Instructional Conventions for Conceptualizing, Graphing and Symbolizing Quantitative Relationships

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1 Orienting to a Problem

We ask readers to think about the context in Table 1 before beginning this chapter and reflect on the reasoning you use to conceptualize and represent the relationships you consider.

What did you think about or imagine when reading the text? Make a drawing to represent the situation, then use that drawing to describe how pairs of quantities, whose values vary, are related and change together. How many distances appear in the description? Try describing each distance so that it is clear which distance you are referencing. What details did you need in your descriptions? Which distances have a value that varies, and which distances have a fixed value (or measure)? Can you verbalize how the distance between the Tortoise and Hare changes during the race? How might you support students in being able to conceptualize, verbalize, and represent this relationship?

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Table 1 One version of the tortoise and hare task (Carlson et al., 2020, p. 17)

A tortoise challenges a hare to a 100-m race and convinces the hare to give him a 60-m head start. They both are moving at a constant speed when the start gun is fired, with the hare running the entire race at a constant rate of 3.6 m/s and the tortoise moving at a constant rate of 0.4 m/s for the duration of the race

2 Introduction

Thompson (2008a) argued that "in the United States, the vast majority of school students rarely experience a significant mathematical idea and certainly rarely experience reasoning with ideas" (p. 31) due to "a systemic, cultural inattention to mathematical meaning and coherence" (2013, p. 57). Most U.S. students experience mathematics only as groups of procedures to memorize and employ (Boston & Wilhelm, 2017; Hiebert et al., 2005; Hill, 2021; Jackson et al., 2015; Laursen, 2019; Litke, 2020; Schmidt et al., 2005; Simon et al., 2000; Stigler et al., 1999; Stigler & Hiebert, 2009; Thompson, 2008a, 2013). In such a setting, students typically do not develop mathematical practices that lead to fluency in solving novel problems; nor do they construct strong meanings for key ideas necessary for success in calculus and STEM fields. For example, early studies of students' understanding of the function concept revealed weak meanings in students' function conception (e.g., Monk, 1992; Sierpinska, 1992; Vinner & Dreyfus, 1989). Studies report that students view (i) a function graph as a picture of an event (Bell & Janvier, 1981; Carlson, 1998; Leinhardt et al., 1990; Monk, 1992) or a static shape with specific properties (Carlson et al., 2002) and (ii) an algebraically defined function as a recipe for getting an answer (Breidenbach et al., 1992) or two expressions separated by an equal sign (Carlson, 1998).

We call for curriculum designers, professional development leaders, course coordinators, and instructors to foster learning experiences that support students in developing productive *ways of thinking* and coherent mathematical meanings essential for understanding calculus and continuing in STEM fields. It is our goal that students become confident and competent mathematical thinkers and problem solvers. Our data supports that students who understand ideas and acquire habits of reasoning meaningfully will be better equipped to spontaneously engage in productive reasoning and access their knowledge when learning new ideas and solving problems. This perspective is aligned with Harel and Thompson's call [as explained in Thompson et al. (2014)] for students to develop the "habitual anticipation of using specific meanings or ways of thinking in reasoning" (p. 13). For example, recent studies have identified ways of thinking that foster the emergence of function graphs as a record of a student's conception of how the value of one quantity varies with the value of another (Carlson et al., 2002; Moore & Thompson, 2015) and function formulas as a record of how pairs of quantities' values are related and vary together (Moore & Carlson, 2012; O'Bryan, 2020a; O'Bryan & Carlson, 2016).

Our call for mathematics instructors to support students in using and trusting their thinking when attempting novel problems and learning new ideas echoes the current recommendations from professional organizations (e.g., American Mathematical Association of Two-Year Colleges, 2018; National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010; Winsløw, 2021). However, even when targeted and sustained professional development training is available, and instructors use research- and inquiry-based materials, researchers report only minor shifts in most teachers' instructional practices, with many instructors continuing to focus on lecture as a means of transmitting knowledge to students (e.g., Baş-Ader & Carlson, 2021; Jackson et al., 2015). Such teacher-centered instruction is not attentive to student thinking, nor does it reveal to teachers the variety of ways students are conceptualizing and reasoning about mathematical ideas. Understanding and assessing student progress in applying their reasoning has been shown to be valuable for informing teacher task selection, questions, and explanations. These *decentering* actions (Baş-Ader & Carlson, 2021; Piaget, 1955; Steffe & Thompson, 2000; Teuscher et al., 2012) are a critical component of responsive teaching whereby instructors act in the moment and adapt their instruction by leveraging students' thinking to successfully make progress toward a lesson's learning goals.

3 The Need for Conventions to Facilitate Changes in Pedagogy and Student Success

Thompson and Carlson (2017) call for introductory undergraduate courses in mathematics to be alert to the static images many students possess for variables, function formulas, and graphs. They call for curriculum and instruction to reengage precalculus level students in trusting and using their reasoning to conceptualize quantities in a problem context, and then illustrate how a student's image of how quantities are related can lead to constructing symbols and graphs that carry meaning for the one constructing them. However, during 15 years of working with and studying precalculus instructors in the Pathways Project, we rarely encountered an instructor who engaged in and valued pedagogical practices for supporting students' development of dynamic imagery related to function relationships and their representations. Most instructors have only experienced traditional curriculum in both their learning and teaching experiences. They need focused professional development to help them reconceptualize mathematical ideas they thought they understood and to reconceptualize effective teaching as focused on and affecting student thinking.

We developed Pathways curriculum materials (Carlson & Oehrtman, 2010) to support instructors in fostering productive reasoning patterns in their students that research has revealed to be essential for students' construction of meaningful function formulas (e.g. Moore & Carlson, 2012; Thompson, 1988, 1990, 1992) and graphs (e.g., Carlson et al., 2002; Moore & Thompson, 2015). These include specific support for: (i) conceptualizing and speaking about quantities and how their values vary together, (ii) representing how two quantities change together using a graph, and (iii) representing quantitative relationships with expressions and formulas. As we refined the Pathways course materials over time, we strove to design for coherence by emphasizing reasoning about and making connections between the three strands of school mathematics that prepare students for calculus: the mathematics of quantity, the mathematics of variation, and the mathematics of representational equivalence (Thompson, 2008a).[1] We included problems, teacher notes, and other resources that we believed would help instructors engage students in constructing strong meanings for ideas from these strands in each lesson and support students in making connections between ideas across multiple strands.

In scaling the use of the Pathways materials, however, we faced persistent challenges in shifting instructors' pedagogical actions and perspectives on student learning to achieve our intended learning goals. Many instructors routinely missed opportunities to conceptualize how the ideas in each lesson were related to one of these three strands or make connections between ideas in different strands. This led us to reexamine the focus of our professional development training and to develop pedagogical conventions that we believed would support both instructor and student learning if enacted. We were also guided by Thompson's (1990) elaboration of the role of quantitative reasoning in students' construction of meaningful algebraic expressions. This led to our introducing instructional conventions over the course of 15 years when working with instructors, including classroom observations, professional development workshops, preservice teacher courses, and training programs for graduate teaching assistants. For example, when students successfully reasoned about applied contexts, we noted instructor moves that supported students in orienting to the problem (Polya, 1957) and engaging in productive problem-solving behaviors (Carlson & Bloom, 2005). In contrast, when students struggled to make sense of ideas or problems, we noted pedagogical actions that were not taken that could have supported students in making sense of relationships in the problem. We also analyzed clinical interview data that suggested actions (such as drawing detailed diagrams) that led to students' successfully conceptualizing quantitative relationships (Moore & Carlson, 2012).

[1] Thompson (2008b) describes these strands as follows. The mathematics of quantity refers to how individuals conceptualize measurable attributes of a situation, create measurement schemes to quantify the attributes' magnitudes, represent the quantities in various ways, and generalize aspects of these attributes. The mathematics of variation refers to how individuals imagine quantities with magnitudes that can vary, how they represent this variation in different ways, and how they draw inferences from noticing what in a relationship remains invariant as two quantities change in tandem. The mathematics of representational equivalence refers to how individuals think of arithmetic and, eventually, algebraic expressions non-computationally as "segues into structural properties of numbers and quantitative relationship[s]" (p. 7). These three strands are interrelated, and opportunities always exist to discuss elements of one strand even within contexts emphasizing another strand. For example, while supporting students in conceptualizing quantitative relationships in some given scenario it can be very natural to explore how two or more of the conceptualized quantities co-vary in tandem. Thompson states that "The three strands in interaction, each receiving appropriate emphasis, and always with the other two in the background, builds a foundation for algebraic reasoning that simultaneously builds a foundation for schemes of meanings that are crucial for understanding the calculus" (pp. 8–9).

Our observations revealed that even when teachers were committed to making their instruction more engaging and meaningful for students, they continued to rely on familiar instructional practices of providing vague explanations and showing students how to find answers. For students to build personal meanings for mathematical ideas requires that instructors create opportunities for them to construct these meanings and engage in productive habits of reasoning as often as possible (Harel, 2008). Our attempts to support instructors in consistently engaging their students in meaningful mathematical activity that begins with their conceptualizing quantities and their relationships as a basis for their graphing and defining activities led to our introducing the instructional conventions described in this chapter. These conventions include underlining phrases in a problem statement that describes quantities, precisely referencing quantities when speaking, constructing a drawing that depicts how quantities are related, physically tracking quantities as their values vary, etc. We introduce them to focus and structure student mathematical activity in support of their meaningful engagement in conceptualizing problems and learning new ideas.

