stuff for 30 years and this book is just like handing me my lunch over and over again.* –Instructor Representative Interview Quote: *And so, one of the challenges was how do we get enough instructors [motivated to use the materials]? Because we did not want to have large classes immediately. So, we wanted to start with smaller classes and then little by little transition to a mix of larger and smaller classes—and then larger classes just so that, you know, you don't have too many confounding factors at first; then you don't know what works and what doesn't work. And so, we designed a way to actually instruct instructors on how to teach this.* –Department Chair	

malleable under conditions in which they are supported and succeed in building stronger understandings of precalculus ideas they teach.

4.4. From the decision to pilot to piloting the innovation and assessing the pilot

After the site initiators carefully reviewed the materials and introduced them to other interested colleagues, the site initiator and one or two other interested faculty piloted the materials at their site.

4.4.1. Making the decision to pilot

The site initiators reported that several factors influenced their decision to pilot the materials. See Table 4.

The site initiators had come to value the Pathways learning goals and believed that PPCPD would achieve greater learning gains for their department's precalculus students. For example, one site initiator said,

> Pathways didn't initially align exactly with what I thought was important for precalculus, but after I spent a little bit of time digging into the materials and visiting [the original development site] to see classes in action, I decided on Pathways. […] I had always taken ideas like average rate of change and constant rate of change for granted. When I began listening to students talk about those ideas, I realized there is a lot that they don't understand that they will need for calculus. And the whole idea of covariational reasoning, what students need to do is think about two quantities changing in relation to each other when defining a function relationship—I thought everybody naturally did that—I wasn't aware of how students were thinking.

It was critical that site initiators convinced departmental leaders to support the pilot. In some cases, the department chair was convinced not just by the developers or the site initiator, but also by attending parts of the workshop and/or speaking to other department chairs at sites already using the materials. One department chair said, "Every department chair I have spoken with says Pathways has the right philosophy; students should learn to figure things out and shouldn't have to be told what formula to use." Departmental leadership's support has proven critical since, at this stage, not all precalculus faculty at a given site valued the Pathways learning goals and/or expressed interest in using the materials. In fact, in all instances, the decision to move forward with a pilot was made by the site initiator and supported by faculty who were in leadership positions and responsible for the success of the course. All sites had some faculty openly opposed to moving away from their current approach, with the level of resistance to considering Pathways varying from site to site. Notably, these faculty, who at one site even comprised the majority of precalculus instructors, could not obstruct other faculty from moving forward to pilot the materials. Faculty frequently resolved their differences by agreeing to collect data in both Pathways and non-Pathways sections to compare students' learning gains, student pass rates, and/or student retention in calculus.

4.4.2. Preparing for a pilot: key attributes of the pathways precalculus workshop

Before beginning the pilot, a Pathways developer led a two-to-three-day workshop for faculty preparing to use the materials. A primary workshop goal was to support faculty preparing to use the materials in reflecting on the thinking needed to complete select Pathways capstone tasks—questions that frequently perturb people with advanced degrees in mathematics and require a strong conceptual understanding of a precalculus idea. Participants worked in small groups to discuss their approaches and produce a written solution on a whiteboard to share with the group. Workshop leaders expressed the expectation that participants explain the ways of thinking and approaches they discussed, followed by one person explaining their solution and the thinking that led to their constructions. In planning and delivering these workshops, the developers were intentional in using tasks that would reveal procedural meanings that had been documented by researchers to be prevalent among students and instructors and limiting in their use. As one example, many students and instructors only conceptualize an average rate of change as the slope of a secant line through two points on a graph. A more productive meaning that generalizes better to a wide variety of contexts is conceptualizing the average rate of a function on an interval of a function's domain as the constant rate of change needed to produce the same net change in the dependent quantity's value on that domain interval (Musgrave & Carlson, 2017; Thompson, 1994; Thompson & Harel, 2021; Yoon et al., 2015).

However, prior experience working with teachers and instructors corroborated similar challenges as Stigler and Hiebert (2009) noted. Namely, that changing instructors' ingrained images of teaching and learning mathematics is challenging. Pathways developers' experiences in working with both undergraduate and secondary precalculus teachers concurred with those reported in the TIMMS study (Stigler and Hiebert, 2009) that it is common for U.S. mathematics teachers' instruction to primarily focus on demonstrating methods for answering questions, followed by student practice, and with little focus on supporting students' engagement in mathematical thinking toward their understanding important mathematics ideas (Baş-Ader & Carlson, 2022; Marfai, 2016; Underwood & Carlson, 2012). As such, the Pathways professional development and research teams have focused on advancing and studying mechanisms for supporting precalculus instructors to shift their instructional practices to have a stronger emphasis on understanding and advancing students' mathematical thinking and understanding of precalculus ideas. Given the short duration of the workshops relative to the developers' goal to shift instructors' images of teaching and learning, early iterations of these workshops concentrated on the following.

- *Encouraged a learning and growth mindset for instructors while creating a non-threatening environment for them to share their thinking.* Pathways workshop leaders repeatedly assured all participants that they might find many of the targeted understandings unfamiliar since these characterizations of what is entailed in understanding and learning key concepts were arrived at through sustained research into knowing and learning ideas. For example, new Pathways instructors commonly require support in conceptualizing the relative size of one quantity compared to another. This reasoning ability is needed to determine and understand what the growth factor, b, represents when modeling exponential growth data. Relative size reasoning also grounds discussions of rational function behavior and understanding radian measure as representing the relative size (or how many times as large) an arc length is as compared to a circle's radius. The leaders shared how their meanings for ideas like average rate of change and exponential growth were primarily focused on calculations and procedures before engaging in sustained research to explore what a rich and meaningful understanding of these ideas entailed. Workshop leaders illustrated the power of constructing more conceptual meanings (e.g., *average rate of change* on an interval of a function's domain as a constant rate of change that has the same gain as the

Table 4
From the Decision to Adopt to Piloting.

Key Themes in the Transition from "The Decision to Pilot" to "Piloting" within the CSSII Framework	Total
Site initiator and departmental leaders value the goals of the innovation and the developers' support	
After initial training and exposure to the materials, the site initiator and some instructors expressed confidence in their ability to use the materials with their students. They expressed that they valued the course learning goals, conceptual scaffolding, and detailed teacher notes and solutions.	12/12
The department chair valued Pathways learning goals or expressed trust in the site initiator's decision to move forward with a pilot.	11/12
The department chair and site initiator valued that Pathways developers would travel to their site to lead a two- or three-day workshop for all faculty and GTAs interested in teaching with or learning about PPCPD.	11/12
Representative Interview Quote: *An adaptive mindset is really crucial for successfully working in communicating with these materials, you know. Being open to saying like, "Okay I've learned or taught in this particular way in the past, [but] how is this new information either challenging for me or new to me? And, you know why? Why is that worth going through and investigating, and how does that deepen my understanding, so I can deepen the students' understanding?"* –Course Instructor	
There is a lack of traction for resistant department members	
Faculty members resistant to or skeptical about using the Pathways materials were not able to obstruct other faculty from piloting Pathways materials.	12/12
Representative Interview Quote: *The department chair is supportive. He understands the need for conceptual understanding. […] He supported us in moving forward with Pathways even though some instructors didn't agree.* –Site Initiator	

function's value on that interval) and more connections that enable the meaningful use of the idea (e.g., making constant rate of change estimations of a function's value over a domain interval). The workshop leaders also assured the participants that the Pathways materials would support them and their students in acquiring more fluency in verbalizing and applying a conceptual meaning based on research; in particular, insights into ideas for which students would need to be repeatedly supported in reasoning in new ways. They also encouraged participants to question the value in only teaching students how to accurately calculate a value (e.g., $\frac{f(6)-f(2)}{6-2}$) without being able to explain what the value represents.

- *Introduced overarching ways of thinking and ideas in the course that bring coherence to the course modules (such as the idea that conceptualizing and defining a function involves conceptualizing how two quantities' values change in tandem).* In more recent years, workshop leaders have promoted the use of instructional conventions (e.g., Carlson et al., 2022) found to be effective in helping instructors support students in understanding functions and using them meaningfully to model quantitative relationships between two quantities with values that vary in tandem, using algebraic symbols and function graphs.
- *Helped instructors understand and practice explaining the most important mathematical ideas targeted early in the course.* A goal was to facilitate early success for the instructors and their students. This often involved working through student workbook investigations *as students* and then revisiting these investigations *as instructors* to understand targeted learning goals and appreciate how the investigations are scaffolded to support students in constructing strong personal meanings for precalculus ideas.
- *Familiarized instructors with curriculum resources.* These resources included detailed instructor notes, pacing guides, homework banks, and PowerPoints linked to each investigation to support the initial implementation and a shift to posing conceptual questions and providing conceptual explanations.

Site initiators expressed in their responses to the written survey that this Pathways workshop, now required of all faculty teaching with the materials and open to all faculty and administrators at a site, was the most important variable influencing an instructor's desire to teach with the Pathways materials. Developers' data further revealed that an instructor's effectiveness in implementing, and desire to continue using the Pathways materials, was related to their participation in this workshop.

