

Chapter 25

Integrating a Models and Modeling Perspective With Existing Research and Practice

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This chapter provides an illustration of how the models and modeling perspective can be used in collegiate mathematics education research and instruction. The models and modeling approach (chapter 1) provides instructional designers with a well-defined structure for creating curriculum. The curricular activities for this approach, referred to as model-eliciting activities (Lesh, Hoover, Hole, Kelly, & Post, 2000), are designed to encourage students to make sense of meaningful situations, and to invent, extend, and refine their own mathematical constructs. The resultant student products reveal students' thinking and provide both teachers and researchers with a powerful lens for viewing students' reasoning and concept development. In this chapter, we discuss how this perspective has influenced both our research and instruction. For the past 5 years we have been engaged in research to investigate undergraduate students' understanding of *rate of change* (Carlson, 1998) and *covariational reasoning* (coordinating two varying quantities while attending to the ways in which they change in relation to each other; Carlson, Jacobs, & Larsen, 2001). These studies have identified aspects of covariational reasoning and have pointed to specific difficulties that students encounter when reasoning about dynamic events. Much of the data from our past research was gathered using specific mathematical tasks. Using the insights gained from this earlier research (Carlson, 1998), we modified these tasks to adhere to the six principles for developing model-eliciting activities (Lesh et al., 2000). We then utilized these new activities in a small-scale study that investigated undergraduate students' covariational reasoning abilities.

First, we present a brief sketch of Marilyn Carlson's research on covariational reasoning and the theoretical framework that resulted from that work. We then describe the process of converting a specific covariation task, discussed in Carlson (1998) to a modeling eliciting activity. This is followed by a description of a study that uses the newly developed model-eliciting activities.

Finally, we conclude this chapter by describing the insights gained into the effectiveness of model-eliciting activities in both revealing students' thinking and facilitating the research process.

REASONING ABOUT CHANGE

The rapid increase of mathematical applications requires that all citizens be fluent in modeling continuously changing phenomena, especially phenomena of dynamic situations (Kaput, 1994). Research has revealed that conventional curricula have not been successful in promoting these modeling abilities in undergraduate students. Several studies (Carlson, 1998; Kaput, 1992; Monk, 1992; Monk & Nemirovsky, 1994; Saldana & Thompson, 1998; Tall, 1992; Thompson, 1994a, 1994b) offer insights for addressing these problems by identifying and describing the complexities encountered by students when representing and interpreting dynamic events.

While investigating young children, Lesh et al. (2000), Kaput (1994), and Confrey and Smith (1995) have observed that, when provided the proper motivation and tools, middle school students are capable of creating and analyzing sophisticated mathematical models. Kaput (1994) has also advocated that with the use of powerful tools (e.g., Simcalc) young children can begin to engage in activities to explore change and accumulation of change, while building a strong conceptual foundation for the major ideas of calculus. These results suggest that similar outcomes may also be possible for undergraduate students.

Consequently, we have responded by working to improve undergraduate students' ability to create and interpret models of dynamic events. Previous research (Carlson et al., 2001) has produced a framework for describing and analyzing the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other (i.e., covariational reasoning). Our initial attempts to improve instruction of covariation have involved the creation of activities to promote preservice secondary teachers' covariational reasoning abilities.

As we have become more informed about the models and modeling perspective and the role of covariational reasoning in building and interpreting mathematical models of dynamic function events (i.e., a functional relationship that denotes a pattern or process of change), the following questions arose:

- Can research related to functions (Carlson, 1998) and covariation (Carlson et al., 2001) inform the design and use of model-eliciting activities?
- What effect does the use of model-eliciting activities have on the development of undergraduate students' covariational reasoning abilities?
- How can the models and modeling perspective inform the refinement of the Covariation Framework?

- How can the models and modeling perspective inform future investigations of students' covariational reasoning abilities?

Motivated by these questions, we began the process of integrating a models and modeling perspective with our covariation research and current instructional practices.