4 Elaborating Quantitative and Covariational Reasoning

We adopt Thompson's theory of quantitative reasoning to explicate the ways of thinking we desire students to construct. According to Thompson (1988, 1990, 1993, 1994, 2011, 2012) *quantitative reasoning* is rooted in a disposition to conceptualize situations in terms of measurable attributes of objects and relationships among them. Quantitative reasoning then is a way of thinking about situations whereby an individual conceptualizes measurable attributes of objects (*quantities*) and organizes relationships between these attributes to form a structured mental representation of the situation. Quantities are mental objects unique to an individual. The way an individual conceptualizes a quantity and the set of quantities deemed relevant provides the space of implications for the reasoning an individual can engage in relative to a given situation (Smith & Thompson, 2007; Thompson, 1994). A key element of quantitative reasoning is *quantification* whereby an individual develops a method for reliably representing a conceptualized quantity's magnitude with a numerical value (that is, its measure). The quantification process is critical for an individual to develop meaningful mathematical models because "[i]t is in the process of quantifying a quality that it [the quality] becomes truly analyzed" (Thompson, 1990, p. 5). The nature of the conceptualized quantity matters a great deal in this process, as does the sophistication of the individual's magnitude schemes (see Thompson et al. (2014) for a discussion of these ideas). For example, the quantity "distance between two people" can be quantified as a multiplicative comparison to some imagined unit (either a standard unit like "meter" or "inch", or a nonstandard unit such as "length of the measurer's foot" or "the length of the measurer's pace"). Some quantities, however, such as the speed of a baseball pitch or the impact force of an automobile striking another automobile arise via conceptualizing a *quantitative operation*, or a mental operation of comparison/coordination of other quantities the individual has already conceptualized (Thompson, 1990, 1994, 2011). The resulting quantity cannot be directly measured in the same way that "the distance between two people" can be directly measured. As a result, quantifying these more complex quantities requires a scheme dependent on the quantitative operations from which they arose (Johnson, 2015; Moore, 2010; Piaget, 1968; Schwartz, 1988; Simon & Placa, 2012; Thompson, 1990; Thompson et al., 2014).

It is also important that students distinguish between quantities with a fixed magnitude and those with a magnitude that can vary. This distinction is key in conceptualizing mathematical models of a dynamic situation as representing the simultaneous covariation of two quantities' values. An individual reasons covariationally when she envisions two quantities' values varying in tandem (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017), and holds in mind a sustained image of the two quantities' values simultaneously (Saldanha & Thompson, 1998).

Thompson's (1988, 1990, 1993, 1994, 2011, 2012) theory of quantitative reasoning "is about a stratum of reasoning that lies beneath both applied arithmetic and applied algebra. It is about people using 'rigorously qualitative' reasoning, where rigor derives from the intention to attend to the quantification of a situation's qualities" (p. 3). The theory is most useful in considering how individuals come to understand quantifying qualities like heat, force, and torque—qualities that cannot be quantified via extensive measurement. However, for individuals to participate in quantifying and using this category of quantities in mathematical modeling they must begin by conceptualizing calculations, variables, and algebraic expressions as tools for representing the quantitative relationships they have conceptualized.

Despite the body of research pointing to the essential role of quantitative and covariational reasoning in students' mathematical development, there is a broad body of research in calculus learning that points to students' failure to conceptualize quantities, how they are related, and vary together as sources of challenges using calculus ideas to advance a problem's solution (Bressoud et al., 2016; Byerley, 2019; Carlson, 1998; Engelke, 2007; Mkhatshwa, 2020; Oehrtman, 2009; Thompson & Harel, 2021; Thompson, 1992; Zandieh, 2000). Our work in the Pathways Project focuses on operationalizing Thompson's theory of quantitative reasoning so that students develop the beliefs, expectations, and ways of thinking necessary to participate in meaningful mathematical modeling for success in calculus and STEM fields. In later sections, we elaborate how these constructs informed our thinking and design of Pathways conventions for supporting precalculus mathematics students' engagement in quantitative and covariational reasoning.

For now, we echo our claim that students' ability to construct meaningful formulas and graphs rely on their using quantitative reasoning to build a structured mental model of quantities (measurable attributes of objects) within a situation. To make this claim more transparent, we ask you to revisit the Tortoise and Hare context and follow-up questions we presented at the beginning of this chapter. Consider again how the following contributed to your ability to describe how the distance between the Tortoise and Hare changed during the race: (i) your conception of the quantities described in the text, (ii) your conception of the quantities to be related and how they vary together; (iii) the clarity with which you conceptualized the quantities (e.g., where a quantity's measurement begins, the direction of the measurement), and (iv) the clarity with which you represented the quantities and their relationships in a drawing.

Data from administering an item that presented the above context and asked students to define the Tortoise's distance (in meters) ahead of the Hare in terms of the number of seconds since the start of the race revealed that very few of over 1000 precalculus students were able to produce a correct answer. Further, data from administering the Mathematical Association of America's *Calculus Concept Readiness* (CCR) exam to 601 students from three different universities during their first week of calculus revealed widespread weaknesses in students' ability to define function formulas and interpret function graphs. In addition, only 28% of these 601 students selected the correct response (out of five multiple choice options) to an item that asked them to define the area A of a circle in terms of its circumference, C (Carlson et al., 2015). CCR data thus suggests widespread difficulties in students' ability to define function formulas to relate two quantities whose values vary as they begin calculus.

5 *Speaking with Meaning*: A Convention for Improving Instructors' Communication

Analysis of video data from instructors' professional learning communities (PLCs) (also reported in Clark et al., 2008) showed instructors being imprecise in referencing quantities and saying what a variable, expression, graph, and function formula represented when communicating with each other. These instructors regularly made vague references to a volume, height, time, etc. without making clear what volume, height, or time they were considering. We also noticed a pervasive use of pronouns that made it difficult for other instructors to understand what the speaker was imagining and conceptualizing when completing a problem. Their inability to be specific in describing and representing quantitative relationships appeared to reveal their weak conceptions and it was common for instructors to pretend to follow incoherent explanations resulting in meaningless exchanges among the instructors.

After our pointing out the difficulties we were experiencing in following their explanations, we collectively negotiated a specific goal to speak more meaningfully when discussing ideas and problems during the PLCs. This led to the project leaders negotiating with the PLC members patterns of speaking that we conjectured would improve communication about the mathematical ideas and how they are learned. We collectively decided to restrict the use of pronouns by requesting that all PLC members be precise in referencing the quantity they were imagining, including the direction of measurement, the starting point for the measurement, and the unit of measure. Our goal was to focus instructors' attention on the coherence of their speaking and how they might be interpreted by others. We were hopeful that reflecting on these issues would motivate them to expend the mental energy to improve in their ability to *speak with meaning*. Since retrospective analysis of the PLC videos revealed that the instructors' classroom explanations mirrored those they provided in the PLCs, we were hopeful that *speaking with meaning* would become normative within their classroom discussions as well.

Our subsequent analysis of the PLC videos, after agreeing on conventions for speaking, revealed instructors gradually becoming more fluent in referencing quantities and describing quantitative relationships. The instructors' language steadily shifted (with consistent reinforcing) to their describing the quantity they were conceptualizing by stating what was being measured, the unit of measurement, the starting point for the measurement, and the direction of the measure (e.g., Juan's distance in feet north of the stop sign). We further noticed that the instructors' ability to precisely reference the quantities and describe how pairs of quantities are related was accompanied by improvements in the instructors' ability to construct formulas and graphs that accurately represented the quantitative relationships described in a problem. When instructors expressed frustration while attempting to *speak with meaning* about a problem they were discussing, they typically had not taken time to conceptualize the quantities described in the problem context.

As our work to support instructors in communicating their thinking to their peers continued, we gradually introduced other conventions for communication, including: (i) the speaker verbalizing her thinking and the rationale for her choices, rather than describing what she did to get an answer; (ii) all members of the PLC attempting to make sense of the thinking of the speaker, instead of only listening to the words being spoken; and (iii) all members of the PLC asking a question if something was unclear, instead of pretending to understand when they were unable to follow. We formalized the convention *speaking with meaning* as a research construct for our continued study by saying,

> An individual who is *speaking with meaning* provides conceptually based descriptions when communicating with others about solution approaches. The quantities and relationships between quantities in the problem context are described rather than only stating procedures or numerical calculations used to obtain an answer to a problem. Solution approaches are justified with logical and coherent arguments that have a conceptual rather than procedural basis. (Clark et al., 2008, p. 297)

According to Yackel and Cobb (1996), considering how others might make sense of explanations requires a shift in perspective from only viewing explanations as something one gives or hears to making the explanations themselves an object of reflection. Thus, when one speaks, they concurrently imagine how their utterances might be interpreted. The capacity for an individual to anticipate how they might be interpreted has been termed *decentering* (Piaget, 1955; Steffe & Thompson, 2000). Our perspective on decentering is elaborated elsewhere (Baş-Ader & Carlson, 2021; Carlson et al., 2004; Teuscher et al., 2012), with these studies revealing that US secondary and university precalculus instructors are generally not oriented to making sense of students' thinking or considering how their explanations might be interpreted by others. This finding is consistent with findings reported in international studies (e.g., TIMSS) and research on teaching (Baş-Ader & Carlson, 2021; Teuscher et al., 2016; Thompson, 2013, 2016). As a result, our data and observations of the impact of instructors adopting the convention that a speaker *speak with meaning* and a listener's attempt to make sense of another's spoken words, revealed cognitive shifts in instructors' conceptions, and substantial shifts in instructors' conversation toward an improved understanding of each other's perspective (Clark et al., 2008).