4.4.3. Enacting the pilot and determining measures of success

Notably, the site initiator almost always (11 out of 12) led the pilot by teaching at least one section of precalculus or college algebra using the Pathways curriculum materials (Table 5). One goal of the pilot was to assess the Pathways materials' effectiveness in supporting student success and advancing student learning at that site. Another goal was for the site initiator and other committed precalculus instructors to experience teaching with the materials and gauge whether they felt confident in preparing their larger instructional team to use them. Furthermore, in looking ahead to potentially shifting from piloting to local scaling, the site initiator and/or the department chair determined what data they would collect to assess and compare Pathways students' success and learning gains with their current approach (all sites). In all cases, this included administering the validated Precalculus Concepts Assessment (PCA) instrument along with tracking student passing rates. Shifts in PCA scores were positive at all sites (e.g., Carlson et al., 2015; Hower et al., 2023; McNicholl et al., 2021), and passing rates improved (or remained unchanged if the passing rates were already high). Developers also encouraged continued or periodic monitoring of measures valued by the department.

In all cases, the instructors attended weekly or biweekly (one site) professional development meetings with other instructors and GTAs teaching with Pathways materials during the pilot. These meetings were one to two hours long. The site initiators reported using these meetings to discuss the learning goals of upcoming Pathways investigations (11 of the 12 sites), discuss anticipated student

Table 5
Key Elements and Concerns of Enacting a Successful Pilot.

Key Elements and Concerns of Enacting a Successful Pilot	Total
The site initiator led the local pilot of the PPCPD materials	11/12
Pathways developers led a two-to-three-day workshop where instructors experienced completing PPCPD materials from a student's perspective and then shifting to understand how to help others learn with the materials from an instructor's perspective	12/12
Educative PPCPD materials modeled conceptual explanations to support answers cited as valuable	8/12
Instructors expressed appreciation for the coherence and connectedness of PPCPD ideas after teaching with materials for a full semester	8/12
Instructor resources in the educative PPCPD materials increased instructor awareness of student thinking	7/12
Representative Interview Quote: *Precision of language is so crucial. [We make] a big deal of that, you know. And the graduate students [during the pilot year] of our regular professional development meetings,] referred to a quantity as "it" then [everyone asked], "what's IT???" you know? In a friendly and fun way. … When instructors are practice teaching [during professional development meetings], we act like we are students and if something's not clear in their explanations we just, in a professional friendly supportive but firm way, we [hold each other's feet to the fire].* –Course Coordinator	

responses to the most challenging questions in upcoming investigations, and practice giving explanations of the reasoning needed to respond to these questions. Discussing course pacing, strategies for keeping students engaged, and formative data on student learning was also typical. In seminars with graduate teaching assistants, the site initiator required that the graduate student instructors complete the student assignments and be prepared to lead a discussion of a specific capstone problem in an upcoming investigation before attending each weekly seminar. Survey responses conveyed that the initial pilot instructors had a positive experience when teaching with Pathways for the first time, with all respondents stating that being supported in understanding fundamental ideas more deeply was rewarding. However, at every site piloting the materials (and even in the original development site), course coordinators report that the first semester(s) teaching with the Pathways materials requires significant adaptations and learning as instructors work to reconceptualize fundamental ideas and how students learn them.[8] Site initiators also commonly expressed (eight out of 12) that only at the end of the semester were instructors able to fully appreciate the coherence and connectedness of ideas in the course. Pathways researchers' data supports that most instructors continue to make mathematical connections as they repeatedly use the materials and that the advancement of their mathematical connections influences their explanations (Tallman & Frank, 2020) and effectiveness in making sense of students' thinking (Baş-Ader & Carlson, 2022).

The initial workshop and ongoing meetings initiated a shift in instructors' meanings for the key ideas targeted by the Pathways curriculum. However, respondents often cited the educative aspects of the materials as key to supporting their own and other instructors' learning and growth, particularly the investigations' conceptual scaffolding. They also pointed to the detailed instructor notes and solutions to all questions in the in-class investigations and written homework as helpful when preparing to teach (all sites). Coordinators conveyed that these resources increased their awareness of possible ways of thinking that students might exhibit (seven of 12) and provided model conceptual explanations to support a solution approach (eight of 12) (see Table 5).

4.5. From piloting to locally scaling the innovation

After an initial pilot, most (nine of 12) sites immediately scaled to all precalculus sections in the department, while others (three of 12) scaled incrementally by only including faculty interested in teaching with the materials in their initial local scaling. Sites with graduate student instructors initially assigned those graduate students to teach Pathways sections only if they demonstrated strong communication skills and/or expressed interest in focusing their teaching on ideas and developing students' mathematical thinking.

4.5.1. Conditions for local scaling: increases in student learning

During the local scaling process, sites used various data sources to document whether the Pathways course successfully promoted student learning gains. The site initiators at all 12 sites, that scaled beyond the initial pilot, reported that Pathways sections achieved larger learning gains on the PCA and usually better passing rates than the non-Pathways sections. During the PPCPD pilot, all sites administered the PCA as a post-assessment to both Pathways and non-Pathways sections, with sections of Pathways mean PCA scores ranging from 12.8 to 15.2, compared to 6.2 to 7.9 for non-Pathways sections. In some cases (four of 12), the improvement in passing rates was dramatic, shifting from less than 50% to greater than 70% (e.g., McNicholl et al., 2021). The sites that compared Pathways and traditional students' success in a traditional calculus course the following semester reported that Pathways students performed significantly better or the same as non-Pathways students.

[8] Nine first-semester Pathways instructors stated they spent between one and 2.5 h preparing to teach with Pathways materials before each class session.

4.5.2. Support for instructors: Pathways workshops and ongoing contact with developers

At all but one site (discussed later), the new Pathways instructors attended both an initial workshop before using the materials, and weekly professional development sessions during the semester. At two other sites, the weekly workshop was discontinued or regressed to discussing only the logistics of using the materials (what homework to assign). In one instance, the site scaled back their use of the materials and, at the other site, the department discontinued using the Pathways materials. Department chairs at these sites conveyed that discontinuing the seminar led to new instructors having diminished confidence in their ability to use the materials, an increase in student complaints, and instructors increasingly requesting a shift to a more traditional textbook. In visiting classes at these sites, the Pathways evaluator and professional development leader observed instructors giving incoherent explanations and displaying an inability or unwillingness to answer students' questions. Rocha (2023) corroborated these observations and documented the role of a teacher's mathematical meanings for teaching an idea on their use of Pathways materials. Rocha's data revealed that instructors had to engage in actions to advance their understanding of course ideas before they could effectively communicate those ideas to their students. Data collected from all sites supports that positive learning shifts occur for groups of students regardless of whether an instructor is motivated to advance their understandings of precalculus ideas and how they are connected. However, student experiences in the class of a teacher who frequently reverts to showing students one way of answering a question are less meaningful and motivating as compared to students in a class with a teacher who engages students in activities to support them in using their thinking to construct strong personal meanings (Marfai, 2016; Rocha, 2023).

4.5.3. Adapting the innovation to meet local needs

It was typical for a site's initiator (seven out of 12) to request (and sometimes collaborate with Pathways leaders to make) minor adaptations and/or additions to the Pathways materials, especially after the pilot semester/year but before fully implementing a locally scaled-up version of the course. These adaptations took on many forms, including unique modifications for specific sites as well as global modifications for all users.[9]

An early adaptation for one site was creating "clicker" questions for use in a large lecture format (McNicholl et al., 2021). At this site, all precalculus sections included three weekly class meetings (two large lecture sessions and one smaller recitation session). Recitation meetings used a collaborative-style format driven by investigations in the student workbook. The site initiator deemed managing collaborative investigations from the student workbook in the large lecture setting impractical. In cooperation with the Pathways developers, the site initiator created a set of conceptually oriented clicker questions aligned with student workbook investigations. These questions were used as "clicker" questions during lectures to provide real-time feedback on students' meanings for important ideas and hold students accountable for working through and discussing the tasks posed during the class session.

To address concerns about the PPCPD's decreased emphasis on algebraic skill practice relative to traditional curricula, some local coordinators requested that developers integrate additional skill practice into the course materials. Some site initiators created their own supplemental materials to support this goal. Pathways developers also created "just-in-time" algebraic skills practice to accompany each course module (unit) based on methods students might need when completing tasks within a module. Most recently, developers have created online "gateway" assignments focused on algebraic skills that university faculty most often request be included in precalculus instruction. While many sites continue to incorporate this additional skills practice into their courses, it is noteworthy that some sites have discontinued it. One site conducted an internal study to determine the impact of incorporating additional skills practice into their Pathways precalculus course. Like other sites using the Pathways curriculum, this site reported meaningful student gains on the PCA and increased student passing rates. Adding additional algebraic skills practice, however, had no impact on these measures of success and, more importantly, no impact on students' success in future math courses. This site cut back on the additional practice they assigned to focus more time on supporting students' repeated reasoning with fundamental ideas.

Another common request after the pilot was to expand the list of topics included in the course. For example, the original course did not cover topics such as conic sections, polar coordinates, parametric equations, and inequalities with polynomial and rational functions. Some sites reported strict content coverage demands from departmental leadership, regional university course standards, or state course standards. Over time, developers worked with several sites to create new investigations and homework tasks to support varied tables of contents. It was important for the developers to participate in this process so that the new investigations emphasized ways of understanding each topic consistent with the overall philosophy and focus of the rest of the Pathways materials. Note that it is increasingly common for the development team to help new site initiators create customized tables of contents and new investigations even when piloting Pathways materials.