A COVARIATIONAL REASONING FRAMEWORK

The Covariation Framework (Fig. 25.1) describes six categories of mental actions that have been observed in students when applying covariational reasoning in the context of representing and interpreting a graphical model of a dynamic function event (Carlson, 1998). Our research findings have revealed that covariational reasoning does not necessarily involve all six mental actions, nor does it consist of a sequential progression from MA1 through MA6, nor do experts always begin their reasoning at MA6 (Fig. 25.1). However, we have evidence that sophisticated covariational reasoning is characterized by the ability to analyze a situation using MA6, together with the ability to “unpack” that mental action by using MA1 through MA5.

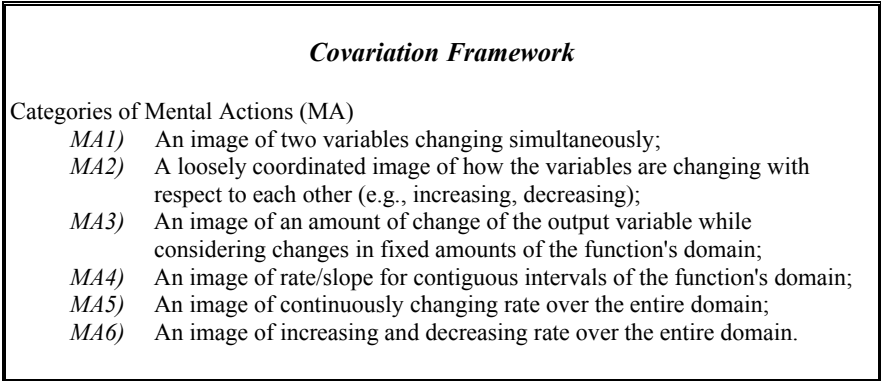


FIG. 25.1. Framework for covariational reasoning.

The following section provides a description of the framework categories in the context of the Bottle Problem, a problem that we used in our earlier research (Carlson, 1998; Carlson et al., 2000). In this chapter, we focus on the graphical representation system, as this was the initial context in which we observed students’ difficulties in applying covariational reasoning (Carlson, 1998).

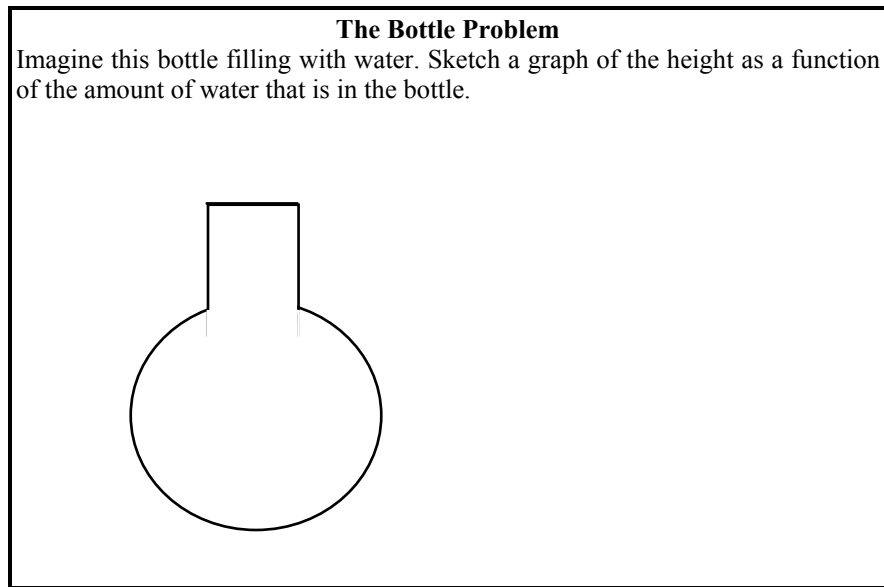


FIG. 25. 2. Earlier version of the bottle problem.

FRAMEWORK DESCRIPTION

MA1) An image emerges of the water level changing while imagining increasing amounts of water in the bottle.

MA2) An image emerges of the height increasing, as the amount of water in the bottle increases.