We should note that in a few cases a PLC leader did not consistently model or reinforce *speaking with meaning*. The conversations and explanations in these PLCs did not shift to become more meaningful or coherent and instructors were observed agreeing with incorrect solutions and illogical explanations. According to Yackel and Cobb (1996), a sociomathematical norm refers to a normative behavior specific to mathematics, such as understanding what constitutes an acceptable mathematical solution or what counts as an acceptable mathematical behavior in a group setting. We reemphasize that the pattern of *speaking with meaning* as a new sociomathematical norm only became normative in settings where the PLC leader was consistent in modeling *speaking with meaning* and consistent in reinforcing speaking with meaning among the instructors.

6 Scaling the Convention of Speaking with Meaning Across the Pathways Project

After five years of research and development of Pathways interventions, the pre-post-gains of student learning using both the validated PCA (Carlson et al., 2010) and Calculus Concept Readiness (CCR) exams (Carlson et al., 2015) were highly significant. The mean PCA scores ranged from 13.5 to 18 (out of 25), representing pre- post-gains of 5–9 points on average. At this stage of the Pathways project, we made the curricular materials and professional development available for other universities, creating an opportunity for us to continue documenting speaking patterns among new communities of Pathways users.[2] The trends described previously among communities of new Pathways instructors, including using a single word to define a variable, providing calculational explanations, etc. were normative at the beginning of all 12 Pathways professional development workshops that proceeded a university deciding to adopt Pathways materials. In Table 2 we provide concrete examples of common speaking patterns and contrast vague speaking with what we considered to be speaking that is more meaningful.

[2] Since our initial introduction of the term *speaking with meaning*, 11 new colleges/universities have participated in Pathways professional development.

Table 2 Examples of *speaking with meaning* compared to statements that show an absence of *speaking with meaning*

Speaking with meaning	
Absence of speaking with meanings	Speaking with meaning
• The graph of the car's distance falls to the right	• The car's distance south of the stop sign (in feet) is decreasing as the number of seconds since the car started moving increases
• *f* (7) is 20 tells me that when I plug in 7 I get 20	• Since *f* (7) is equal to 20, the tank had 20 gallons of water 7 min after the tank started draining
• I multiplied 1.08 by $2000 to get my answer	• Since the amount I must pay is 8% more than the price, the amount I must pay is 1.08 times as large as $2000
• 24 divided by 5 is 4.8 because 5 goes into 24 4.8 times	• Since I need to determine how many times as long 24 in. is as compared to 5 in., I must divide 24 by 5. My answer tells me that 24 in. is 4.8 times as long as 5 in.
• Since the graph curves up the distance is getting larger from 0 to 5	• During the first 5 s of the race, the distance travelled by the runner over successive fixed amount of time increases. This also means that the runner is speeding up during the first 5 s or the race

6.1 Speaking with Meaning in Instruction and Curriculum

We leveraged the insights from our study of PLCs (Clark et al., 2008) in our initial workshops with instructors preparing to use Pathways materials at a particular university. In doing so, we consistently modeled *speaking with meaning* and asked for clarification when workshop participants were unclear about what quantity they were referencing. Imprecise quantity references that failed to describe the starting point of the quantity's measurement such as "time elapsed" or "hours passed" were called out for clarification (e.g., "the number of minutes since 9 am", the number of seconds since the car left home). When a workshop participant used pronouns or made imprecise statements like, "its distance is getting closer" the workshop leader might ask questions like, "What is getting closer?" "What distance?" "Closer to what?", hoping to raise the instructor's awareness of the need to be more specific in describing the distance the instructor was imagining. Our persistent probing typically generated a response like, "the car's distance north of the intersection is decreasing". The reinforcement of *speaking with meaning* in the initial workshop made the criteria for *speaking with meaning* public among all workshop participants. It is also noteworthy that workshop participants began to ask each other for clarification when speaking among themselves.[3] The Pathways curriculum materials further supported

[3] In universities where a commitment to *speaking with meaning* became a social mathematical norm and was highly valued by the local coordinator in our initial Pathways workshop, we have since documented the persistence of *speaking with meaning* in this local community of instructors.

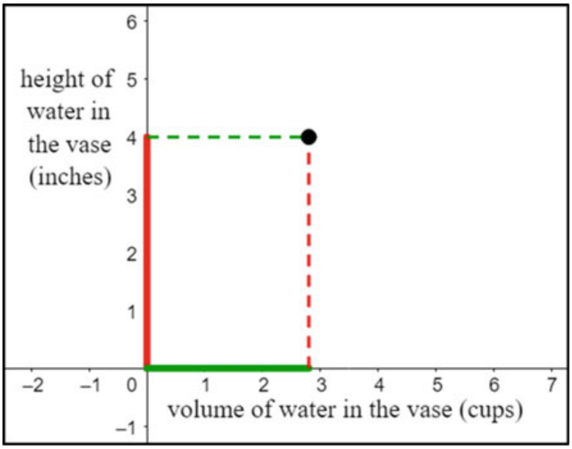
an instructor's shift to *speak with meaning* by providing open-ended questions, opportunities for students to select and clearly define variables of interest, and requests for students to explain and justify their reasoning. The instructor materials also provide detailed solutions and explanations that model *speaking with meaning*, giving instructors clear examples of the preciseness in speaking that is needed when responding to specific problems and questions. Our observations of Pathways instructors' classrooms reveal gradual shifts in their effectiveness in using and reinforcing *speaking with meaning* with variation in their: (i) consistency in modeling *speaking with meaning*; (ii) commitment to making *speaking with meaning* a classroom convention all students adopt; and (iii) consistency in asking for clarification when a student produces vague descriptions and/or explanations.

6.2 *Emergent Shape Thinking* and Conventions for Meaningful Graphing Activity

Researchers have documented students' impoverished conceptions of graphs, including their conceiving of a graph as a picture of an actual event (e.g., Bell & Janvier, 1981; Carlson, 1998; Kaput, 1992; Leinhardt et al., 1990; Monk, 1992). For example, Carlson (1998) reported that high performing precalculus students interpreted the speed-time graphs of two cars as the paths on which the cars were driving and the intersection of the two graphs as a collision location. In the same study she reported that high performing second semester calculus students could say nothing more detailed about a graph's inflection point than it being the location on a graph where the graph changes concavity, and when pressed to explain what the concavity conveyed about the quantitative relationships they responded with comments about the curvature of the graph. In contrast, more recent studies have revealed five levels of student reasoning as they attempt to construct a graph of two quantities as their values varied together in a non-linear pattern (Carlson et al., 2002) and have demonstrated the utility of students' graphing activities as emerging from their conceptualizing two quantities' values varying in tandem while imagining how the two quantities' values are changing together (e.g., Carlson et al., 2002; Moore & Thompson, 2015).

Moore and Thompson (2015) classified and contrasted students' ways of thinking about graphs in terms of the thinking they used to construct the graph. They say that a student is engaging in *static shape thinking* if the student conceptualizes the "graph as an object in and of itself, and as having properties that the student associates with learned facts." For example, a student who constructs a graph by plotting points and applying algebraic methods to identify roots, inflection points, maximum/minimum values would be engaging in *static shape thinking*. In contrast, a student is said to be engaging in *emergent shape thinking* when the graph's trace emerges from the student considering two quantities' values as they vary together. A student engaging in the first three mental actions described in the Carlson et al. (2002) covariation framework would be engaged in *emergent shape thinking*. The mental actions as characterized

Fig. 1 A conception of a point as the simultaneous values of two quantities (a multiplicative object)



in the context of a student constructing a graph of the height of water in a vase in terms of the volume of water in the vase entails: (i) conceptualizing two quantities' values varying together (i.e., the volume of water in the vase and the height of the water in the vase) (MA1); (ii) conceptualizing the two quantities' values varying simultaneously and continuously, while considering the direction of the variation of each quantity's value (as the volume of water in the vase increases the height of the water in the vase increases) (MA2); (iii) conceptualizing how the values of the two quantities vary together by imagining a successive fixed amount of variation in one quantity while considering the amount of variation in the other quantity (considering how much the water's height varies while considering successive fixed increases in the volume of water) (MA3).

When the student has linked together two measurable attributes of the vase (the volume of water in the vase and the height of the water in the vase), they have conceptualized a *multiplicative object*, an object that simultaneously combines the attributes of two conceived quantities (Saldanha & Thompson, 1998; Thompson, 2011). A student who has conceptualized the two quantities as a multiplicative object may find it useful to sometimes consider the variation in one quantity only; however, when doing so the student will have a persistent awareness that the other quantity's value is also varying (Thompson & Carlson, 2017) (see Fig. 1). The volume of water in the vase (in cups) and the height of water in the vase (in inches) vary in tandem, and a point is used to represent the simultaneous correlated values of each quantity.

7 Pathways Conventions for Graphing

The Pathways Project uses specific conventions to support students in conceptualizing a graph as a record of how two quantities' values vary together. Prior to introducing graphing, we engage students in using what we call the *quantity tracking tool.*

While observing a dynamic event (e.g., someone walking across the room from one wall to another) or an applet that displays a dynamic event (e.g., a vase filling with water) in which at least two quantities' values are varying, students are prompted to move their index fingers to track the variation in two quantities' values. The *quantity tracking tool*, as first described in Thompson (2002), supports students in conceptualizing graphs as emergent traces produced from the coupling of values for two co-varying quantities. According to Thompson (2002), conceptualizing graphs as a record of simultaneous variation requires having students.