A final set of modifications to the original curriculum involved the creation of online homework and, eventually, online lessons compatible with the goals of the Pathways investigations and online textbook. Many universities use online homework systems to help manage due dates and grading and to provide students with automated feedback on their course progress. Rather than trying to coordinate the use of available homework systems (e.g., ALEKS, MyMath Lab) with philosophies of learning incompatible with the Pathways developers' philosophy, the Pathways developers created online homework sets to provide students individual practice in using their reasoning and applying their understanding of the ideas introduced in the student workbook investigations. Users' response to these resources was so positive that the Pathways developers expanded the Pathways online supplements to include fully scaffolded

[9] Site-specific modifications, such as clicker questions to accompany lectures, were concurrently made available to all users, but frequently these modifications remained local to that site. It is worth noting that developers typically made additional annual modifications to student investigations and course resources based on qualitative studies in the project's context and monitoring student assessment data to improve learning relative to specific important ideas at the core of the course.

online lessons compatible with the Pathways curriculum. These online lessons include interactive applets, text, videos, and open-ended questions developed using the iMathAS (Lippman) learning system. It is noteworthy that each PPCPD adaptation provides more options for new Pathways adopters going forward.

We provide data from the surveys and interviews supporting key themes in the transition from piloting the innovation to scaling the innovation in Table 6.

4.6. From scaling to sustaining the innovation

Even prior to Cai et al.'s (2020) call to reconceptualize the goals of research on scaling, Pathways developers realized that disseminating Pathways to new sites would require adapting the PPCPD resources and professional development to local sites' content requirements, class sizes, and instructional teams. They embraced the challenge of accommodating local conditions and welcomed the opportunity to learn about the adaptations and their impact on instructors and students.

In this section, we report the results of our analysis of a purposive sample of sites we selected because they reflect key themes in the transition from scaling the innovation to sustaining the innovation (the importance of continuity of leadership and goals, adaptation to local needs, meaningful professional development for instructors, and continuous learning by developers, the local coordinator, and local leadership). Each example site we discuss highlights a subset of these themes, although elements of all themes are present at each site, particularly the importance of continuous learning by all stakeholders.

4.6.1. Sites A and B: the importance of continuity in leadership and instructional goals

At Site A, the innovator reported results consistent with those reported at many sites (e.g., McNicholl et al., 2021), including a dramatic improvement in passing rates (from less than 50% to greater than 70%) as well as seeing students using the Pathways curriculum for precalculus go on to pass calculus at a higher rate than students using a different curriculum. The site's department chair at the time of the pilot and scaling process strongly supported adopting and sustaining the Pathways innovation. However, after five years of using the Pathways curriculum for their precalculus courses, the site returned to using a more procedurally oriented textbook.

Like other sites, Pathways developers initially led workshops at Site A to train instructors and GTAs using Pathways materials. At the first workshop the site initiator (who was also the coordinator and teaching a section using the Pathways materials) attended the initial workshop along with the other instructors and GTAs. However, in subsequent years, the site initiator moved on to a different project and another coordinator was hired who had not taught using the materials or attended any Pathways workshops. This did not raise concerns at the time since the developers continued to lead the initial three-day training for instructors and GTAs. However, the new coordinator attended only one day, of the three-day workshop Pathways developers led, prior to teaching with the materials and coordinating the course. It is also noteworthy that there was no overlap in coordinating responsibilities. The new coordinator expressed that they valued and understood Pathways learning goals. However, it later became clear that they did not follow the instructional model set forth by the Pathways developers and site initiator. After working with this coordinator for several years, it appeared that they equated good instruction with providing clear explanations of how to work problems. In addition, they discontinued the weekly seminar and relied on new instructors and GTAs to individually prepare to use the Pathways materials. Without adequate support in preparing to use Pathways materials, the quality of instruction diminished, resulting in increased student frustration and complaints to the new department chair. Without a strong local leader mentoring the development of new instructors and overseeing the quality of the Pathways implementation, the quality of implementation at the site degraded. As a result, the new department chair decided that returning to a more traditional curriculum created less resistance than identifying a new coordinator and reinstating mandatory professional development.

Site B was the only site that did not scale the Pathways materials after the initial pilot.[10] Like Site A, Site B dropped Pathways following a new coordinator taking over the course. When the new coordinator was assigned, the site initiator no longer oversaw precalculus instruction at the site. Also, like Site A, the new coordinator at Site B did not fully participate in professional development workshops prior to teaching with the materials. The first semester with the new coordinator was intended to be a second pilot semester, but by the end of the semester the new coordinator was no longer using the Pathways materials, and the site's initiator and administrators decided to discontinue using the materials. When asked why the site chose not to scale the PPCPD to other instructors at this site, the site initiator stated that "we really didn't have any champions (no coordinator) to keep it going." The site initiator noted that they did not collect comparison PCA data from non-Pathways sections. In contrast, all other sites collected comparison data during the pilot, and the initiators at those sites continued teaching with the materials and leading the site's professional development after the initial pilot semester.

Sites A and B demonstrate what we see as common challenges for efforts to improve mathematics instruction at universities, particularly the importance of sustaining the initial learning opportunities that were critical for the site's successful piloting of the innovation. As a new faculty member assumes leadership of an innovation, that site's continued success depends on the faculty member receiving adequate support for embracing and implementing the project's learning goals. Based on experiences with sites A

[10] The initiator at Site B was not asked to complete the same survey as other site initiators since the site did not attempt to locally scale the innovation. Instead, the site initiator completed a separate survey via email focusing on their experiences. It is worth mentioning, however, that during the initial pilot the site had some of the largest PCA pre-post gains of any site and a student passing rate over 70%, and that the site initiator expressed satisfaction with the materials and with student success rates.

Table 6
From Piloting to Scaling the Innovation.

Key Themes in the Transition from "Piloting" to "Scaling the Innovation" within the CPSSII Framework	Total
Measures of learning and passing rates improve	
Quantitative measures of learning and passing rates increased with PPCPD	12/12
Adaptation occurs to meet local needs	
Initiator requests minor adaptations to PPCPD materials to meet local needs	7/12
Regular meetings of instructors using the innovations increases understandings of new mathematics and pedagogy	
PPCPD coordinator attends the initial departmental workshop and is (or has) taught with the materials.	11/12
Continuation of PPCPD instructor workshops to provide content and pedagogical support for instructors	11/12

Representative Interview Quote: W*e were concerned that [students] do well in the subsequent course and I think I sent you data that supported that and they were, people were impressed that we did that. You may have noticed if you looked at that, [Pathways] students did slightly better. … They did hugely better on the PCA. But we [also] wanted to make sure that they succeed in the next course.*
–Course Coordinator
Representative Interview Quote: *The most helpful thing is giving instructors as much opportunity as they can to think on their feet and be flexible. That's what tends to blindside people the most. And it seems to me to be what people are the most afraid of—having to think on their feet. So, creating scenarios [during professional development meetings] where instructors have to do that is really helpful.*
–Course Instructor

and B, the Pathways leadership team is now more proactive in helping to transition a new coordinator into their position. They also share their strong recommendation that faculty taking over coordinator duties at a site have previously taught with Pathways materials (preferably at least two times). They further recommend that Pathways coordinators provide yearly updates to their departmental chair/administrators, including reminders of Pathways research-based learning goals, student learning/continuation data, and inquiring about their concerns.

4.6.2. Site C: the importance of meaningful professional development

As mentioned, and as outlined by Stigler and Hiebert (2009), the pervasive image of teaching and learning mathematics in the United States involves a top-down, teacher-led approach in which students reproduce algorithms and methods for obtaining answers. As a result, it is not reasonable to expect that instructors and GTAs will make a rapid shift to view teaching as supporting the development of students' mathematical thinking and understandings. We have observed that a teacher's image of teaching and their mathematical meanings for teaching specific ideas influence their instructional choices and the quality of discourse in their classes, including how they interpret and respond to students' questions and explanations (Baş-Ader & Carlson, 2022; Carlson, O'Bryan & Oehrtman, & Moore, 2007, 2022; Teuscher et al., 2011). In cases when developers observed a teacher becoming curious about their students' thinking and then responding by posing questions aimed at advancing students' thinking, the teacher had participated in sustained professional development. The workshop leader held the instructors accountable for reviewing course materials for the upcoming week and engaged them in considering productive and unproductive ways of thinking related to learning ideas in the upcoming lessons. Our studies of teachers revealed that weekly one-hour to ninety-minute professional development sessions with a knowledgeable leader were essential for sustaining the local learning community and acceptable use of the Pathways research-based materials.[11] The insights gained from examining Site C's scaling and implementation revealed the impact of an agreed-upon normative way of acting—this significant learning experience led developers to adapt the Pathways professional development model.

During developer-led workshops at Site C, workshop leaders attempted to emphasize the sociomathematical norm (Yackel & Cobb, 1996) of *speaking with meaning*. This phrase was a reminder of the goal "that responses are conceptually based, conclusions are supported by a mathematical argument that can be understood by the intended audience, and explanations, when appropriate, refer to actual quantities and how they are related" (Clark et al., 2008, p. 298; Carlson et al., 2022). However, the degree to which Site C's initiator, instructors, and GTAs embraced this norm was unique to that site. The initiator at Site C said that:

> During the initial Pathways workshop, the instructors practiced leading a mini-lesson on an idea or problem—this was after working through student level problems in the Pathways investigations and explaining their thinking in a small group. This was really helpful—the instructors experienced how challenging it is to explain why and to describe the thinking needed to justify an answer. Otherwise, it's way too easy to nod as if they already understand. But having to get up in front of peers and actually lead a lesson and have the Pathways leader and me [the coordinator] ask a bunch of questions prompting the instructor to reference quantities and explain why—not just how—this was good for them; they realized they needed to fine tune their understandings and explanations, and this definitely helped them to teach more conceptually.