MA3) A fixed amount of water is imagined being added to the bottle, while concurrently constructing an image of the height of the water inside the bottle. This step is repeated by imagining the same fixed amount of water being added to the bottle with the construction of a new image of the amount of change in the height until the bottle is imagined as being full.

MA4) An image of the slope/rate of height change with respect to an imagined fixed amount of water is constructed. As successive amounts of water are imagined, the rate of change of the height with respect to the amount of water is imagined and adjusted. This process is repeated until the empty bottle becomes full. Representing these changing rates on a graph involves the construction of successive line segments with the slopes adjusted as each new amount of water is imagined.

MA5) An initial continuous image of the slope/rate of height change with respect to volume is formed. As the volume of water is imagined to change continuously, the rate of change of height with respect to volume (e.g., slope) is continuously adjusted. In the context of a graph, this results in the construction of a smooth curve.

MA6) An initial image of changing rate emerges. While imagining the filling of an empty bottle, distinctions are made between the decreasing rate in the bottom half of the spherical part, the increasing rate in the top half of the sphere and the constant rate in the straight neck. In the context of the graph, this results in the formation of a concave down construction, followed by a concave up construction, followed by a straight line. Inflection points are interpreted as situations where the rate changes from increasing to decreasing, or decreasing to increasing.

This framework both guided the development of covariation activities and served as a lens for analyzing and describing students' thinking. (See Appendix A for a more complete description and explanation of the Covariation Framework.)

EARLY ATTEMPTS TO TEACH STUDENTS TO REASON ABOUT DYNAMIC EVENTS

Curricular activities were designed to promote the development of 16 preservice secondary teachers' covariational reasoning abilities. Worksheets were written to assist these students in acquiring the ability to reason about dynamic events, as described in the covariation framework. This curriculum included activities that asked students to represent, in real time, various dynamic events and to produce graphs of specific dynamic behaviors. Students were prompted to verbalize the rationale for their constructions (e.g., Explain why the generated graph is smooth. In your own words, explain what this graph conveys about the changing rate of this situation. Discuss the nature of the changing shape of the graph in the context of the dynamic event.) This initial instructional intervention, although based on a research foundation, did not employ the models and modeling perspective. In particular, the activities were not designed to adhere to the six principles of model-eliciting activities. Pre- and post-instructional assessments indicated only moderate movement in these students' covariational reasoning abilities. For instance, only a few of these preservice teachers showed significant improvement in their ability to construct and represent images of slope/rate and changing slope/changing rate while imagining continuous change in the amount of water (MA5 and MA6). Although specific students appeared to demonstrate some improvements, evidence of incomplete understanding and blurred concepts (Monk, 1992) continued to persist among most of the participants in this earlier study.

The insights gained from these early instructional interventions provided both the foundation and motivation for developing covariation-based model-eliciting activities. As a first step in creating these new activities, we modified the Bottle Problem to adhere to the six principles of model-eliciting activities.

INTEGRATING A MODELS AND MODELING PERSPECTIVE

In the first chapter of this book, Lesh and Doerr describe the defining properties of model-eliciting activities (p. 6). In this section, we describe how we have used these guidelines to develop activities to promote and reveal acts of covariational reasoning.

Model-eliciting activities involve local conceptual development, a property that is especially relevant for achieving our goals. According to Lesh and Doerr (chap. 1, this volume),

...model-eliciting activities can be designed so that they lead to significant forms of learning (Lesh, Hole, Hoover, Kelly, & Post, 2000). Furthermore, because significant forms of conceptual development occur during relatively brief periods of time, it often is possible to observe the processes that students use to extend, differentiate, integrate, refine, or revise the relevant constructs. Consequently, to investigate cognitive development, it is possible for researchers to go beyond descriptions of successive states of knowledge to observe the processes that promote development from one state to another. (Lesh, 1983, p. 21).

If activities can be designed to promote significant development of covariational reasoning in a short period of time, they have the potential to provide valuable insights for researchers and curriculum developers. Activities that promote local conceptual development may be useful for practicing teachers as well. In addition to the instructional value of the activities, the teacher is able to observe and analyze subtle aspects of each student's mathematical development, rather than only the more obvious problems and insights of a small and vocal group. When working through a model-eliciting activity, students are asked to produce descriptions, explanations, procedures, and constructions. These types of products reveal much of the process that leads to their development. In essence, the students leave a paper trail outlining, in their own words, the reasoning they used when completing the activity.

