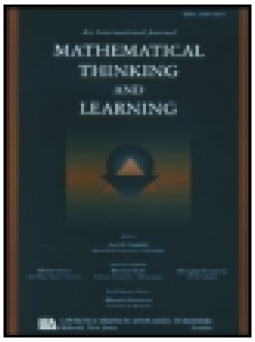

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Decentering framework: A characterization of graduate student instructors' actions to understand and act on student thinking

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ABSTRACT
This study examined the spontaneous teacher–student interactions that occurred in three graduate student instructors' [GSIs] precalculus classrooms. We classified these interactions relative to these instructors' actions to make sense of and use student thinking to inform their in-the-moment interactions with students. Our characterizations of teacher–student interactions fell into five levels, ranging from no decentering, the instructor showing no interest in the student's thinking, to the instructor inquiring into and building a mental model of the student's thinking, then using that model productively when interacting with the student. Our descriptions of the instructors' mental actions and behaviors are summarized in our decentering framework. These descriptions may be useful for other researchers, curriculum developers, and professional developers working to study and advance teachers' ability to adapt their teaching based on student thinking.

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Decentering; teacher education; teacher-student interaction; responsive teaching; radical constructivism; graduate student instructor

Introduction

As teachers respond to calls for classrooms to function as communities of learning, choices for fostering productive discourse will require teachers to adapt their responses and actions while teaching (Ball, 1993; Jacobs & Empson, 2016; National Council of Teachers of Mathematics [NCTM], 1991, 2014; Sherin, 2002; Staples, 2007; Staples & King, 2017). Teachers are being called on to listen to students' ideas as they are expressed and make moment-to-moment instructional decisions "concerning what mathematics to pursue and how to pursue it" (Sherin, 2002, p. 122). The spontaneous and interactive nature of this type of teaching has been documented to be challenging for teachers to enact (Jacobs & Empson, 2016; Johnson & Larsen, 2012; Speer & Wagner, 2009). Researchers have begun to investigate the nature of these challenges using constructs of *responsive teaching* (Jacobs & Empson, 2016) and *adaptive style of teaching* (Sherin, 2002).

In recent years, researchers have made progress in identifying aspects of responsive teaching, with some studies highlighting the types of knowledge needed by teachers to enact teaching that is responsive to student thinking (Johnson & Larsen, 2012; Sherin, 2002; Speer & Wagner, 2009). Other studies have focused on teachers' *follow-up* on students' explanations to engage them in doing mathematics (Franke, Webb, Chan, Ing, Freund, & Battey, 2009). Researchers studying teaching in elementary teachers' classrooms have used the construct of *noticing* to characterize teaching that is responsive to students' mathematical thinking (Jacobs & Empson, 2016; Jacobs, Lamb, & Philipp, 2010; Sherin & van Es, 2009), while researchers in undergraduate mathematics (Speer & Wagner, 2009) have investigated a teacher's *analytic scaffolding* or "use of those [students'] ideas to keep the discussion moving in a mathematically productive direction" (p. 531). There have been significant advances in understanding and characterizing attributes of responsive teaching. However, more work is needed to understand the mechanisms by which teachers engage in responsive teaching, including

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the mental actions involved in inquiring into and acting on students' thinking for the purpose of advancing students' understanding of mathematical ideas (Teuscher, Moore, & Carlson, 2016).

The current study built on and extended the literature related to teacher attention to students' thinking by using Piaget's (1955) idea of *decentering* as a lens to examine graduate student instructors' [GSIs] interactions with students when teaching precalculus using a research-based and conceptually focused curriculum (Carlson, Oehrtman, & Moore, 2016). Decentering is broadly defined as taking actions that adopt the perspective of another. We aimed to characterize and contrast GSIs' behaviors and perspectives relative to their effectiveness in making sense of and using their students' expressed thinking when teaching. The research question that guided the study was: What mental actions and associated behaviors do graduate student instructors exhibit when interacting with their students when teaching precalculus using a research-based and conceptually focused curriculum?