"Speaking with meaning" became a standard by which all individuals in this community of Pathways instructors held each other

[11] We do not want to convey that it is impossible to shift instructors and GTAs, only that it can take time and experience with the curriculum to support the shift.

accountable. The Pathways coordinator at Site C reported that these Pathways instructors *elected to* continue holding each other accountable in their subsequent weekly professional development meetings. They frequently asked for clarification when an explanation was vague and/or a colleague failed to reference quantities when speaking (e.g., "it is increasing") until their colleague's explanations were clear in referencing quantities and describing how relevant quantities in the problem context vary together (e.g., "the car's distance from the stop sign decreased as the number of seconds since the car started moving increased"). This commitment translated to consistently implementing the pedagogical action of posing questions responsive to students while fostering productive refinements in their explanations and thinking. When bringing on a new group of instructors each year, the local coordinator elected to continue introducing and reinforcing the convention of "speaking with meaning" during both the summer and weekly professional development meetings. The site coordinator reported that consistently asking students to identify and reference quantities, and describe how they are related and vary together, led to meaningful discourse around problem solutions and ideas.

In addition, the Pathways evaluator visited this site in their fourth year of using the Pathways materials to observe the workshop's impact on first-time Pathways teachers' views of effective teaching. The project evaluator reported that the community of returning Pathways instructors held each other accountable for referencing quantities when constructing and interpreting graphical and algebraically defined functions. They also regularly used the phrase "speaking with meaning" as a gentle reminder to be clear in providing conceptually oriented explanations. Our studies of student learning have repeatedly corroborated the observation that students' success in defining and interpreting function formulas and graphs improves when instructors reinforce "speaking with meaning" and other conventions that foster identifying and conceptualizing how two quantities' values are related and vary together. The *Pathways Conventions for Supporting Quantitative Reasoning* include making a drawing to represent quantities in a problem context visually, and always being specific in defining a variable; that is, saying what attribute of the quantity is being measured, the starting point and direction of the measurement, and the units for measuring. Carlson et al. (2022) elaborate on these and other conventions for supporting quantitative reasoning.

Prior to working with Site C, Pathways developers knew that embracing the norm of speaking with meaning was important. However, they struggled to foster experiences where participants truly internalized the norm to the point where it consistently translated to pedagogical actions in the classroom and refinements in the instructors' ways of thinking for teaching an idea (Carlson et al., 2022). After their experiences with Site C, developers have attempted to establish this norm in subsequent workshops by contrasting vague explanations with ones in which the speaker clearly references quantities and describes how they are related. They also engage workshop participants in explaining their solutions, with explicit directives to attempt to *speak with meaning*, and to listen carefully to the explanations of others while posing questions if an explanation is unclear. Workshop leaders adapt the approach for each workshop group as they work to make "speaking with meaning" a norm that workshop participants value, while being viewed as personally productive for making one's explanations more coherent and understandable by others. According to the Site C local coordinator, the instructors' consistent attempts to "speak with meaning" make them active participants in transitioning to improve their teaching. In her words, "they cannot just nod as if they already understand" but must be active in listening and attempting to give explanations that will be understandable by their students.

4.6.3. Site D: the importance of knowledgeable site leaders, and adapting to local needs.

Site D's initiator was committed to Pathways goals and taught with the materials during the first pilot semester. Then, the initiator shifted their focus to studying the effectiveness of other instructors' use of the materials. As the site scaled locally, they hired adjunct faculty (high school and community college instructors working part-time) to teach most of the site's sections using the Pathways materials. These instructors spent little time on campus and their contracts excused them from professional development during the semester. In addition, the math department hired most of these adjuncts in the period between the three-day Pathways developer workshop and the first day of classes. As a result, these instructors did not attend professional development prior to using the materials. Also, while the faculty member leading the site's weekly professional development meetings valued Pathways learning goals, they had not taught with the materials.

At the end of the first semester after scaling the innovation locally, the project initiator alerted a Pathways developer that the adjunct faculty struggled to implement the materials. The department chair was also receiving complaints from students. Fortuitously, the Pathways developer and project evaluator had a planned visit to Site D as part of the Pathways evaluation. General findings from the external evaluator visiting four Pathways classes and conducting clinical interviews with five instructors revealed that they did not understand the Pathways learning goals and preferred to teach with traditional materials. They also expressed that they were following directives from the Pathways initiator about which problems to assign students during class and believed that students would achieve the intended learning goals just by answering these questions.[12] The project evaluator's and project developer's visit to Pathways classes further revealed that the instructors' explanations focused on showing steps for determining an answer, and at times, instructors overlooked incorrect student answers and were unable to answer students' questions.[13]

During this visit, the Pathways lead developer and project evaluator learned that the local coordinator was not supporting the adjunct faculty in understanding the Pathways learning goals. Instead of following the model of working through problems and

[12] The site department chair stated that "[t]hese instructors also refused to give students an answer to a question—they thought that inquiry-based meant independent learning and the materials would do all of the work."

[13] It is natural for instructors to show deficiencies in providing conceptual explanations when initially using the Pathways materials. However, if teachers have not worked through the student-level tasks before using them with their students, observations suggest that this can lead to instructors providing incorrect and incoherent explanations to students.

discussing specific learning goals relative to ideas in the investigations, the weekly professional development meetings involved the instructors in discussing assigned readings focused on general teaching practices. During this visit, Site D's professional development leader conveyed that they underestimated the support new instructors would need to understand the Pathways learning goals. The professional development leader further expressed their belief that the instructor solutions and notes would be sufficient for instructors to prepare to use Pathways materials. The site initiator also conveyed that instructors did not regularly collect homework. These conversations revealed problems with the implementation that might explain why instructors struggled to manage conversations and provide conceptually focused explanations.

At the end of the visit, the site's initiator invited the project evaluator and Pathways developer to share observations and recommendations. After this conversation, the Pathways developers met with the site's initiator and the newly hired department chair to create a plan of action. The Pathways developers volunteered to create more online support for students, including computer-graded online homework for each in-class investigation. This made it possible for instructors to assign homework more frequently and for students to receive immediate feedback. This addition to the Pathways materials increased student participation and provided the instructor and project initiator with regular feedback on student learning. The site initiator also made adjustments that began with recruiting adjunct faculty earlier in the summer and requiring that the faculty attend both the two-day pre-course workshop and the weekly one-hour professional development seminar. Faculty teaching the course assumed responsibility for the weekly professional development seminar. Thus, professional development training shifted to align with the model the developers had proposed—that of initially engaging the instructors in completing tasks to advance their understanding of Pathways learning goals and the rationale for the conceptual scaffolding in the in-class investigations. As instructors taught with the materials repeatedly, Pathways developers continued to engage with the site's leaders to share new seminar materials. The materials included descriptions of practices for engaging students' thinking and for holding them accountable for explaining their reasoning.

Site D's initiator conveyed that making it possible for the instructors to participate in professional development focused on understanding Pathways learning goals resulted in the adjunct faculty having more positive experiences using Pathways materials. Notably, the adjunct instructors all decided to return to the position the following year. The department chair also began advocating for the value of the Pathways course with the site's other stakeholders, including the department's calculus instructors, faculty from other departments, and campus advisors. To increase student enthusiasm for the course, the Pathways developers shared strategies for communicating the benefits of a course that supports the development of students' mathematical thinking and ability to communicate their ideas to others. After implementing these changes, the site reported increased student achievement and improved passing rates in line with those reported at other sites. The department chair and site initiator also conveyed that students' satisfaction with the course improved.

Like at Site A, the story of scaling at Site D revealed the importance of remaining closely in touch with a site's implementation, including the focus of professional development and instructors' homework policies. Detrimental modifications to the PPCPD model included such things as not assigning homework regularly, not holding students accountable for attending class, not mandating instructor attendance at professional development training, and not supporting advances in instructors' understanding of ideas targeted in the curriculum. When visiting Site D, our observations revealed that such modifications can decrease the curriculum's effectiveness and endanger the program's continuation at a site. Had the Pathways developers and leadership at Site D not worked to make the changes described above, site leaders would likely have discontinued the Pathways innovation. As of this writing, the site is now in its eighth year of using Pathways materials.

Notably, the size or status of a college or university considering Pathways did not appear to have any bearing on whether Pathways was considered, successfully scaled, or sustained. Some sites had Research I status and others were primarily teaching colleges. There were as few as four precalculus instructors at some sites, and at other sites there were as many as 30. The consistent variable across sites that considered Pathways was an individual who, after learning about the Pathways learning goals, came to value them, then took the initiative to implement the materials in their teaching and subsequently led the gradual scaling of the materials to other precalculus sections at their site.

5. Continued challenges and current directions

The PPCPD provides an example of how a research-based mathematics instructional innovation can advance precalculus students' learning and increase precalculus passing rates. An instructor's use of research-based instructional resources, that embody productive ways to engage students' reasoning and advance their conceptions, can result in students experiencing instruction that is more effective in advancing their thinking and learning. The local coordinators at most sites also reported that teaching with the Pathways curriculum has a positive impact on Pathways instructors', including mathematics PhD students', understanding of foundational ideas and conceptions of good teaching. For example, one initiator said:

> Teaching with the curriculum increases graduate students' understanding of the mathematics—just recently a graduate student said he now understood inverse functions much better as a result of teaching with the Pathways materials.