When working with traditional curricular activities and assessments, teachers frequently find themselves trying to figure out what a particular solution tells them about a student's thinking. In most cases, they can say nothing more than the student did or did not do the problem correctly. They are often unable to say anything about what the student does know or what they can do. However, when we examine solutions to model-eliciting activities, it is easier to observe how students are thinking about a mathematical situation and to find out what they can do. The products, themselves, allow us to observe their processes (e.g., reasoning, verifying, and justifying), and not just the failure to produce an expected answer.

CONVERTING THE BOTTLE PROBLEM TO A MODEL-ELICITING ACTIVITY

The Bottle Problem has been a valuable research tool for revealing students' covariational reasoning abilities. However, by incorporating model-eliciting principles into this problem, we believe it is possible to create an activity that revealed even greater insights into the student's thinking, and one that was more effective in developing students' covariational reasoning. The new model-eliciting version of the bottle activity (Fig. 25.2) is presented in Fig. 25.3, and is followed by a discussion of the new activity in light of the six principles for developing model-eliciting activities.

Bottle Model-Eliciting Activity

Dear Math Consultants,

Dynamic Animations has just been commissioned to animate a scene in which a variety of bottles will be filled with fluid on screen. We need your help to make sure this scene appears realistic.

We need a graph that shows the height of the fluid given the amount of fluid in the bottle (a height/volume graph). Below, we have provided a drawing of one of the bottles used in the scene. Please provide a graph for this bottle and a manual that tells us how to make our own graph for any bottle that may appear in this scene.

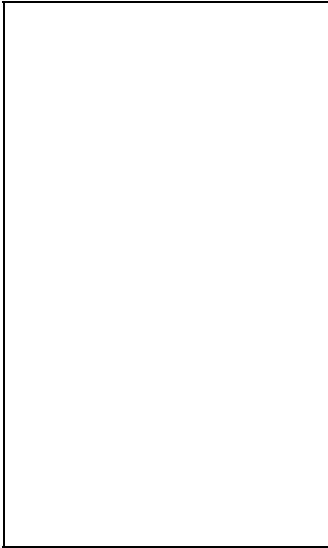


FIG. 25. 3. The model-eliciting version of the bottle problem.

INCORPORATING THE SIX PRINCIPLES

The process of transforming the Bottle Problem into a model-eliciting activity was guided by the six principles (Lesh et al., 2000). We approached our task by discussing aspects that would be desirable in the new activity and made choices about the task-design using both these criteria and the six principles. An elaboration of this process follows.

The Reality Principle. Will students make sense of the situation by extending their own knowledge and experiences? Perhaps the most challenging part of adapting a traditional activity to a model-eliciting activity is satisfying the requirements of the reality principle. For this problem, we needed to devise a context that motivated or created a need to apply covariational reasoning. The context that we created involved a production studio that needed assistance in producing realistic animations of bottles filling with liquid.

This context required the use of covariational reasoning because such a studio would need to coordinate the fluid level of the bottle with time. Because we wanted to compare our results with earlier findings, we decided to require students to coordinate the fluid level (height) with the amount of fluid in the bottle (volume), though a truly realistic task would involve the coordination of height with time. Additionally, our research goals were focused on covariational reasoning in the graphical representation because this representation is so crucial in mathematics and has been shown to be problematic for students. Therefore, our task focused on representing the situation graphically, although it is likely that a real production studio would use a different approach.

These decisions limited the role of the realistic context of the activity. However, as is conveyed below, the activity is consistent with the other five principles, resulting in a task that is both thought revealing and model eliciting.