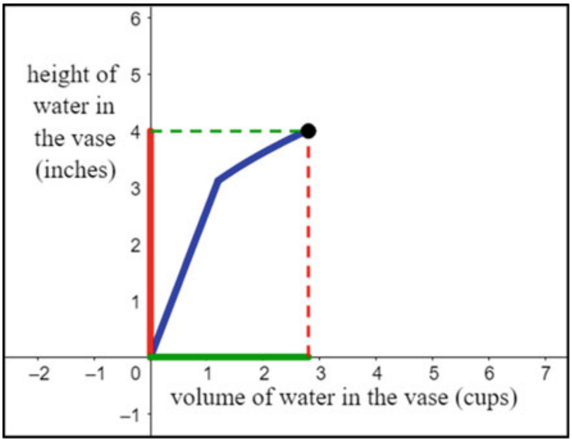
> internalize their perceptions of two quantities whose values vary, making that variation experientially concrete. To make covariation of quantities values experientially concrete, it is essential that they envision a single quantity's variation as itself having momentary states and therefore that the attribute whose value varies has momentary values. (p. 206)

When introducing the *quantity tracking tool*, the instructor negotiates a location for students' index fingers that represents a measurement of 0 units for each quantity either on (or beside if negative values are being tracked) each student's desk. The instructor also engages the students in deciding on a direction for moving each index finger when representing increasing and decreasing values.[4] The instructor then prompts students to use one index finger to track the variation in one quantity's value (e.g., the height of water in a vase in inches) as they observe the dynamic movement or applet. This is repeated several times, with the instructor prompting specific students to describe what they were imagining as they moved their finger. Students next pick another quantity (in the same problem context) whose value is varying in tandem with the first quantity (e.g., the volume of water in the vase in cups or the number of seconds elapsed since the waterspout was turned on). After the class has decided on a direction for the measurement, they track the quantity's value as they again observe the dynamic event or applet. The instructor may also stop the event or applet and prompt students to say what they are imagining as they move their index fingers. The instructor then prompts students to use both index fingers to simultaneously track the two quantities' values while observing the situation unfold. To ensure that students are engaging in quantitative reasoning, it is important that the instructor prompt students to explain what their finger movement represents, what the initial location of one or both of their index fingers represents, how they know what direction to move their fingers, etc. (see the next section for a detailed example of using the *quantity tracking tool*).

It is noteworthy that use of the *quantity tracking tool* requires students to both conceptualize and track a quantity's measurement, including where the measurement begins and the measurement's direction, prior to moving both index fingers to track the simultaneous value of the two quantities as their values vary. The convention that each finger be moved up or down (or right or left) from a common starting point keeps students' focus on the magnitude of the quantity's value in contrast to

[4] When initially using the *quantity tracking tool* students typically decide that a positive value of one quantity is represented by a distance upward from a starting point and that positive values of a second quantity are represented by a distance to the right of the same starting point, while negative values are downward and to the left respectively.

Fig. 2 Graphs emerge as traces



tracing a pre-imagined shape. We also see the *quantity tracking tool* as providing a meaningful foundation for conceptualizing coordinate axes as measurement tools for two quantities' values. Ours and others' studies (e.g., Frank, 2017a, 2017b) suggest that students typically lose sight of the fact that each axis is a measuring tool for one quantity, and that every point on a graph represents the simultaneous measurement of two quantities. The Pathways convention of drawing dashed lines from the point back to the axes (as in Fig. 1) are one attempt to reinforce the conception of a point as the simultaneous value of the two quantities at an instance during the unfolding of the dynamic event.

We encourage instructors to leverage the ways of thinking supported in the *quantity tracking tool* as a basis for constructing graphs in their lessons. As the unfolding of a dynamic event is displayed in an applet or animation, instructors support students in seeing the graph materialize with the leading point on the emergent trace projected back to the axes with dashed lines (Fig. 2) and the point representing the simultaneous values of two quantities (with values determined from the axes). In this way attributes of the relationship between the covarying quantities also emerge as "properties of covariation" (Moore & Thompson, 2015, p. 786), such as whether one quantity increases or decreases as the other quantity increases and where and why this behavior may change.

8 Implementing the *Quantity Tracking Tool*

We designed an applet to assist students in conceptualizing the quantities in the Tortoise and Hare context. Recall that the Hare agrees to give the Tortoise a 60-m head start (Fig. 3).

As the instructor moves a slider smoothly to vary the number of seconds since the start of the race (or hits "play" and allows the slider to move automatically), each

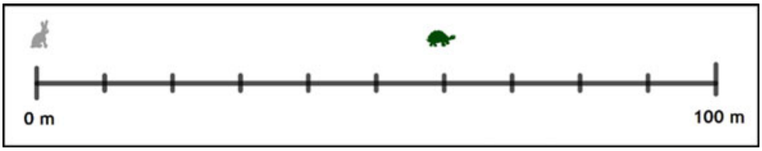


Fig. 3 The initial positions of the tortoise and hare

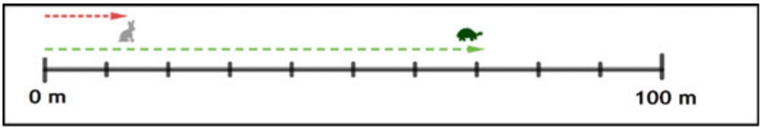


Fig. 4 Tracking each animal's distance from the starting line

animal's distance from the starting line is represented as a dashed line with an arrow pointing to the right (Fig. 4).[5]

As the applet plays, *the elapsed time since the start of the race* varies as does *the amount of time remaining until the Tortoise finishes the race* and *the amount of time remaining until the Hare finishes the race*. There are also five distances with values that vary as the race plays out. Let us assume that the instructor has begun exploring this context with her class and is now interested in exploring how *the distance (in meters) the Tortoise is ahead of the Hare* varies with the *elapsed time (in seconds) since the start of the race*. The conceptions and imagery students construct while the teacher engages them in a conversation using the applet and *quantity tracking tool* is impacted by the instructor's effectiveness in focusing students' attention on conceptualizing the relevant quantities and how they vary together. To illustrate this point, we provide a brief example of hypothetical instructor and student actions for promoting quantitative reasoning in students (see Table 3[6]). Note that the questions posed by the instructor in the context of the applet and *quantity tracking tool* continue to promote *speaking with meaning*.

Constructing a graph that carries meaning for the one constructing it relies on the individual having in mind a goal to represent the simultaneous variation of two quantities' values as they vary together. When enacting the *quantity tracking tool*, students' attention is focused on coordinating the magnitudes and directed measures of quantities with values that vary in tandem. As students make decisions about the starting position for their index fingers, they consider the starting point (0 value) for each measurement. As they begin to track a quantity's value by moving their

[5] The instructor selects a specific button to indicate which two quantities to isolate when exploring the concurrent variation in two quantities' values.

[6] The convention of *quantitative drawing*, as explained later in this chapter, can further support students in conceptualizing the quantities we want them to coordinate with the *quantity tracking tool*. The conventions in this chapter all support each other to maximize students' learning opportunities.

Table 3 Utilizing the *quantity tracking tool* in a class setting with students to help them conceptualize relevant quantities and relationships between quantities

	Instructor actions/statements to students	Student actions	Follow-up questions/observations
1	"Place your right index finger on your starting position for measuring the elapsed time since the race began."	Students choose a starting position for their fingers	Pose questions to specific students: "What does the position of your finger represent?" If student say "time," ask them, "What time?" Continue probing students until your weakest students can verbalize a precise description of the quantity (e.g., "The number of seconds elapsed since the start of the race.")
2	"As I move the slider on the applet, move your right index finger in a way to represent the number of seconds elapsed since the start of the race." (repeat if some students don't participate)	All students move their right index finger to the right from the designated starting point for measuring the number of seconds elapsed since the start of the race. The motion is smooth and continuous. See Fig. 5	Pose questions to specific students: "What were you imagining as you moved your finger? What does the starting position of your finger represent? What does the ending position of your finger represent? Should your finger be moved smoothly? Explain."
3	"To represent that the tortoise is a *distance of 0 m ahead of the hare*, position your left index finger at the same initial position as your right index finger. Positive values will be above this position."	Students position their left index fingers at the same initial position as their right index finger	Students often overlook exploring how the distance that the tortoise is ahead of the hare varies during the running of the race. Ask students to point out where they "see" this quantity's magnitude represented on the drawing
4	"Now, place your left index finger on a starting position that represents the approximate distance that the tortoise is ahead of the hare at the start of the race."	Students place their left finger some distance above the starting reference point	Note that some students will likely place their finger at the position that represents 0 m from the starting line. Ask these students to say what a 0 value represents. Ask questions like, "Where is the tortoise relative to the hare at the start of the race?"
5	"As I move the slider on the applet, move your left index finger in a way to represent the approximate distance between the tortoise and hare."	All students move their left index downward from their designated starting point. The motion is smooth and continuous. See Fig. 6	If some students do not participate or move their index finger in the wrong direction, prompt them to explain their thinking; then replay the race being run

(continued)

Table 3 (continued)

	Instructor actions/statements to students	Student actions	Follow-up questions/observations
6	Prompt students to model *the tortoise's distance ahead of the hare* in terms of *the number of seconds since the start of the race* **using both index fingers** at the same time	All students coordinate the motion of their right and left index fingers to represent the relationship between the two quantities' values. The motion is smooth and continuous. See Figs. 7 and 8	Since textbooks and instructors commonly make requests for students to represent one quantity "in terms of" another. It is important to introduce this language (and make sure students are clear on what it means) so students can attach this request to the mental operations of covarying *the tortoise's distance ahead of the hare* with the *amount of time that has elapsed since the start of the race*

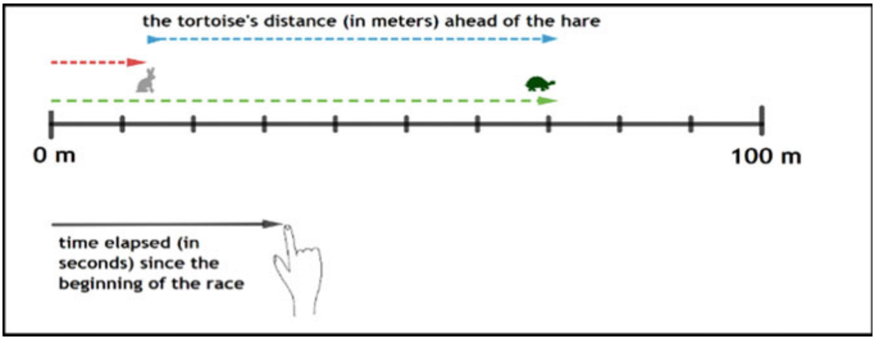


Fig. 5 A student's right hand as she uses her index finger to sweep out a duration of elapsed time while the animation plays

index finger, they associate a direction of movement for positive measurements and a direction for negative measurements (Figs. 5 and 6).