Literature review

Characterizations of responsive teaching: Instructional practices

Professional Standards for Teaching Mathematics (NCTM, 1991) called for teachers to orchestrate classrooms in which "reasoning and arguing about mathematical meanings is the norm" (p. 35). Several studies elaborated on this call by attempting to identify instructional practices to help teachers manage productive discourse (Jacobs & Empson, 2016; Staples, 2007; Staples & King, 2017; Stein, Engle, Smith, & Hughes, 2008). As two examples, Speer and Wagner (2009) and Staples (2007) proposed instructional moves that scaffold the production of student ideas by posing questions and prompts that are attentive to and tailored to the student's perspective.

In the area of undergraduate mathematics teaching, Speer and Wagner (2009) reported that a mathematician with extensive teaching experience using an inquiry-oriented curriculum faced challenges in using analytic scaffolding to move class discussions toward the lesson's mathematical goals. Their findings suggested a specialized content knowledge, separate from mathematical knowledge, as one variable that may impact a mathematician's analytic scaffolding, while calling for studies that perform a "fine-grained analyses of teachers' practices and reasoning" (p. 560).

These studies have made valuable contributions for classifying different categories of teacher actions that were responsive to student thinking; however, they have not described the thinking or mental actions of a teacher when enacting these behaviors. Researchers have argued that assessing and advancing teachers' effectiveness in making in-the-moment instructional decisions will benefit from understanding what teachers are thinking when they take student thinking into consideration when deciding on instructional actions (Teuscher et al., 2016).

Characterizations of responsive teaching: Cognitive aspects

In recent years some researchers have explored cognitive aspects of teachers' ability to attend to and act on student thinking in the moment of teaching (Jacobs & Empson, 2016; Jacobs et al., 2010; Sherin & van Es, 2009). As one example, research on teacher noticing (i.e., ability to attend to and interpret classroom interactions) revealed that teachers who learned to notice student mathematical thinking in a video-based professional development environment began to pay closer attention to students' thinking by attempting to make sense of students' ideas during their teaching (Sherin & van Es, 2009).

Jacobs et al. (2010) further reported that teachers' ability to make in-the-moment instructional decisions based on children's thinking, referred to as *professional noticing of children's mathematical thinking*, entailed an effective integration of three components: (a) attending to children's strategies, (b) interpreting children's understandings behind their strategies, and (c) deciding how to respond based on children's understandings when teaching. However, these researchers cautioned that a teacher's effective integration of these three skills "does not necessarily translate into effective execution of the response [to children's verbal and written strategy explanations], because execution requires yet another set of complex skills" (p. 192). Jacobs and Empson (2016) extended this research to characterize responsive teaching. They extended professional noticing to include four general categories of teaching moves used by an experienced teacher while assisting students when completing story problems.

These studies provided considerable insights into cognitive aspects of teachers' ability to be responsive to student thinking when teaching. However, the theoretical constructs that these studies highlighted do not have the power to explain teachers' mental actions that inform the teachers' moment-to-moment decisions and interactions with students (Thompson, 2013). Researchers from the radical constructivist perspective claim that the process of teachers' building models of their students' thinking and using these models during their interactions with students should be in the foreground when characterizing effective teaching (Hackenberg, 2005; Teuscher et al., 2016; Thompson, 2013; Ulrich,Tillema, Hackenberg, & Norton, 2014). In this study, we used the construct of decentering as a lens for characterizing instructors' model building processes and elaborated how varying degrees of instructors' decentering actions influenced their interactions with students.

Conceptual framework

This study was situated in the radical constructivist theory. A basic tenet of this theory is that knowledge must be constructed by every person through her[1] experiences (Steffe & Thompson, 2000). This implies that knowledge cannot be transferred from one person to another. Knowledge construction can be fostered through social interactions, including the construction of inter-subjective knowledge in which an individual attempts to understand utterances of another and respond meaningfully to her understanding of another's meaning(s) (Steffe & Thompson, 2000). We followed Thompson, Carlson, Byerley, and Hatfield's (2014) distinction between understanding and meaning. An understanding is an in-the-moment cognitive state resulting from assimilation. In contrast, a meaning is the space of implications of an understanding – the images, actions, or schemes that become readily available as a result of the assimilation. When a person assimilates a current experience to a scheme, the scheme is the space of implications of that understanding. Meanings both constrain and support the possible ways in which the individual might act.