The Model Construction Principle. Does the task immerse students in a situation in which they are likely to confront the need to develop (or refine, modify, or extend) a mathematically significant construct? Does the task involve constructing, explaining, manipulating, predicting, or controlling a structurally significant system? As previously mentioned, the original activity partially met these criteria by requesting that students produce a graph of the situation. However, because the original activity only requested that students produce a graph for one bottle, we modified the activity by prompting students to develop a general model (i.e., a manual) for analyzing this type of dynamic situation. (This activity actually required the production of two models, the graph and the instruction manual. However, our reference to the model refers to the instruction manual because this was the primary product that we requested.)

The Self-Evaluation Principle. Does the activity promote self-evaluation on the part of the students? This principle was addressed by requesting that students produce a manual for any bottle. This request provided students criteria for assessing the quality of their model. In particular, the students needed to determine whether their instruction manual was an effective guide for producing graphs of different-shaped bottles.

The Construct Documentation Principle. Will the question require students to reveal their thinking about the situation? This was the primary motivation for developing a model-eliciting version of the Bottle Problem. In order to move our research forward we needed greater insight into the thinking involved in graphically representing a dynamic situation. The students revealed how they thought about the situation by creating an instruction manual.

The Construct Generalization Principle. Does the model provide a general model for analyzing this type of dynamic situation? The requirement that students produce an instruction manual prompted students to generalize their covariational reasoning in the context of the bottle (Fig. 25.2) to create a model that is applicable to a wide variety of different-shaped containers.

The Simplicity Principle. Is the situation simple? The situation made a specific request for students to represent graphically the filling of a bottle. As such, the situation was not particularly complex. Further analysis is necessary to determine whether this situation is simple enough to allow it to play the role of a prototypical problem.

A SMALL STUDY EMPLOYING THE MODELS AND MODELING PERSPECTIVE

Following the creation of the model-eliciting activities, a small study with 22 preservice elementary teachers, enrolled in the course “Mathematics for Elementary School Teachers” at a large public university, was conducted. The class was taught using a student-centered approach with students regularly working in groups while making their mathematical constructions. All students had completed a course equivalent to college algebra, with four students also completing introductory calculus at the university. The students were paid for their participation and were informed that their willingness to participate required that they make a strong effort to think through three model-eliciting activities. Students were videotaped outside of class and each group of two to four students spent approximately 4 hours completing the tasks. The subjects were comfortable conversing with one another because the groups were essentially the same as their class groups.

During the first session, students were asked to physically model the graphs of seven different functions using calculators and CBRs (Computer Based Rangers; these devices consist of motion detectors linked to graphing calculators. As the students walk, their motion is represented on the calculator screen in the form of a distance/time graph.) In addition to actually producing each graph, they were asked to describe the movement that would generate a specific graph. After completing this task, they were given a model-eliciting task that required them to write a strategy guide to prepare an individual to physically model any graph without experimentation.

CBR Model-Eliciting Activity

Dr. Erikson's physics class is preparing for a lab exam about motion. One of the things they will be required to do is to use a CBR to reproduce (by walking) distance-time graphs that Dr. Erikson draws on the board. They do not know what graphs they will be asked to produce in advance. Please write a strategy guide that will prepare Dr. Erikson's students to reproduce any possible graph. Below are two graphs that Dr. Erikson has used in the past. Note: the students will be allowed only one attempt to produce a graph.

FIG. 25.4. The CBR model-eliciting activity.

The second model-eliciting activity involved a distance/time graph in the context of an airplane flight. The final activity was the model-eliciting version of The Bottle Problem described earlier (Fig. 25.3).

SELECT RESULTS

Analysis of students' manuals and their discussions while writing the manuals revealed new information about students' covariational reasoning abilities, as well as valuable insights regarding the effectiveness and usefulness of the activities. Select results are presented to illustrate the ways in which these activities revealed students' thought processes. (Student solutions are available for all groups for both the CBR activity and the Bottle Problem. See Appendixes B and C.)

Were the activities thought-revealing? Consider the following solution that was produced by one group for the model-eliciting version of the Bottle Problem (Fig. 25.5).