Another Ph.D. student reported that teaching with Pathways materials fostered advances in her understanding of trigonometric functions, contributing to a breakthrough in her mathematics Ph.D. research project. The Pathways coordinators also reported that mathematics graduate student instructors regularly convey appreciation for the opportunity to revisit and advance their understanding of fundamental ideas. However, this does not mean the process is simple, foolproof, or that success at a new site adopting the innovation is guaranteed. Pathways developers continue to find that their greatest challenge is shifting instructors' mindsets away from a

primary focus on content coverage and practicing methods for answering questions to supporting the development of students' mathematical thinking and understandings. Nevertheless, it is not the only challenge. Even experienced mathematics instructors can have unproductive meanings for key ideas in precalculus and can end up promoting students' construction of these same unproductive meanings in their students. In a study of 19 mathematics graduate students' understanding of average rate of change, Musgrave and Carlson (2017) found that, prior to teaching with the Pathways materials, most conceived of average rate of change as being related to a secant line between two points with vague meanings of what the slope of the secant line represented. They further report that teaching with the Pathways materials resulted in positive shifts toward their understanding and teaching the idea of average rate of change meaningfully, as the constant rate of change on an interval of a function's domain that results in the same net gain as the function on that domain interval.

As already discussed, Pathways focuses on developing students' engagement in quantitative and covariational reasoning (conceptualizing quantities, or measurable attributes of objects, and focusing on how quantities' values change in tandem) when developing algebraic and graphic function models. Unfortunately, it is uncommon for U.S. teachers to reason quantitatively and U.S. precalculus curricula does little to foster quantitative reasoning in their students (e.g., Byerley & Thompson, 2017; Musgrave & Carlson, 2017; Thompson, 2016; Thompson & Milner, 2018). Instead, the focus is more on learning methods for constructing a graph's shape (Moore & Thompson, 2015) and recognizing algebraic forms of a function type (e.g., exponential growth curves up to the right (O'Bryan, 2018; Ström, 2008)). This inattention to quantities and relationships between covarying quantities makes planning for and delivering coherent instruction difficult. One Pathways initiator said that:

> During the weekly workshops the GTAs and instructors are asked to explain ideas. I was shocked when a GTA thought a function graph with a positive concavity meant the function value was increasing—he was confusing the ideas of increasing rate with increasing function.

While Pathways workshops support instructors and GTAs in constructing more productive personal meanings for key mathematical ideas, the Pathways materials are designed to support instructors in advancing their conceptions and connections among ideas and how students learn those ideas. Studies have revealed that it is when preparing for teaching that many instructors are most motivated to conceptualize the ideas and how students learn them (Baş-Ader & Carlson, 2022; Marfai, 2016; Rocha, 2023; Rocha & Carlson, 2020). At the same time, developers have observed diverse approaches and wide variations in instructors' integrity in preparing for a lesson and in reflecting on the effectiveness of a lesson after teaching (Rocha, 2023). They have also observed that although many Pathways instructors eventually shift to engage their students' mathematical thinking, others maintain a predominant focus on explaining how to answer specific problem types.

The first implementation of an innovative curriculum such as Pathways can be challenging for most instructors. However, a project and local support system can effectively assist instructors in preparing for and overcoming unexpected challenges. This includes steps such as ensuring that all instructors receive pedagogical and content-focused training before attempting to teach with the materials. It is also critical for a dedicated and knowledgeable coordinator who is teaching (or has taught with) the materials to work closely with instructors to support their reconceptualization of foundational ideas, especially during an instructor's first two times of teaching with the materials. It is also advisable that a knowledgeable coordinator continue leading a weekly seminar for all instructors, with more experienced Pathways instructors assisting in mentoring new instructors (e.g., during the seminar they lead demonstration lessons and listen to the new instructor's explanations and pose questions aimed at fostering advances in their thinking). Other targeted professional development training may include a request for instructors to reflect on the effectiveness of their instructional choices immediately after teaching, with their documenting (and then sharing in the upcoming weekly seminar) choices that were more (or less) effective for engaging and advancing their students' thinking. Another might include asking instructors to reflect on their consistency in implementing the Pathways conventions for fostering students' use of quantitative reasoning (see Carlson et al. (2022)). The Pathways developers have also noticed that a site's level of commitment to the curriculum rises as faculty at a site engage in their own research relative to the curriculum's impact on their students' learning.

In a course such as precalculus, that often has broad ownership among departmental faculty and administrators, it is important to accommodate local demands and constraints by making site-based adaptations. We also recommend comparing students' learning and course success rates in the research-based course with those of students taking the course using their current approach. Adaptations such as developing materials for other topics, including additional skills practice and developing aligned online-homework can also broaden the innovation's appeal to other sites. This approach is consistent with Cai et al.'s (2020) call for approaching scaling as an *ongoing process of adapting and learning* within new educational settings.

6. Concluding remarks

Our findings detail a process that led to scaling and sustaining (or not sustaining) a research-developed and refined precalculus curriculum and professional development innovation. During the phase of considering the PPCPD, it was important that the site initiator and other precalculus faculty engage with the Pathways materials (through a research talk or short workshop), and subsequently express enthusiasm for the Pathways learning goals. It was also critical that instructors preparing to use the materials participate in professional development that focused on advancing their understanding of the precalculus ideas they were preparing to teach, while also considering how Pathways materials support students' learning of these ideas. At times during the sustaining phase, the Pathways site coordinator turned their attention to other projects. This resulted in the site's new Pathways instructors not being mentored in understanding Pathways learning goals and preparing to use the materials. After using the Pathways materials these

instructors expressed a desire to return to traditional materials. This resulted in sites either discontinuing (one site) or scaling back the use of Pathways materials (two sites). In instances when the site only scaled back the number of sections using Pathways materials, the Pathways professional development was reinstated and use of Pathways materials was continued, with one of these two sites now, as of this writing, using Pathways materials in all the site's precalculus sections.

However, important aspects of supporting teachers in implementing the PPCPD effectively require more research attention. We have described how the developers introduced the PPCPD to the site initiator, departmental administrators, and faculty intending to pilot the materials. We also described instructor and department motivations for seeking a research-based solution to local concerns (e.g., low passing rates or low persistence rates into calculus). However, we do not fully understand the complex aspects of shifting a department to break with traditional student learning goals, nor do we understand why some instructors embrace research-based learning goals and are willing to rethink what is involved in understanding and learning a course's ideas, while others do not. We also need a better understanding of the transitions that take place with individual teachers who initially value procedurally focused materials and subsequently shift to embrace and use the Pathways materials with their students.

Local coordinators conveyed that faculty teaching with the Pathways materials the first time are regularly perturbed to rethink what understanding and learning precalculus ideas entails. However, we need better insights into how or if (and under what conditions) Pathways instructors who repeatedly use the materials evolve in their instruction. In other research projects that studied teachers using Pathways materials over multiple semesters, we observed that teachers' mathematical meanings for teaching precalculus ideas advanced as they taught the course repeatedly (Duncan et al., 2017; Marfai, 2016; Musgrave & Carlson, 2017). Researchers also observed that teachers supported in reflecting on what is entailed in learning precalculus ideas, while also being supported in considering their students' thinking, continued to advance in their ability to provide conceptually coherent explanations (Rocha, 2023). Reflections on students' thinking and how students learn ideas also foster advances in a teacher's pedagogical choices and ability to interact productively with students during instruction (Rocha, 2023; Underwood & Carlson, 2012). As of this writing, Pathways developers are creating additional resources based on their insights into the most productive innovations for supporting shifts in teachers' *mathematical meaning for teaching an idea* (Carlson et al., 2024) and their instructional practices. The developers are also working on professional development interventions (e.g., video case studies) to engage Pathways instructors in inquiring into students' thinking (Rocha, 2021; Rocha & Carlson, 2020). However, we have much to learn about how such interventions impact Pathways instructors' conceptions and strategies for supporting their students' learning.

The PPCPD was successfully scaled and sustained at 12 sites (for a minimum of five years), with the sites reporting positive gains in students' learning and generally improved passing rates compared to their prior approach. Two of the 12 sites also documented that Pathways students were more likely to continue on to calculus than non-Pathways students (e.g., Hower et al., 2023; McNicholl et al., 2021). However, we have only anecdotal accounts of the PPCPD's long-term impact on students' mathematical thinking and development. We call for research to investigate if and how Pathways students continue to engage in quantitative reasoning when devising algebraic and graphical models of dynamically changing phenomena in later calculus and other STEM-related courses required for their degree.[14] Investigations into the link between Pathways instructors' meanings for precalculus ideas, their instructional actions, and their students' learning might shed new light on the mathematical meanings for teaching specific precalculus ideas that are productive for supporting student learning.

The Pathways research group's fine-grained cognitive studies for characterizing what is entailed in understanding and learning precalculus ideas informed (and continues to inform) each iteration of the course materials and professional development. For example, as studies revealed specific ways to engage students in using quantitative and covariational reasoning (see Carlson et al., 2022), these insights informed adaptations to the professional development and instructor resources. Our researcher team's early cognitive studies also contributed to the theoretical framing for the Precalculus Concept Assessment (PCA) (Carlson et al., 2010). Through the systematic adaption of these theories for learning precalculus ideas, Pathways developers have been able to reliably assess the impact of their materials on student learning. The PCA data collected at each site when piloting the Pathways materials was often a primary factor in a site's decision to scale the PPCPD. As already discussed, PCA results also revealed specific ideas and reasoning abilities for which a low percentage of Pathways students selected the correct answer. This resulted in the developers making targeted curriculum modifications for these topics. As other mathematics researchers move forward to design and disseminate research-based course materials and assessments, we recommend beginning by conducting cognitive studies aimed at characterizing what is entailed in understanding and learning the ideas in the course. The results of these studies can produce novel insights for establishing and refining learning goals, thus providing an initial theoretical framework for a quantitative assessment and the materials' design.