Actions that involve one adopting the perspective of another were first identified by Piaget (1955), who referred to them as decentering. In his work on children's cognitive development, Piaget (1955) introduced the idea of decentering to describe a child's transition from her egocentric thought to the capability of adopting the perspective of another. Steffe and Thompson (2000) and Thompson (2000, 2013) extended the notion of decentering to characterize communication relative to whether the individuals engaged in communication were expressing "thoughts that are directed at another" (Thompson, 2013, p. 63). Thompson (2000, 2013) extended the idea of decentering to characterize two types of interaction between people: (a) reflective interaction and (b) unreflective interaction. In the case of teacher–student interaction, if the teacher acts reflectively, she observes and is aware of the student's contributions to the interaction. In contrast, the teacher who acts unreflectively does not attempt to understand the student's perspective, and according to Thompson (2000), does not attempt to decenter (see Figure 1).

If a teacher interacts with a student unreflectively, she is constrained to use only her first-order model (her meaning of an idea) when making decisions about how to act (Teuscher et al., 2016). First-order models are "the hypothetical models the observed subject constructs to order, comprehend, and control his or her experience (i.e., the subject's knowledge)" (Steffe, von Glasersfeld, Richards, & Cobb, 1983, p. xvi). For instance, a teacher who plans a lesson based on how she learned an idea is using her first-order model to plan the lesson.

If during interaction with a student, a teacher attempts to understand the perspective of the student and uses her understanding of the student's thinking to guide her actions (i.e., decentering), we say that the teacher is acting reflectively and the result of this reflection is a second-order model of the student's thinking. The teacher may then use that second-order model to inform her future actions. According to Steffe et al. (1983) second-order models are "the hypothetical models observers may

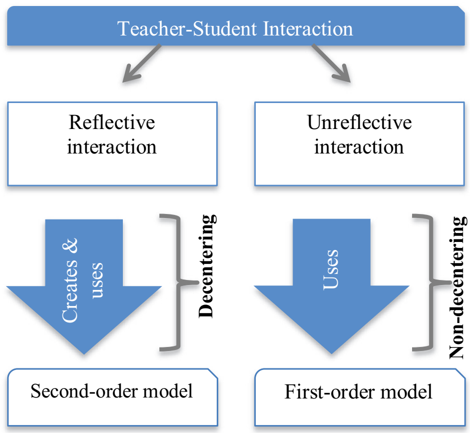

Figure 1. Two types of teacher–student interaction (Thompson, 2000, 2013).

construct of the subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (p. xvi). In the case of teacher–student interaction, for example, the teacher considers how the student might be interpreting her utterances when attempting to convey her way of thinking to the student.

By engaging in conversations, posing questions, observing the student's actions, etc., the teacher updates her second-order model of the student's thinking. As the teacher's model of the student's thinking advances to include insights about how the student is conceptualizing an idea, for example, the teacher is able to make better decisions about future actions for advancing the student's thinking (Teuscher et al., 2016; Thompson, 2013).

Decentering

The construct of decentering has been used to analyze the nature and quality of mathematical discourse (Carlson, Bowling, Moore, & Ortiz, 2007; Teuscher et al., 2016). Carlson et al. (2007) used the decentering construct to examine teacher interactions in the context of a teacher-led (facilitator) professional learning community (PLC) of secondary mathematics teachers. Findings of this research demonstrated that the facilitator's conscious attempt to model the other teachers' thinking (i.e., decentering) resulted in meaningful conversations about knowing, learning, and teaching the key mathematical ideas and often led to advances in the teachers' understanding of the ideas being discussed. Carlson et al. further presented general descriptions of five decentering moves that were exhibited by the PLC facilitator. Teuscher et al. (2016) extended this research on facilitator decentering moves by investigating the relationship between the level of a teacher's decentering actions and the quality of her interaction with students during classroom instruction. They illustrated how the teacher's decentering actions (or lack of these actions) assisted (or constrained) the teacher in making in-the-moment instructional decisions that led to advances in students' thinking.

Researchers have also used the construct of decentering in the context of instructional innovations designed to advance student mathematical thinking (Hackenberg, 2005; Ulrich et al., 2014). In this context the researchers designed a sequence of tasks, questions, and interactions they conjectured would support the student in constructing desirable mathematical meanings and connections. In addition to analyzing the teaching episodes using the construct of decentering, they performed a retrospective analysis of the teacher–student interactions after each teaching episode (Ulrich et al., 2014). The researchers' updated second-order models informed the refinement of the hypothetical models of the student's thinking. The updated models were then used to inform the design of new tasks and future interactions aimed at fostering advances in the student's thinking and learning (Ulrich et al., 2014).