This solution provides interesting insights into how this group thought about the situation. While their thinking (Fig. 25.5) appeared consistent with the general characterization of MA4 of the covariational framework, it differed from the actual description provided in the framework. Recall that MA4 involves the construction of an image of the slope/rate of the change in height with respect to an imagined amount of water. In this case, however, the students did not imagine the rate of change in height with respect to an imagined fixed amount of water. Instead, their image of rate appeared to depend on the width (cross-sectional area) of the bottle at a given height, and was determined by comparing the volume and height for a small disk of water. This observation was also substantiated by analyzing the videotaped interactions among the members of this group.

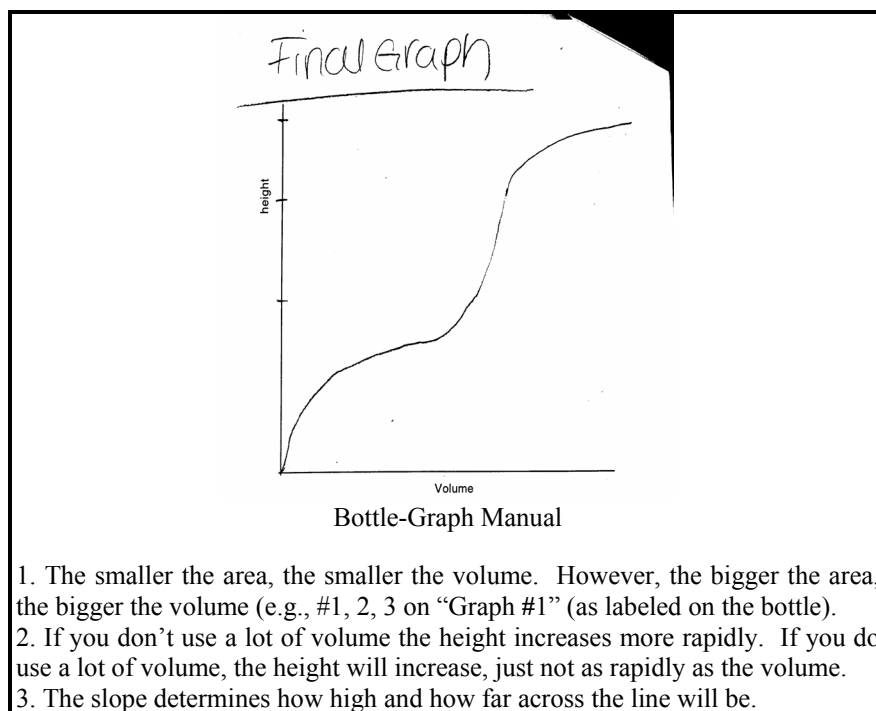


FIG. 25.5. An example of the students' solutions.

Mary: "How do you explain that?"

Chris: "Like when you have a little bit of volume, you don't need a lot volume but you have a lot of height. And when you have a big area and less height, you are going to have more volume and its going to go flat."

Joni: "Or as the unit goes more parallel to each other, whether if its this way or this way, it's going to be less height."

Mary: "What if we said it this way, on a graph the greater the height, the steeper the graph will be. The wider it is, the more level it is. Do you know what I am saying? How do we say that?"

Chris: "The reason the graph is like this is because there is so much volume needed for little area."

Joni: "As the sides go out, it requires more fill."

Chris: "Requires more volume verses height."

Joni: "So if the unit is not so wide, it requires less volume and more height..."

FIG. 25. 6. Transcript excerpt.

Figure 25.6 provides excerpts of students' conversations as they responded to this task. As is revealed, the process of negotiating one final product encouraged individuals to justify and verbalize their responses (e.g., the reason for the graph is because there is so much volume) and to compare their own ideas with those of the group. These negotiations provided additional insights into each individual's concept development and reasoning patterns. As can be observed (Fig. 25.6), these students appeared to express different justifications for the graph's shape (i.e., Mary: "The greater the height, the steeper the graph the wider it is, the more level it is." Chris: "When you don't need a lot volume, you have a lot of height." Joni: "If it is not so wide, it requires less volume and more height."). Not only do these expressions deviate from the language of the framework, but like the group's manual, they express conceptualizations that also differ from the ideas of the framework (i.e., the students are not coordinating the output variable, height, while imagining changes in the input variable, volume). This example is especially significant because it provided implications for the refinement of the Covariation Framework. In particular, it suggests that the language used in describing MA4 is not general enough to describe the various ways that a student might think about rate of change of one quantity with respect to another.