The high resistance among instructors to transition their mathematics instruction from a primary focus on teaching methods for working problems to using cognitively scaffolded investigations is more challenging than the Pathways developers initially anticipated. Even though we believe the PPCPD has made progress toward this end through the project's sustained research program focused on instructor's conceptions and their teaching with the Pathways materials, we have only scratched the surface in understanding the complexity of this endeavor. Our findings as of this writing suggest that an instructor's transition is best fostered when they are supported and motivated to devote time and intellectual resources to participate in a sustained journey of exploring, reflecting, and reimagining their teaching and their students' learning. Research into incentivizing faculty and researchers to take on this challenge might be a useful starting point.

[14] A recent study (Bettersworth, 2023) of students having completed a multivariable calculus course reported that quantitative reasoning was key for students' construction of algebraic functions and graphs in the context of modeling dynamically changing quantities in a three-dimensional context.

As we have demonstrated in this paper, developing, refining, and supporting the scaling of a research-based mathematics curricular innovation is challenging but possible. The Pathways approach to making local adaptations that respect a site's local constraints while fostering greater learning for students (implementation integrity) led to successfully scaling the PPCPD at all but one site. We hope that others who have already developed research-based curriculum materials will be inspired to develop instructional resources and professional development to support disseminating their materials. As we have reported, a key first step in disseminating research-based instructional materials is to ensure that the student materials and instructor resources can be used effectively by a small group of instructors. Once the research-based materials and professional development model stabilize to produce significantly greater student learning gains and course retention when used by other faculty at the developer's site, disseminating the materials to other sites is possible. Recall that the only criteria for working with a site to consider and pilot the Pathways materials is a site-based initiator who values the innovation's learning goals and who, after reviewing the materials and attending an initial workshop, decides to pilot the materials with their students. Throughout the consideration and piloting phase, it is also important to engage administrators and other faculty in learning about the research-based materials and their impact on students' learning. Having support for the innovation among departmental administrators and other faculty can positively impact whether a site scales research-based materials to more sections. Future studies on scaling mathematics educational innovations in a math department would benefit from investigating the role of mathematics departments' power structures and their influence on the innovation's initial adoption and successful scaling and sustaining.

We presented a model for the concurrent development of curriculum and theory for disseminating research-based precalculus instructional materials to other universities. We also remind the reader that our results are not generalizable since we employed a case study methodology when designing and implementing our study. However, we hope that our CPSSII Framework, and our descriptions of the processes we engaged in during each phase from considering to sustaining the Pathways model, will be informative for other researchers working to scale their research-based materials. We urge other researchers to consider the mechanisms by which their research insights and materials for advancing students' learning might be disseminated—if not you, then who?

Our reflections on the journey of designing and disseminating the PPCPD suggest that research insights gained during each phase of the PPCPD design were critical for informing each phase of its dissemination. Thus, we hypothesize that it is researchers who are best equipped to lead efforts to disseminate their research-based mathematics education innovation—especially if the innovation requires a major reconceptualization of the course content and how it is learned. Translating knowledge of ways of thinking and conceptions that are gained by years of research into the teaching and learning of a course's ideas is complex because it requires new ways of thinking and acting on the part of the teacher and the student. In the case of Pathways, the developers are now (as of this writing) attempting to support precalculus instructors in acquiring research-based understandings of precalculus ideas and how they are learned through their use of Pathways materials and participation in Pathways professional development. As instructors advance in their reconceptualization of the learning goals of a research-based course (that includes insights into productive ways to engage students for advancing their thinking), they are better equipped to provide conceptually focused explanations, understand how a student is thinking, and pose questions to advance students' thinking. As instructors inquire into how their students are thinking when interacting with them in the context of learning or using an idea, they construct stronger conceptions of how to support students' learning of that idea. Carlson et al. (2024) identified this symbiotic relationship between a teacher's mathematical meanings for teaching an idea, and their inquiry into a student's thinking, as impactful in advancing an instructor's ability to act productively in moments of teaching. We hypothesize that consistent efforts to support teachers to focus their instruction on advancing and understanding students' thinking will pay high dividends for both getting and keeping teachers on a continually adapting journey toward supporting all their students in advancing their mathematical understandings and thinking. We believe that more fine-grained studies of instructors' as they engage and teach with research-based materials (e.g., Rocha, 2023) will be productive for future efforts to scale research-based mathematics curriculum and professional development.

Declaration of Competing Interest

Marilyn Carlson and Alan O'Bryan are authors of Precalculus: Pathways to Calculus, the innovation being studied. At the time of submission, Alan O'Bryan is employed full-time supporting sites using the curriculum, including leading professional development training, creating new instructional supports, and helping adapt the materials to meet sites' local needs. Jess Hagman was the project evaluator for the NSF grant that funded research into this project. All other authors declare that they have no conflicts of interest.

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Appendix A. Pathways Local Coordinators Survey

Directions:

1. Provide your answers directly into this word document.
2. Please answer all questions as fully as possible, while staying focused on the question being posed.
3. If the question does not pertain to you, respond by answering N/A.
4. If you encounter a question for which a prior answer applies, please copy the prior answer and paste it into that question.
5. Please send your survey responses to Jess Ellis at: jess.ellis@colostate.edu

Initial Survey Questions:

1. What is your university position?
2. What is/was your role with Pathways?
3. How did you learn about Pathways?
4. In your view, how did your goals for precalculus teaching align with the goals of Pathways prior to using/adopting the Pathways materials?
5. How aligned are the Pathways learning goals and materials from the goals of the curriculum you were using in precalculus/college algebra prior to using Pathways? Provide 2 or 3 specific examples of the differences.
6. What made adopting Pathways materials challenging?

Adopt/Pre-pilot

1. Describe the process you engaged in that led to your decision to use Pathways materials.
2. To what degree were the following considered when deciding to use Pathways materials.
   a. The fact that Pathways materials are based in and refined by research on student learning in precalculus and calculus.
      (1. Strongly, 2. Somewhat, 3. Very little, 4. Not at all)
   b. The availability of Pathways resources for instructors.
      (1. Strongly, 2. Somewhat, 3. Very little, 4. Not at all)
      [If so, which resources? What about them?]
   c. The professional development provided for your local instructors.
      (1. Strongly, 2. Somewhat, 3. Very little, 4. Not at all)
      What about Pathways professional development was valued?
3. How were university and/or departmental administrators involved in deciding whether to pilot Pathways materials?
   a. How involved were your precalculus instructors in deciding whether to use Pathways materials?
      (1. Very, 2. Somewhat, 3. Very little, 4. Not at all)
      To what degree was buy-in from your current precalculus instructors considered when deciding to pilot Pathways materials?
      (1. Very much, 2. Somewhat, 3. Very little, 4. Not at all)
      If so, how were these instructors engaged in learning about Pathways materials?
   b. What criteria did you and your departmental administrators use when deciding whether to adopt Pathways materials?
   c. What were the concerns that you and/or your departmental administrators had when considering Pathways?
4. What obstacles did you and/or your departmental administrators confront when considering Pathways?
5. How were your initial Pathways instructors selected? What is the general background and professional rank of Pathways instructors in your department?(Approximate percent in each category: GTAs, part time instructors, teaching faculty, tenure track faculty.)

Piloting:

1. What is the general background and professional rank of the person who led the Pathways pilot at your university?
2. What was the structure of your pilot [number of sections, qualifications/background of pilot instructors, class size, structure of class (e.g., small section/large lecture with recitation)]?
3. Did instructors use other materials to supplement Pathways during your pilot? If so, explain.
4. What data was collected during the pilot?
5. What professional development was provided for the instructors during the pilot?

Scaling:

1. What criteria influenced the decision to scale Pathways beyond the initial pilot?
   a. Improvements in student learning [If so, how was student learning measured?]
   b. Student success equal to or greater than success rates prior to using Pathways?
   c. Passing rates or percentage of students receiving an A, B, or C? [Percentage of students continuing on to/passing calculus?]
   d. Buy-in of precalculus and/or tenure track faculty and/or departmental administrators.
   e. Other
2. What aspects of the materials were considered and/or influenced the decision to scale Pathways beyond 3 sections? Please elaborate.

3. To what degree was Pathways professional development valued by: a) the Pathways instructors? b) your departmental chair and/or other involved administrator(s)
   (1. Strongly, 2. Somewhat, 3. Very little, 4. Not at all)
4. To what degree did the Pathways professional development influence the decision to scale Pathways?
   (1.Strongly, 2. Somewhat, 3. Very little, 4. Not at all)
5. If you have not scaled Pathways to all sections of precalculus/college algebra at your university, what are the primary reasons for not scaling?

Sustaining/Not Sustaining:

(Answer these questions based on what has/is happening during the sustaining phase. If you are not in the sustaining phase, provide a response of N/A.