A point on a graph in the coordinate plane is then viewed as the co-occurring values of the two quantities (a multiplicative object) at an instance as the two quantities' values vary together (Figs. 7 and 8).

Given the broadly documented difficulties students encounter in creating and interpreting graphs (e.g., Carlson, 1998; Monk, 1992) and the commonly held view that a point on a graph is a result of a sequence of actions (count over 4 and down 5), shifting students to conceptualize graphs as an emergent trace of the values of two covarying quantities requires repeated reinforcement. Tracking the simultaneous variation in the two quantities' values by physically moving one's index fingers together promotes students' conceiving of the two quantities' values as coupled (a

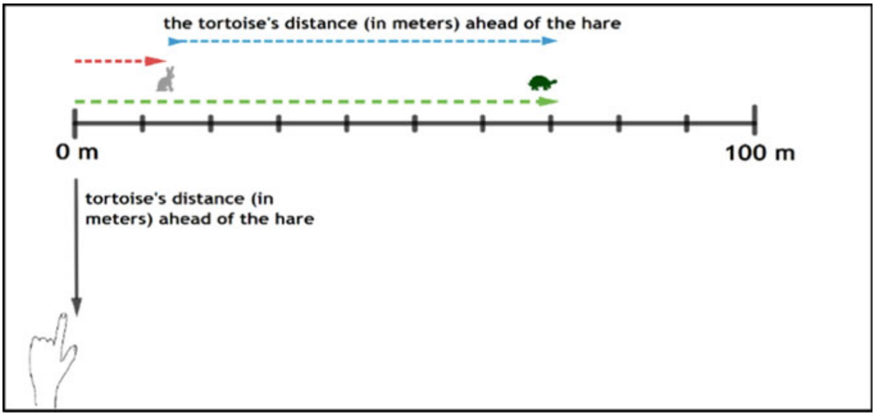


Fig. 6 A student's left hand as she uses her index finger to sweep out the distance the tortoise is ahead of the hare while the animation plays

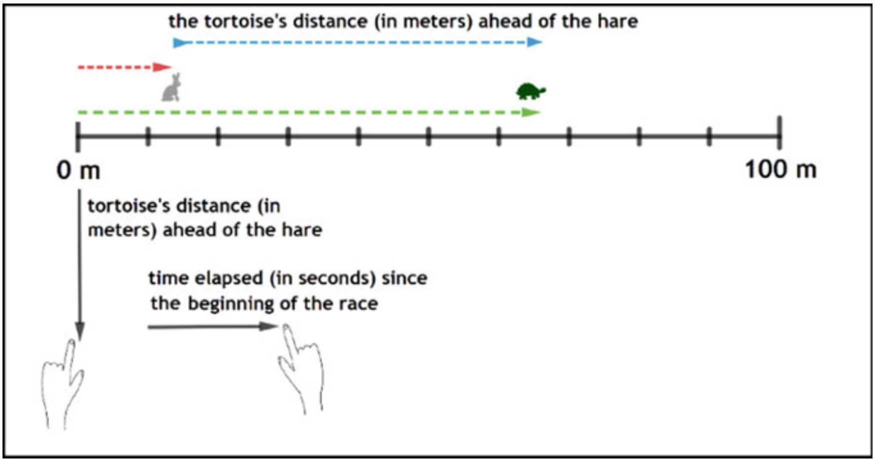


Fig. 7 A student using her index fingers to coordinate time elapsed (in seconds) since the beginning of the race with the Tortoise's distance (in meters) ahead of the Hare

multiplicative object) (Fig. 8), while considering how the values of the two quantities vary together.[7]

[7] The conceptualizations for enacting the *quantity tracking tool* parallels the thinking for constructing a graph of two quantities' values as they vary together.

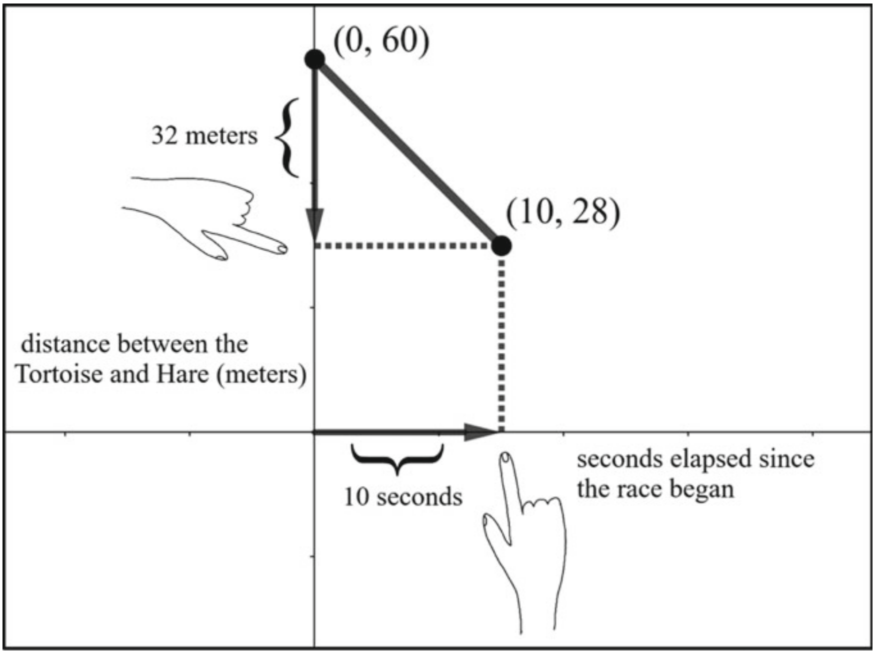


Fig. 8 The imagery we want students to develop for a graph as an emergent trace while using the *quantity tracking tool*

9 *Emergent Symbol Meaning* and Conventions for Meaningful Symbolization Activity

Generating a meaningful algebraic representation of a relationship between quantities' values relies on individuals organizing quantities they find relevant in a structured mental model. Constructing a structured mental model of the quantitative relationships depends on the individual's ability to conceptualize new quantities by relating two other quantities. As an example, one might conceptualize the relative size of quantity A with respect to quantity B and represent this new quantity as a quotient, (size of quantity A)/(size of quantity B), where the result, as well as any expression representing that result, is understood to be that relative size measurement. Representing a quantitative structure in a drawing can be useful for advancing an individual's image of a problem's quantitative structure while laying a foundation for producing a meaningful algebraic representation that relates two quantities' values as those values vary.[8]

[8] We repeat that the conventions and ideas described throughout this chapter support each other. *Speaking with meaning*, the *quantity tracking tool*, and *quantitative drawing* all support students in using symbols meaningfully and making connections between different representations of the same relationship.

Thompson (1990, 1993, 2011) draws a careful distinction between quantitative operations and arithmetic operations. A person uses arithmetic to calculate quantities' values, but the choice of operations is based on the quantitative operation conceptualized—the way the individual has formed a new quantity in their mind as a relation involving two other quantities. For example, "difference" as a quantitative operation is an additive comparison[9] between two quantities. However, a difference is not always evaluated using subtraction. Thompson uses the following problem to demonstrate this idea with a difference evaluated via division: "Jim is 15 cm taller than Sarah. This difference is five times greater than the difference between Abe and Sam's heights. What is the difference between Abe and Sam's heights?" (Thompson, 1990, p. 11). This example illustrates why the common pedagogical approach of training students to key on specific words like "difference" when determining the operation for combining two quantities to represent a new quantity can be problematic and sometimes leads to constructing symbols that do not represent the quantitative relationships described in the problem.

It is also our experience that instructors often focus on the "sameness" of the solutions produced by multiple solution paths instead of highlighting and emphasizing the unique reasoning, variety in quantities and structures conceptualized, different conceptualizations of problem goals, and so on. They also tend to ignore parallels between steps in a reasoning process (including the evaluation of various quantities) and the steps in a numerical or algebraic solution process (including highlighting which quantities are evaluated/represented at each step). Instead, they defer to algebraic "equivalence" even when the solutions represent unique ways of conceptualizing the context. The impact of this is that students receive the message that, even though there are multiple solution paths, there is one preferred path and algebraic form of a solution, which often does not foster students' confidence in their own mathematical reasoning and sense-making.