The thought-revealing nature of these activities is particularly important to us because we are introducing these activities in our courses for preservice secondary teachers. We are encouraging these future teachers to use these activities to gain access to the developing understandings and reasoning patterns of their students. The implications for our students' future teaching practices are significant, because they are being introduced to an efficient and powerful means of gaining regular access to their students' thinking.

IMPACT OF THIS STUDY ON OUR FUTURE RESEARCH

In the previous section, we provided one example of the impact of this study on our ongoing refinement of the covariation framework. In this section, we describe some other implications of this study on our continuing investigation of covariation reasoning. The written solutions and the videotape transcripts enabled us to capture many instances of covariational reasoning, spanning the entire range of the Covariation Framework. However, the students' expressions of these mental actions sometimes differed qualitatively from the descriptions provided within the framework. These observed differences point to questions for further research as well as potential refinements of the covariation reasoning framework.

Students often treated height as the input variable. In the case of the bottle activity it was expected that students would treat the height as a function of volume because they were asked to create a height versus volume graph. However, the models produced by the students revealed that they frequently thought about the volume as the dependent variable. This suggested that our

interpretation of the framework in situations where the function underlying the dynamic situation is invertible needs to be more flexible.

Students often treated time as the input variable. This was revealed in the students' written products and was evidenced by their use of terms and phrases like *faster*, *slower*, *quickly*, *longer*, and *more time to fill*. Their choice to replace the independent variable (volume) with time appeared to reduce the cognitive load for the students by allowing them to focus their attention on the changing nature of one variable or quantity. However, there was evidence that the students who used this approach did not completely understand the relationship between the height versus time graph and height versus volume graph. When asked, one group indicated that if the water were not poured at a constant rate, their *height* versus *volume* graph would be affected (a misconception). These findings suggest that the role of time in covariation reasoning needs to be further explored.

The groups did not describe a process in their instruction manuals that involved either point plotting or curve smoothing. All of the groups were able to produce correct smooth graphs for the given bottle. However, none of the groups described a process that would lead to a smooth curve by point plotting or the refinement of line segments representing average rates of change (even when their initial constructions used one of these two approaches). Instead, they gave instructions that focused on determining the rate of change at any point on the bottle and described the rate of change as an object that they could move along the domain. This observation should provide important implications for our continued investigation of covariational reasoning. In particular, it suggests a possible description of the development needed for students to form a continuous image of rate of change.

CONCLUSIONS

The students' final products, reasoning abilities, and persistence exceeded our expectations for the study. All 22 of these preservice elementary teachers provided a reasonable graph for the bottle activity. Factors that appeared to contribute to their successes include: the requirement that students verbalize their reasoning and both receive and give feedback to their peers (a feature of the model-eliciting activities); the requirement that students continue refining their solutions until a reasonable answer was produced (also a feature of the model-eliciting activities); and, the fact that they were allowed to take whatever time they needed to complete the tasks. In retrospect, these activities appeared to have the effect of placing each student in a role similar to that of a teacher. They were expected to provide a clear, logical, and defensible rationale for their solution, and like good teachers, they rose to the challenge.

The results of this study suggest that informal exploration may have promoted greater engagement and sense-making on the part of these students. Further, the unexpected successes exhibited by these preservice elementary

teachers suggests a need for continued investigation of the power of these activities in transforming context-specific notions into more general models.

As a follow-up to this study, we have begun developing model-eliciting activities to promote students' development and understanding of the major conceptual strands of introductory calculus. We call for others to join us in investigating the effectiveness of model-eliciting activities for developing other mathematical concepts in undergraduate mathematics. We also call for additional investigations into the effectiveness of model-eliciting activities for promoting undergraduate students' reasoning, communication and problem solving abilities.

ACKNOWLEDGMENTS

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