1. Describe the professional development offered for:
   a. continuing Pathways instructors;
   b. new Pathways instructors.
      (Be specific in describing how faculty are engaged during the professional development)
2. How many instructors/GTAs are involved in teaching Pathways at your university?
3. Approximately what percent of your Pathways faculty team are teaching Pathways for the first time each year? If you have the data to provide exact numbers of new instructors vs. returning instructors for the past 2-3 years, please do so.
4. How are new Pathways instructors/GTAs supported in understanding the mathematical goals of the Pathways curriculum?
5. What general structure is imposed on Pathways instructors and implementation?
   a. Common exams [If so, who creates them?]
   b. Common homework [If so, who creates/selects them?]
   c. Consistent homework deadlines If so, what deadlines are imposed? [For example, homework is submitted by students weekly, before each exam, between each class meeting. or individual faculty decide on their own homework deadlines.]
   d. A coordinator or leader determines workbook questions for class use.
   e. Students complete workbook questions in groups during class
   f. Other.
6. Which among student success in precalculus, student learning in precalculus, student views of the course, student continuation into calculus, student success in calculus, or other measures were tracked during the sustaining phase? [Explain what was/is being tracked if anything.]
7. Are the departmental administrators updated about Pathways student performance, satisfaction, success, or learning? If so, how are they updated?
8. To what degree does your current department chair value the goals of Pathways?
   (1. Strongly, 2. Somewhat, 3. Very little, 4. Not at all)
   Explain.
9. In your view, approximately what percent of your Pathways students value their experience in Pathways at the end of the course? (Please include any data you have or describe the basis for your approximation.)
10. Are instructors supported in getting students to value the more conceptual focused aspects of Pathways? If so, how?
11. Has the Pathways coordinator changed at any point during the sustaining phase? If so, has this impacted the Pathways intervention? Explain.
12. Has the departmental chair changed since Pathways was adopted? If so, has this impacted the Pathways intervention? Explain.
13. If Pathways has been scaled back or discontinued at your university, what primary factors do you conjecture to have influenced this decision?

The Pathways Coordinator(s)

1. What is the professional rank of Pathways coordinator(s)?
2. What are the responsibilities of Pathways coordinator(s)?
   ● What are the coordinator(s)' teaching responsibilities? Research responsibilities? Administrative duties?
3. Does or has the Pathways coordinator(s) at your university taught a section of Pathways at your university. [Please elaborate on the number of sections and frequency. For example, one course per semester for two semesters when adopting Pathways, but discontinued after that.]
4. Does the Pathways coordinator and/or local professional development leader hold regular meetings with Pathways instructors? If so,
   a. What is/are the primary goal(s) of these meetings?
   b. If possible, include a meeting agenda and/or specific task(s) that characterizes how the coordinator engages the instructors during the meeting.
   c. How often and how long are the meetings?
   d. Are Pathways instructional faculty required to attend?
   e. Are these meeting led by the Pathways coordinator or someone else? Who?

5. What 2-3 pieces of advice do you have for a new Pathways coordinator?

The Pathways Instructor

1. In your view, what does it mean to be a "highly effective" Pathways instructor?
2. How are your instructional faculty supported in becoming a more effective instructor?
3. How are new instructors/GTAs supported in preparing to teach Pathways for the first time? In your view, which of the supports have been most effective?
   a. To what degree are your instructors being supported in understanding the ideas and learning goals of the Pathways materials?
      (1. A lot, 2. Somewhat, 3. Very little, 4. Not at all)
      Please elaborate.
   b. Describe supports and/or strategies you believe have been most effective in advancing your instructors' knowledge of the course and how they are learned?
4. What advice do you have for Pathways leaders/local coordinators for improving faculty willingness to use the Pathways materials?
5. What advice do you have for Pathways leaders/local coordinators for improving faculty effectiveness in using the Pathways materials?
6. How (if at all) are your instructors supported in gaining insights into student learning, during the semester? at the completion of each semester?
7. How (if at all) are your instructors supported in providing more conceptual explanations?
8. How (if at all) are your instructors supported in asking questions that promote and/or focus on understanding student thinking? Explain.
9. A cross-cutting goal of Pathways PD and materials is to support instructors in promoting the *Pathways Problem Solving Practices*, including their ability to support students in conceptualizing quantities in applied problems and representing them in a drawing prior to trying to write a formula or construct a graph. How (if at all) are your instructors supported in promoting the Pathways Problem Solving Practices in their classroom?
10. What 3 pieces of advice do you have for a new Pathways instructor to assure their success during their first semester of teaching with Pathways materials?
11. Pathways workshop leaders promote "speaking with meaning" (e.g., referencing quantities, explaining why rather than just showing how). To what degree and how are Pathways instructors at your university encouraged to support their students in "speaking with meaning"? (1. A lot, 2. Somewhat, 3. Very little, 4. Not at all)

The Pathways Materials

1. What do you like most about the Pathways materials? [Name specific attributes and say why you like them/how they have impacted you.]
2. What are the primary shortcomings of the Pathways materials? What modifications would you like to see in:
   a. the workbook?
   b. The online materials?
3. What aspects of the Pathways materials have been most helpful for
   a) improving your conceptions of what it means to learn and/or understand specific ideas of precalculus
   b) improving your classroom practices, including your questioning, explanations, ability to make sense of student thinking/perspective? Explain.
   c) improving students' learning
   d) improving student successful completion of the course
   e) improving students' mathematical self-confidence
4. What is your view of how Pathways materials prepare students for learning calculus?
   Please elaborate.
5. Have you (or someone else at your university) supplemented the Pathways materials. If so, who and how?
6. Have you (or someone else at your university) adapted the Pathways materials? If so, how?
7. Describe how you (and your instructors/GTAs) use the:
   a) workbook investigations
   b) applets
   c) PowerPoint slides
   d) online homework
   e) online textbook
   f) videos
   g) homework in the workbook

Appendix B. Department Chair Interview Goals and Protocol

Goals of Interview:

1. Understand the department chair's goals for the department's precalculus course. Understand the degree to which the department chair understands and supports the goals of Pathways.
2. Understand the role of the Department Chair in adopting, scaling and sustaining the Pathways materials (degree and nature of support provided to the course coordinator and instructors; the degree to which local data was collected, analyzed, and shared with the department chair).

Interview Questions:

- What is your criteria for determining if an undergraduate course such as precalculus and calculus is generally effective? effective for students?
  - o What are your primary goals for student learning in precalculus? College algebra?
- In what courses were the Pathways materials used in your department?
  - o Who teaches this course in your department? What instructional format is used? Who decides what materials are used in a precalculus level course in your dept.?
- What is your understanding of the Pathways project?
  - o Were you aware that Pathways materials are based in/informed by research into student learning of calculus and precalculus concepts? If so, did this impact your decision to pilot Pathways? What is your understanding of how research on student learning informed Pathways development?
- Were you chair when the Pathways materials were adopted in your department? If so, what role did you play in the adoption of the Pathways materials?
  - o What factors led to your department deciding to pilot Pathways Precalculus?
  - o What factors led to your department deciding to scale Pathways to more sections?
  - o Is your department currently using Pathways materials and professional development in precalculus or college algebra? If not, what factors led to your department discontinuing the use of Pathways materials? If so, has the number of sections remained stable, declined, or increased over the past several years? If the number of sections has declined, in your view, what caused this decline?
- What (if any) data were collected during the pilot phase? During the scaling phase? Currently? How does/did your student passing rates (and PCA scores if collected) in Pathways precalculus compare with prior approaches?
- Have you reviewed the Pathways materials? If so, what do you like most about these materials? least about these materials? What additions/adaptions would you like to see in the Pathways materials?
- What challenges did Pathways present for you and other members of your department?: i) during the initial adoption of Pathways?; ii) during the scaling phase?; iii) during the sustaining phase?
  - o Did instructors express concerns? If so, how did you address their concerns?
  - o Did you receive complaints from students? If so, what was the nature of the complaints? How did you address the complaints?
- In your view, what are the attributes of a highly effective instructor in precalculus and beginning calculus?
- Who are the instructors for precalculus in your department? How are/were Pathways instructors/GTA's selected in your department?
  - o How are new Pathways instructors prepared prior to teaching with Pathways? That is, what is the nature of the professional development they receive (if any) before and during the semester when teaching Pathways for the first time?
  - o GTA's and mathematics instructors commonly report that they have not previously been supported in thinking about what is involved in understanding and learning precalculus ideas (e.g., average rate of change, exponential growth, linear growth, unit circle trig.) and beginning Calculus. Is this surprising to you?
    - ■ How do you expect that GTA's and precalculus instructors with no prior professional development focused on understanding and learning these ideas approach their teaching of these ideas?
    - ■ Do you see value in having Pathways instructors attend professional development to help them prepare for teaching with Pathways materials?
  - o In your view, has teaching with Pathways materials had an impact on Pathways instructors' teaching effectiveness when they teach other courses (e.g., Calculus)? Explain.
- In your view, what do students need to learn in precalculus so they are prepared for calculus?
  - o To what degree did/does Pathways goals for student learning align with your view of what students need to learn in precalculus at the university level?
  - o What is your view of skill based precalculus courses and their effectiveness in preparing students for calculus?
  - o Are you aware of Pathways focus on helping students develop ways of thinking that are needed to work applied problems, including the ability to model quantitative relationships with graphs and algebraically defined functions? Is this an ability you value? What ways of thinking, understandings, abilities do you believe are most important for students to learn in precalculus?

Appendix C. Course Coordinator Interview Goals and Protocol

Primary goals of the interview with the Course Coordinator:

- Elaborate details/expand answer to questions from the written survey.
  - o The interviewer should adapt this protocol as needed to expand survey answers.
- Gain insights into their understanding of the Pathways goals and how they align with their (past and present) goals for student learning in precalculus
- Gain insights into their role as a course coordinator.
- Understand their conception of the Pathways materials and how/if they have impacted their conceptions of precalculus ideas and how they are learned
- Understand their conception of the goals of the Pathways professional development and what they do in the weekly professional development.
- Gain insights into how/if they monitored student learning and success during and after the semester.