10 Quantitative Reasoning and Algebra

It is the attempt to generalize the quantification process for a quantitative relationship where algebraic expressions enter the picture and where distinctions between quantitative and arithmetic operations become critical. Thompson (1992, 1996) describes a study where fourth-grade students were free to develop their own notational methods for communicating their reasoning about decimal place value within activities he designed to encourage students to move back and forth between a situation and how the individual wanted to express that situation with notation. The result was that

[9] An "additive comparison" is the answer to the question, "By how much does the magnitude (or value) of one quantity exceed the magnitude (or value) of another quantity?" In contrast, one category of "multiplicative comparisons" is the answer to the question, "One quantity is how many times as large as a second quantity?".

students talked about their images of the situations presented "*as they spoke about notational actions* [italics in original]" (Thompson, 1996, p. 16).

As a precursor to seeing an algebraic expression as representing the value of a quantity (in underdetermined form), two understandings are paramount. (1) The individual has a productive conceptualization of the quantity for which he is attempting to represent the value. (2) The individual sees the expression that evaluates the quantity, as well as its numerical value, as representing the quantity's measurement. For example, imagine dropping an object from a tall building. Between two moments in time, the object's height above the ground changes from 42 to 29 ft. In order to evaluate the quantity, *the change in the object's height above the ground*, the individual must understand that the change in the object's height above the ground is a *difference*, and moreover there is a frame of reference (the 0 value is the ground and a positive value represents a distance above the ground) such that the difference from an initial value to a final value is a directed change that indicates the direction of the movement (upward or downward). With this conceptualization, he can understand that the expression "29–42 ft" represents the value of this change (directed difference), as does "– 13 ft". Individuals with both understandings are poised to understand how the variable expression "h – 42 ft" (where h represents the object's height above the ground in feet) represents the change in the object's height above the ground (from its initial height) at any moment during its fall. Thus, it is important that students see unevaluated expressions as representing a new quantity that is the result of performing a designated quantitative operation, in addition to the final numerical value (Thompson, 2011) (see Table 4). *Note that the table describes basic quantitative operations for sum, difference, product, and quotient. This list is not exhaustive. Again, we emphasize that student conceptualization of quantitative operations is what determines their choice of operation—not key words. And more complex quantities such as measurements of force, torque, work, and so on often require multiple levels of comparison and coordination.*

Within Thompson's theory, there is no assumption that proficiency with quantitative reasoning (or a disposition to reason quantitatively) is a natural outcome of participating in mathematics courses. In fact, research reports frequently point out how instruction in the United States produces students (and instructors) without this proficiency (e.g., Moore & Carlson, 2012; O'Bryan, 2020b; Thompson et al., 2017; Yoon et al., 2015). As we mentioned earlier, professional development training designed to introduce instructors to the usefulness of quantitative reasoning and provide support for implementing activities to support quantitative reasoning in the classroom often failed. Introducing Pathways course materials have helped make shifts in students' and instructors' meanings possible, but it remains an ongoing challenge to help instructors create lessons and consistently engage students in discourse focused on helping students strengthen their quantitative reasoning skills. O'Bryan and Carlson (2016) report on one instructor who did create such a learning environment for her students. The primary finding was that this instructor had internalized a set of expectations about engaging in mathematical reasoning (especially in how that reasoning connected to algebraic representations of mathematical relationships). It was these expectations that drove her use of quantitative reasoning and the

Table 4 Quantities to be evaluated along with the expressions that represent those quantities' values

Concept	Specific example	General example
Combining quantities additively	Mario is 6.5 years older than his sister Lexi. If x represents Lexi's age in years since she was born, Mario's age is $x + 6.5$ years	Quantity A is measured in some unit and has a measurement of a in that unit. Quantity B is measured in a compatible unit and measures b in that unit. The sum $a + b$ represents the measurement of the quantity formed by combining Quantity A and Quantity B additively
Additive comparisons of two quantities	Michael is 62 in. tall and Maria is 55 in. tall. Michael is 62–55 in. taller than Maria	Quantity A is measured in some unit and has a measurement of a in that unit. Quantity B is measured in a compatible unit and measures b in that unit. The difference $a – b$ represents the amount by which the measurement of Quantity A exceeds the measurement of Quantity B
Combining quantities multiplicatively	The radius is 1.7 in. long and the arc length is 2.8 times as long as the radius. The length of the arc is (2.8)(1.7) times as long as the radius	Quantity A is measured in some unit and has a measurement of a in that unit. Making n copies of Quantity A produces a resulting quantity that is n times as large as Quantity A, with measurement $n \cdot a$ in the unit of Quantity A
Comparing quantities multiplicatively	The height of water in an empty pool increased 5 in. in 3 h as water flowed into the pool at a constant rate of change. The quotient 5/3 tells us the relative size of the height (in inches) of the water in the pool is, as compared to the number of hours since the water started flowing into the pool	Quantity A is measured in some unit and has measurement a in that unit. Quantity B is measured in some unit and has a measurement b in that unit. The quotient $a/b = \mathrm{c}$ is the relative size of a and b, with the quotient c representing how many times as large a is than b (and is usually associated with conceptualizing average rates of change)

activities she designed for students. O'Bryan (2018, 2020a) called these beliefs and expectations *emergent symbol meaning* (see next section).

As students develop meanings for ideas like relative size, differences, change in a quantity's value, etc., it is critical that instruction and activities encourage a "dialogue" within the student about the quantities conceptualized and the numerical methods for calculating the quantitative relationship's value. It is also critical to emphasize how symbolic methods for representing this value as an unevaluated expression such that the order of operations parallels the calculations performed mirrors the steps in the reasoning process that motivated those calculations. The algebraic expressions then represent the same reasoning from which the quantities were conceptualized, but where the quantity's value is undetermined (Thompson, 1990, 2011).

10.1 Emergent Symbolization

As mentioned earlier, Moore and Thompson (2015) coined the term *emergent shape thinking* to describe reasoning about graphs so that (1) they emerge as traces of how two quantities change together in tandem, (2) individuals see already-complete graphs as having been generated as emergent traces and they can imagine the coordination that produced the graph, and (3) individuals conceptualize properties of the relationship between the co-varying quantities through this emergent trace. They argue that this way of thinking about graphs is useful because "students thinking about graphs emergently are positioned to reflect on their reasoning to form abstractions and generalizations from their reasoning… not constrained to a particular labeling and orientation" (pp. 787–788). We emphasize that *emergent shape thinking* describes both how someone could reason about generating a graph but also describes how that person could interpret a graph produced by someone else. In other words, the individual's graphing scheme contains an expectation about what it means to reason about graphs as well as actions related to producing graphs.

We argue for a similar idea related to productive reasoning about generating and interpreting algebraic formulas that relate two or more quantities' values in a situation. O'Bryan (2018, 2020a) described *emergent symbol meaning*[10] as a similarly productive set of expectations, beliefs, and meanings related to generating and interpreting quantitatively meaningful algebraic statements. Individuals may have any combination of these expectations to varying degrees of sophistication, and their expectations may be situation dependent. The following list is expanded from the original definition.

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[10] Note that O'Bryan (2020a) uses *emergent symbol meaning* to describe a set of meanings and expectations that may guide an individual's algebraic symbolization activity and interpretation of the algebraic symbols that others generate. He uses *emergent symbolization* when describing the actions an individual engages in that are motivated by these meanings and expectations.

1. An expectation that performing calculations or generating expressions should reflect a quantification process for quantities that the individual conceptualizes.
2. An expectation that demonstrating calculations and producing expressions are attempts to communicate an individual's meanings. Thus, when given a set of calculations or an expression/formula, we can hypothesize how the individual conceptualized a situation based on analyzing the products of their reasoning.
3. An expectation that the order of operations used to perform calculations, evaluate expressions, and solve equations reflects the hierarchy of quantities within a conceptualized quantitative structure.[11]

Like emergent shape thinking, we emphasize that emergent symbol meaning describes the motivations and goals for how an individual could reason about the process of developing algebraic statements *and* how that person could reason about the algebraic statements someone else produces. As with emergent shape thinking, students with these expectations "are positioned to reflect on their reasoning to form abstractions and generalizations from their reasoning" (Moore & Thompson, 2015, p. 787).

O'Bryan (2018, 2020a) provides examples of how students with alternative sets of beliefs and expectations for their mathematical activity tended to be inattentive to quantities in choosing and justifying numerical operations and algebraic representations. It is worth mentioning that emergent symbol meaning is not just about framing a productive set of beliefs and expectations for students. We again point readers to O'Bryan and Carlson's (2016) report on how an instructor who internalized these beliefs and expectations was positioned to support productive mathematical discourse and to develop tasks that allowed her to decenter relative to her students' thinking. This is why we believe that introducing emergent symbol meaning as an explicit element of the theory of quantitative reasoning is useful. It can help orient researchers and curriculum designers to something important in attempts to foster quantitative reasoning—the beliefs and expectations students and instructors have regarding their mathematical activity. Without altering these beliefs and expectations we have found little success in shifting instructors or students to valuing and utilizing quantitative reasoning. See Table 5 for some examples of less and more productive beliefs about calculations and symbolization.

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[11] Numbers 1 and 3 in this list might seem quite similar, but there is a different intent. The first item focuses on the expectation that all calculations or parts of expressions should represent an evaluation process for some quantity in the situation, and thus each calculation or part of an expression can be quantitatively justified (and, if the individual cannot justify it, then it provides a motivation to reconsider how she has conceptualized the situation). The third item is about how mathematically equivalent expressions with different orders of operations reflect different ways of understanding the situation and that manipulations, including "simplifying" or rewriting an equation to solve for a different variable, may require reconceptualizing the quantitative structure to make sense of the result.

Table 5 Some beliefs and expectations we and others have found to be unproductive for students along with a similar list of more productive beliefs and expectations grounded in Thompson's theory of quantitative reasoning

Some beliefs and expectations guiding mathematical activity: calculations and symbolization	
Examples of less productive beliefs and expectations	Examples of productive beliefs and expectations (emergent symbol meaning)
Variables always stand in for an unknown value to be solved for (Jacobs, 2002; Lozano, 1998)	A variable represents the value of a quantity when that value is not fixed. We also use variables to allow us to express the value of one quantity in terms of the value of another quantity when those values change in tandem
Numbers and letters are paired together by looking for key words like sum, difference, and product and should match the form of examples in the current textbook section or instructor demonstrations	Calculations and algebraic expressions reflect the relationships between quantities' values as conceptualized by the individual. The first steps in mathematical reasoning are to make sense of the problem context and identify quantities and their relationships
The equal sign in a statement is an indication that something must be "solved for" or calculated	An equal sign indicates that you have expressed the value of a quantity in two ways (and thus the expressions on each side of the equal sign represent the value of the same quantity), including the possibility that one of those is as a targeted constant value. Equal signs thus express equality in both value and meaning (or equality in value between two like quantities)
If the answer to a question is an algebraic expression, equation, or formula, then students should always and immediately simplify their answers as much as possible	The order of operations for evaluating an algebraic expression reflects the quantitative structure the individual conceptualized. Mathematically equivalent statements do not necessarily indicate equivalent reasoning, and much information can be gained from examining and discussing non-simplified expressions (practicing and understanding the purpose of simplification is a separate mathematical idea)

10.2 *Emergent Symbolization* in Instruction and Curriculum

Part of supporting the development of emergent symbol meaning, and thus a propensity to reason quantitatively and analyze others' reasoning quantitatively, involves instructors endorsing and highlighting key differences in students' reasoning. An instructor should look for opportunities to discuss the thinking that led to a student's choice of operations; she could also prompt students to explain the thinking that determined the order in which calculations were performed. As students produce algebraic expressions the instructor should ask them to explain what specific terms and expressions represent. The instructor might ask, "What quantity's value is being represented on each side of the equal sign?" An instructor's overarching goal is to support students in conceptualizing quantities and how they are related as a habitual way of acting for constructing symbols that carry meaning for the student. The instructor can foster students' habitual use of quantitative reasoning by consistently requiring students to make a drawing that represents the quantitative structure of a problem. Our challenge has been to get instructors to consistently adopt these practices in both their teaching and own reasoning. We are developing new approaches aimed at gradually shifting instructors' commitment to viewing symbols as emerging from their conceptions of quantities and how they are related (emergent symbol meaning) as a perspective for modeling dynamic situations in mathematics and science with function formulas.

11 An Example of Unproductive Beliefs in Action

We do not have space within this paper to provide multiple examples of how students without the expectations described by *emergent symbol meaning* operate when working in mathematical contexts. However, we present one brief example from O'Bryan (2020a). O'Bryan found that students tended to produce linear models for contexts where exponential models were expected. It appeared that students were trying to translate English goal statements using mathematical symbols without attending to the quantities involved or the meaning of the expressions they produced. For example, in trying to model the height of a plant that was 7 in. tall when first measured and that grew by 13% per week, a large majority of students provided the result shown in Fig. 9.

Students' explanations revealed a failure to notice that $0.13t$ did not represent a number of inches. One student, when trying to represent the height of a different plant that grew by 50% for two weeks and was four inches when first measured, produced the answer shown in Fig. 10. Even though his answer "8 in." is incorrect based on the conventional meaning of percent change, what is most interesting is that (1) the expression he wrote, $4 + 2{\cdot}\frac{1}{2}$, fits the pattern in Fig. 9, (2) he noticed and verbalized that the expression did not produce the value he claimed it did, and

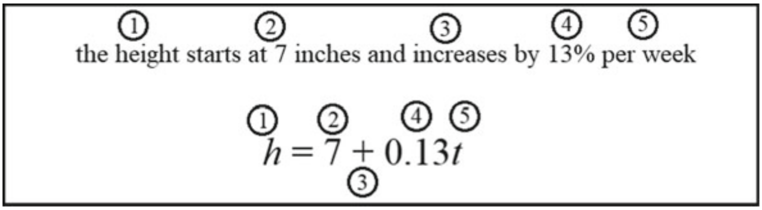


Fig. 9 Students appeared to produce answers as literal translations of English words into mathematical symbols (O'Bryan, 2020a, p. 453). The circled numbers are for emphasis only and were not written by students

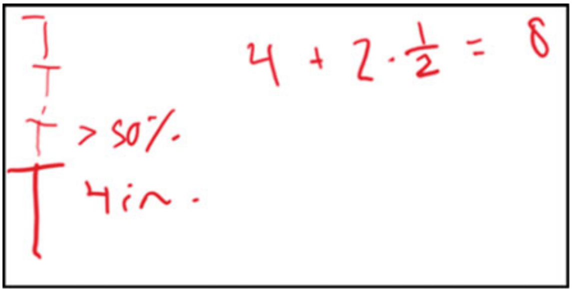


Fig. 10 A student's work justifying a plant's height after two weeks if its first measured height was four inches and it grew 50% per week (O'Bryan, 2020a, p. 451)

(3) he chose not to try to rewrite the incorrect statement since he was confident his final answer was correct.

According to Thompson (1996),

> the expression of an idea in notation provides [a student with] an occasion to reflect on what she said, an occasion to consider if what she said was what she intended to say and if what she intended to say is what she said. To act in this way unthinkingly is common among practicing mathematicians and mathematical scientists. Behind such a dialectic between understanding and expression is an image, most often unarticulated and unconsciously acted, of what one does when reasoning mathematically. This image entails an orientation to negotiations with oneself about meaning, something that is outside the experience of most school students. […] [T]he predominant image behind students' and instructors' notational actions seems to be more like 'put the right stuff on the paper.' (p. 12)

The effect of students' unreflective combining of symbols leads to students encountering significant barriers in their future math classes. For example, the students' work in Fig. 10 is likely the product of many years in mathematics classes without an emphasis on the quantitative significance of notational activity. Focusing instruction on promoting emergent symbol meaning and emphasizing important conventions such as speaking with meaning, emergent shape thinking, quantitative drawing, careful variable definitions, etc. helps students focus their thinking on conceptualizing quantitative relationships. This provides the necessary foundation to help shift students to viewing function formulas and graphs as two ways of representing how two quantities' values vary together.

11.1 Quantitative Drawing and Building Imagery for Quantitative Relationships

As noted earlier, a major initial challenge in supporting precalculus instructors to maximize the impact of the Pathways research-based materials has been advancing their understanding of the course's key ideas and how they are learned. This includes their acquiring productive conceptions of ideas of variable, function, function composition, constant rate of change, exponential growth, etc., and their viewing function graphs and formulas as ways of representing the constrained covariation of two quantities' values. A second major challenge has been to support instructors in shifting their teaching to have a primary focus on developing and leveraging student thinking toward the goal of supporting their students in relying on their reasoning as a foundation for emerging as confident and competent mathematical thinkers.

To provide a concrete example of an instructional shift we are trying to achieve, see Table 6. We contrast two Pathways instructors' approaches to helping students respond to a request to define the distance that the Tortoise is ahead of the Hare in terms of the time (in seconds) since the start of the race. Note that we would describe Instructor B as engaging in *quantitative drawing* because conceptualizations of quantities and their relationships to each other are foregrounded in the conversation and highlighted in representations. We once more emphasize that the convention of *quantitative drawing*, with practices that include writing clear variable definitions and using vectors to represent a quantity's magnitude when that magnitude can vary, becomes even more powerful when the instructor also emphasizes *speaking with meaning*, *emergent shape thinking*, and *emergent symbol meaning* (most of which are practices within these exchanges).

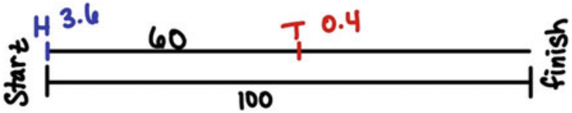
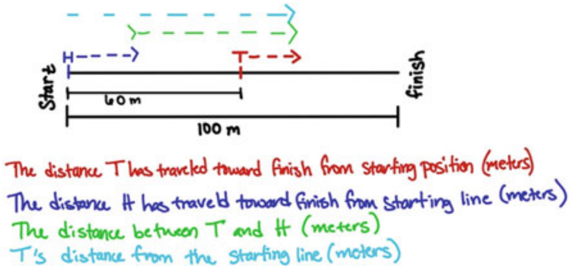
The contrasting approaches in Table 6 illuminate how an instructor's commitment to quantitative reasoning influences her instructional orientation, discussions, and questions. What we found to be surprising (and now predictable over time) is the strong commitment new Pathways instructors have to showing students steps for obtaining answers and how little value they place on helping their students use their own reasoning to make sense of a problem context prior to trying to write formulas and construct graphs. As examples of how these views surface during instruction, notice that Instructor A appears to be the one doing most of the thinking and her questions to students are almost exclusively focused on how to find the answer, what to write, or what to do (e.g., What is the formula for the Hare's distance? What did you do to get $60 - 3.2t$?). In contrast, Instructor B consistently makes requests and poses questions to engage students in identifying and conceptualizing the quantities in the problem context and considering how they are related (e.g., What are you imagining measuring? How should I represent the length of the race?) prior to making requests for students to represent quantities' values with symbols. It is also noteworthy that Instructor B consistently *speaks with meaning* when interacting with students and is careful to specify the quantity when defining variables.