Interview Questions: Course Coordinator

- What are your primary goals for student learning in precalculus? College algebra? What do students need to learn in precalculus so they are prepared for calculus?
  - o To what degree did/does Pathways goals align with your view of what students need to learn in precalculus at the university level?
- What is your criteria for determining if Pathways is effective at your university?
- In what courses did/do you use the Pathways materials?
  - o Who teaches this course in your department? Who decides what materials are used in a precalculus level course in your department? To what degree are your precalculus level instructors equipped to support student learning (and understanding) of key ideas of precalculus, such as exponential growth, unit circle trig., average rate of change? In your view, has Pathways professional development (prior to and during the semester) been useful for helping instructors and GTA's use the Pathways materials?
- What is your understanding of the Pathways project?
  - o Were you aware that Pathways materials are based in/informed by research into student learning of calculus and precalculus concepts? If so, did this impact your decision to pilot Pathways? What is your understanding of how research on student learning informed Pathways development?
- Were you coordinator when the Pathways materials were adopted in your department? If so, what role did you play in the adoption of the Pathways materials?
  - o What factors led to your department deciding to pilot Pathways Precalculus?
  - o What factors led to your department deciding to scale Pathways to more sections?
  - o Is your department currently using Pathways materials and professional development in precalculus? If not, what factors led to your department discontinuing the use of Pathways materials? If so, has the number of sections remained stable, declined, or increased over the past several years? If the number of sections has declined, in your view, what caused this decline?
- What (if any) data were collected during the pilot phase? During the scaling phase? Currently? How does/did your student passing rates (and PCA scores if collected) in Pathways precalculus compare with prior approaches?
- What challenges did Pathways present for you, your chair, other instructors in your department i) during the initial adoption of Pathways?; ii) during the scaling phase?; iii) during the sustaining phase?
  - o Did Pathways instructors or GTA's express concerns? If so, how did you address their concerns?
  - o Did you receive complaints from students? If so, what was the nature of the complaints? How did you address the complaints?
- In your view, what are the attributes of a highly effective instructor in precalculus and beginning calculus?
- Who are the instructors for precalculus in your department? How are/were Pathways instructors/GTA's selected in your department? How are/were Pathways instructors/GTA's prepared to teach with the Pathways materials?
- Describe your role as the Pathways coordinator. In your view, what expertise do you have that led to you being selected as the Pathways coordinator?
  - o How are new Pathways instructors prepared prior to teaching with Pathways? That is, what are the goals?, What is the nature of the professional development they receive? How many days? Who leads?
  - o How are new Pathways instructors supported during the semester/term when teaching with the Pathways materials? Do you have weekly meetings? If so, what is your primary focus/goals for your weekly meetings? Describe a typical weekly meeting with Pathways instructors/GTA's. Do all Pathways instructors/GTA's attend weekly meetings during the semesters when they are teaching with Pathways materials? If not, what criteria are used to determine that an instructor/GTA no longer needs to meet the Pathways instructional team
  - o Pathways research has revealed that many instructors and GTA's in mathematics are not equipped to teach precalculus ideas such as exponential growth, unit circle trig., average rate of change, etc. meaningfully/conceptually to students. Is this surprising to you? Explain. In your view, to what degree did/does your professional development support your Pathways instructional team in preparing to teach precalculus ideas meaningfully? Explain.

  - How do you expect that precalculus instructors (with weak understandings of what it means to understand an idea) address these ideas/cope when teaching?
  - In your view, has teaching with Pathways materials had an impact on Pathways instructors' teaching effectiveness when they teach other courses (e.g., Calculus)? Explain.
- What do you like most about the Pathways materials? least about Pathways materials? What additions/adaptions would you like to see in the Pathways materials?
- In your view, what do students need to learn in precalculus so they are prepared for calculus?
  - To what degree did/does Pathways goals align with your view of what students need to learn in precalculus at the university level?
  - What is your view of skill based precalculus courses and their effectiveness in preparing students for calculus?
  - Are you aware of Pathways focus on helping students develop ways of thinking that are needed to work applied problems, including the ability to model quantitative relationships with graphs and algebraically defined functions? How do other precalculus curricula and/or approaches compare with Pathways in this regard?
- Describe your experience as a Pathways coordinator.
  - What have you liked/disliked about being the Pathways coordinator at your university? What advice do you have for new Pathways coordinators?
  - To what degree have you impacted the teaching effectiveness of the Pathways instructors? Explain.
  - To what degree (if at all) have the Pathways materials impacted your (and the other Pathways instructors/GTAs in your department) understanding of what is entailed in understanding and learning precalculus ideas? Explain.

Appendix D. Instructor Interview Goals and Protocol

Primary goals of the interview with Pathways Instructors/GTAs:

- Gain insights into their goals for student learning in precalculus and the degree to which the Pathways and their goals for student learning are aligned.
- Gain insighted into how they were supported in preparing to teach with Pathways materials, both prior to and during the semester/term when using the materials.
- Understand how they perceived and used specific Pathways materials/tools.
- Identify difficulties they experienced and how they responded/adapted.
- Gain insights into how they supported student thinking and learning during class and through homework.
- Understand how they approached their teaching (e.g., held students accountable for staying engaged, supported students in conceptualizing quantities and how they are related)
- Understand if/how they monitored student learning.
- Understand how their conceptions of precalculus ideas and how they are learned changed (if at all) as they attended PD and used the materials.
  - Identify aspects of the Pathways materials and PD that were effective for supporting advancements in their teaching and conceptions of precalculus concepts and how they are learned.

Questions for Pathways Instructors/GTA's:

- What are your primary goals for student learning in precalculus? College algebra? What do students need to learn in precalculus so they are prepared for calculus?
  - To what degree did/does Pathways goals align with your view of what students need to learn in precalculus at the university level?
  - Were you aware that Pathways is based in research on student learning in calculus and precalculus? What is your understanding of how research on student learning informed Pathways development?
- What course(s)/recitation sections have you taught using Pathways materials? When did you teach these courses? What was the instructional format (e.g., large lecture, small class, recitation)?
- Did you attend meetings and/or participate in professional development prior to using the Pathways materials? If so, what was the nature of these meetings/professional development? What aspects of this professional development was most helpful?
  - To what degree did you feel prepared to use Pathways materials after attending these meetings/professional development? Explain.
- Were you supported during the semester/term when teaching with the Pathways materials? Did you have weekly meetings? Describe a typical weekly meeting with Pathways instructors/GTA's. For how many semesters/terms did you attend weekly meetings when teaching with Pathways materials?
- Prior to participating in Pathways professional development, have you participated in other professional development that focused on understanding, learning, or teaching precalculus level ideas? If so, explain the nature of this professional development.
- Most university mathematics instructors and graduate students in mathematics have not been mentored in thinking about what is involved in understanding or learning ideas in precalculus. Did teaching with the Pathways materials improve your understanding of what is involved in understanding and/or learning any ideas in precalculus? Explain.

- What was your initial opinion of the materials? How has your opinion changed over the course of teaching with the materials for multiple semesters?
- Did teaching with Pathways materials have an impact on your image of effective teaching? If so, how?
- Describe your teaching. How much and when do you lecture? Do you have students work in groups? Do you call on students to explain their thinking? If a student provides an incoherent explanation, what do you do?
- To what degree do you try to understand your students' thinking when teaching? Explain. What specifically do you do (if anything)?
- To what degree do you hold your students accountable for: i) speaking with meaning? Ii) staying engaged during class? Explain. What specifically do you do (if anything)?
- In your view, what does it mean to be a highly effective instructor of precalculus?
- How did you use the Pathways materials (investigations, Powerpoints, applets, online homework) in your teaching?
  - How often did you assign homework to your students? Was the homework graded?
  - What was the nature of the homework you assigned (Pathways online, investigation questions, homework questions in the workbook, questions you developed)?
- Did you give quizzes or assessments to check your students' learning? If so, how did you use the data you generated? Did you adapt your teaching if you noticed student difficulties in learning a specific idea? Explain.
- What obstacles/difficulties did you encounter when teaching with Pathways? Did you seek help? Did you get help? If so, describe the nature of the help.
- When teaching with Pathways, did you feel supported by your course coordinator? department chair? Explain.
- What (if anything) makes teaching with Pathways materials more challenging for instructors?
- What (if anything) makes teaching with Pathways materials more enjoyable for instructors?
- What are the attributes of a teacher who emerges as a "highly effective" Pathways instructor?
- Approximately what percent of your Pathways students who were officially enrolled in your course received a grade of A, B, or C?
- Would you teach again with Pathways materials? Why/why not? What changes do you suggest for improving an instructor's experience in teaching with Pathways?
- Pick one idea in precalculus that you now teach differently after using Pathways materials? Explain how you teach this idea now as compared to how you taught it previously?

- What is your view of the Pathways focus on supporting students to conceptualize quantities in a problem context and how they change together? Explain.
  - In your view how (if at all) is giving students time and support in learning to initially conceptualize quantities in a problem context helpful for student learning and ability to define (and interpret) function formulas and graphs?
  - Can you explain what you would say and show what you would draw to illustrate how you help students conceptualize the quantities in the following problem context.
    - A Tortoise challenges a Hare to a 100-meter race and convinces the Hare to give him a 60-meter head start. They both are moving at a constant speed when the start gun is fired, with the Tortoise running the entire race at a constant rate of 3.6 meters per second and the Tortoise moving at a constant rate of 0.4 meters per second for the duration of the race.
  - To what degree (if at all) is it useful to ask/require students to: i) make a drawing to represent the quantities in a problem context prior to attempting to define a formula, answer a question, etc.; ii) be specific in defining their variables (include starting point of the measurement, direction of the measurement, units); iii) label their drawing with variables and expressions?, iv) be specific in referencing quantities when speaking; v) be specific in explaining the thinking they used when constructing a formula, drawing, graph, etc.
    - In what ways, if at all, are these conventions useful for student learning?

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