Table 6 Comparing instructor actions when those instructors value quantitative reasoning or do not value quantitative reasoning

Instructor A: doesn't value QR	Instructor B: values QR
Focuses students' attention on finding an answer • Reads the problem statement once • Picks out values explicitly listed in the problem statement	The instructor focuses students' attention on reasoning about the situation in the problem statement • Reads the statement of the problem multiple times • Asks students what they can imagine measuring • Directs students to underline phrases that describe attributes that have or can assume a measurement value
Poses questions for the purpose of getting an answer • Any ideas on how to solve this problem?	Poses question for the purpose of gaining insight into how the students are thinking about the problem • How are you imagining this situation? • Can you make a drawing to represent how you are visualizing the situation?

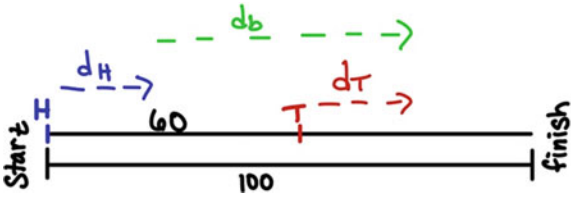
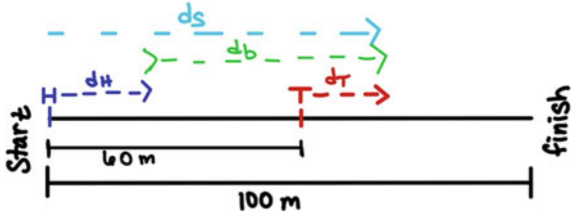
(continued)

Table 6 (continued)

<table>
<tr><th>Instructor A: doesn’t value QR</th><th>Instructor B: values QR</th></tr>
<tr><td>Makes a drawing and labels it with values stated in the problem context

</td><td>Uses the students’ interpretation of the problem statement to represent each quantity in the situation. Prompts students to consider the following:
• How should I represent the length of the race?
• Where should I place the hare? The tortoise?
• What values change during the race? How should I represent distances that vary?
• The instructor develops the drawing, one quantity at a time (labels picture carefully), with input from students

</td></tr>
<tr><td>Defines variables generically
• t = time
• d_T = Tortoise’ distance
• d_H = Hare’s distance
• d_b = Distance between</td><td>Defines variables carefully
• t = time in seconds (or number of seconds) since the start gun was fired
• d_T = the tortoise’s distance traveled (in meters) since the starting gun was fired
• d_s = the tortoise’s distance (in meters) from the starting line
• d_H = the hare’s distance (in meters) from the starting line
• d_b = the distance (in meters) that the Tortoise is ahead of the Hare</td></tr>
</table>

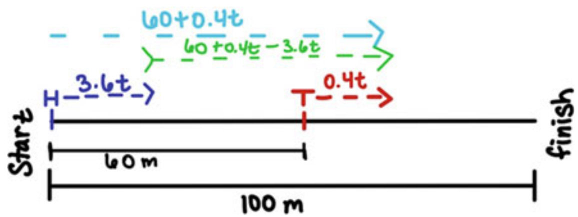
(continued)

Table 6 (continued)

Instructor A: doesn't value QR	Instructor B: values QR
Updates drawing on the board  	Updates drawing on the board 
Poses questions for the purpose of getting an answer • What is the formula for the hare's distance? • What is the formula for the tortoise's distance? • What is the formula for the distance between the tortoise and the hare?	Poses question for the purpose of gaining insight into how the students are thinking about the problem • What is the tortoise's distance from the starting line at the beginning of the race? • As the time (in sec) since the start gun was fired increases, how is the Tortoise's distance (in meters) from the starting line changing? • As the time (in sec) since the start gun was fired increases, how is the Hare's distance (in meters) from the starting line changing? • As the time (in sec) since the start gun was fired increases, how is the distance between the Tortoise and the Hare (in meters) changing?

(continued)

Table 6 (continued)

Instructor A: doesn't value QR	Instructor B: values QR
Writes expressions on the board without asking students to explain what 60, – 3.2t, or 60 – 3.2t represent $d_b = =60 - 3.2t$	Written expressions emerge from students' responses to instructor questions. As students share their algebraic expressions the instructor prompts them to explain what each term and each expression represents. (e.g., What does 3.6t represent in this situation?) As the instructor writes expressions, she calls on a student to label their term/expression on a drawing displayed to the class on a whiteboard • 60 + 0.4t – 3.6t 
Poses questions that focus students' attention on the calculations that led to the answer • What did you do to get 60 – 3.2t? • Did anyone perform different calculations to get to the answer?	Poses questions that focus students' attention on the reasoning that led to the answer • What thinking led you to writing 60 – 3.2t? • Why did you subtract? • What thinking led you to writing 3.2t? • Did anyone think about the problem differently?

12 Discussion

Studies of precalculus instructors' pedagogical practices have revealed a predominant focus on instructor-led demonstrations of methods for obtaining answers, with instructors constructing incoherent drawings and doing the majority of the speaking in class (e.g., Carlson & Bas-Ader, 2019; Teuscher et al., 2016). Supporting precalculus instructors to commit to engaging students in quantitative reasoning (and all that this entails), and be equipped to do so, is more complex and challenging than we initially imagined 15 years ago. We have been successful in some contexts, and less successful in others, and continue to explore and investigate approaches (such as the conventions described in this chapter) for supporting precalculus faculty to make this shift.

Making the problem more difficult is (1) that many precalculus instructors possess relatively weak meanings of fundamental mathematical ideas (e.g., Baş-Ader & Carlson, 2021; Musgrave & Carlson, 2017; Tallman & Frank, 2018) and (2) instructors with weak conceptions of ideas they teach are unable to engage their students in conversations that leverage and advance their students' thinking (Carlson & Bas-Ader, 2019). For example, during one of our professional development workshops with 25 secondary precalculus instructors, the majority expressed that any statement with an equal sign was an equation that needed to be solved. In another context, we asked this same group of instructors to explain how solving an equation and evaluating a function formula differed. After a relatively long wait for a response, one instructor said she saw no difference since both are equations that need to be solved. As one more example, prior to intervention, many Pathways instructors conceive of a constant rate of change as a description of the "slantiness of a line" rather than the relative size of the changes in two quantities' values. Our observations are corroborated by Thompson's research group's studies (e.g., Byerley & Thompson, 2017; Yoon & Thompson, 2020) of U.S. instructors' mathematical meanings. Elaboration of the difficulties these impoverished conceptions create for students are discussed in Thompson and Carlson (2017).

Instructors who have only experienced traditional curricula in both their learning and teaching need sustained and focused support to reconceptualize mathematical ideas they thought they understood and to reconceptualize effective teaching as focused on and affecting student thinking. Our work with instructors continues to provide us with confidence that the culture of mathematics teaching in the U.S. can change and that such a change benefits students. We have observed that as instructors become more interested in understanding and affecting their students' thinking, and reflecting on their effectiveness in doing so, they (over time) acquire more robust images of diverse ways of thinking that students present and improved insights into what productive thinking entails (Carlson & Bas-Ader, 2019; Rocha & Carlson, 2020). As instructors' mathematical meanings, images of student thinking, and images of effective teaching develop, they are more able to effectively adjust their lessons and instruction to be more meaningful and coherent for students (O'Bryan & Carlson, 2019; Rocha & Carlson, 2020; Underwood & Carlson, 2012). Instructors who became committed to implementing the Pathways conventions for supporting quantitative reasoning are individuals who attended our workshops and shifted to value our focus on conceptualizing and relating quantities as a foundational way of thinking for generating meaningful formulas and graphs.

It is our goal to support all instructors in engaging in meaningful reflection about the impact of their teaching on students' learning. The Pathways research, development, and professional development teams are collectively committed to quantitative reasoning as providing a unifying lens for advancing and studying growth in instructor knowledge and instructional practices. Our commitment to this perspective emerged from many other attempts to improve precalculus and calculus students' learning, and consistently recognizing that students' difficulties in understanding ideas, constructing meaningful function formulas and graphs, etc. were rooted in their failure to conceptualize quantities in a problem context and then to consider how pairs of quantities are related and change together.

13 Concluding Remarks

We continue to study our effectiveness in supporting instructors' construction of strong conceptions of the key ideas taught in the Pathways curriculum. Silverman and Thompson (2008) argue that an instructor must become aware of the mental processes and operations that constitute coherent mathematical understandings for reorganizing their mathematical knowledge and engaging in effective teaching practices. Quantitative reasoning will not become a meaningful part of an instructor's teaching practices until she has an image of the conceptual affordances of this way of reasoning for students' learning. We have evidence that the Pathways conventions for representing quantitative relationships, if implemented consistently, lead to advances in students' mathematical thinking, including their expectation that symbols are useful for representing quantities and relationships between quantities and that graphs emerge as a trace of an individual's conception of how two quantities' values vary in tandem. Our work has revealed that an instructor's consistent implementation of the Pathways conventions results in both students and instructors constructing more robust meanings for specific mathematics ideas, and instructors showing greater interest in understanding student's reasoning. These shifts were frequently accompanied by an increased attention on quantitative reasoning in instructors' lesson design and delivery. The result is a profound shift in how students and instructors approach applied problems; in particular, their actions to conceptualize quantitative relationships, as a foreshadowing of their construction of function formula and graphs that are personally meaningful to the one constructing them.

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