In summary, the body of literature related to responsive teaching has focused on the nature of teacher–student interactions during instruction, with analytic scaffolding, teacher noticing, and following-up being identified as useful constructs for exploring these interactions. As researchers have moved to understand the mental processes that lead to teachers' engaging in responsive teaching they have attempted to understand the source of effective teacher–student interactions, including the behaviors of an individual when attempting to understand and be understood by another.

The current study built on and extended this body of literature by examining the mental actions graduate student instructors exhibited when interacting with students. We examined whether the instructors were attentive to their students' thinking, and if so, whether they attempted to understand and build a mental model of how a student was thinking. We further examined the degree to which the instructors used their models of a student's thinking to inform their utterances, actions, and instructional choices in the moment of teaching. In particular, we used Piaget's construct of decentering to: (a) characterize a GSI acting reflectively with a student, (b) describe reflections and mental actions that enabled a GSI to construct a mental model of a student's thinking, and (c) describe how a GSI might use her model of a student's thinking to respond to the student in an anticipatory way.

Methodology

The context of the study: The Pathways project

This study was conducted as a part of the second author's *Pathways project*, an ongoing 12-year research and development project aimed at advancing the project's GSIs' precalculus meanings and teaching practices to be more responsive to student thinking (Musgrave & Carlson, 2017). There were 14 GSIs participating in the Pathways project at the university where we conducted our study.[2] The GSIs all used the project's conceptually focused *Precalculus: Pathways to Calculus* (Carlson et al., 2016) student curriculum and accompanying teacher resources. The Pathways materials were informed and refined by research studies, including research into student learning of the function concept in precalculus level mathematics (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, 1998; Moore & Carlson, 2012). The GSIs were mentored in developing students' mathematical practices, including students' ability to conceptualize and represent quantitative relationships (Moore & Carlson, 2012).

The *Pathways Professional Development Model* that is used to support GSIs includes in-class investigations with conceptually focused questions scaffolded to support students in constructing strong meanings for the ideas central to a lesson. Teacher versions of each investigation describe the understandings that are targeted in the investigation, including productive reasoning that supports students' solutions. Prior to using the Pathways materials GSIs attend a 2–3 day workshop led by Pathways authors or professional development leaders. During the semester GSIs also attend a weekly 90-minute professional development seminar that engages them with the ideas of upcoming investigations, while exploring approaches for supporting their students' learning of the ideas central to the upcoming lessons. The seminar is led by an instructor who values and is aligned with the project goals and experienced in using the Pathways materials. This professional development leader engages the GSIs in discussions of productive and unproductive ways of thinking their students might exhibit and encourages the GSIs to pose questions to their students for the purpose of revealing how they are thinking.

Participants in the study

Participants were three graduate student instructors, who were also mathematics PhD students, Karen, Dave, and Greg.[3] They were teaching precalculus in class sizes of 25–30 using the Pathways curriculum at a large public university in the United States. Greg was in his fourth year of teaching with the Pathways materials and Karen and Dave were in their third year. The subjects were purposefully selected to participate based on our goal to identify 3 subjects who represented diversity relative to: (a) whether they engaged their students during class when observing their teaching during the prior year and (b) the strength of their mathematical meanings, as expressed during the professional development seminar the prior year. Accordingly, Greg consistently engaged with his students and expressed strong meanings of the ideas of the course, while Karen and Dave sometimes struggled to provide conceptual explanations about the course's ideas during discussions with their peers in the weekly professional development seminar.

Data collection

Our data collection included classroom video, notes taken during classroom observations, and clinical interviews. These data sources were chosen with the aim of answering our research question. The video data and classroom observations for all three subjects took place during the fifth and sixth weeks of the spring 2017 semester. During this 2-week period all three subjects were using the Pathways Exponential Functions module. The camera was focused to capture the GSI's actions, explanations, and conversations with students. The instructor wore a lapel microphone. The researcher (the first author) was responsible for videotaping and taking field notes to capture instances of teacher–student interactions during the lessons.

We also conducted an interview with each GSI subsequent to our initial review of the classroom videos (a total of three interviews). The purpose of the interviews was to gain insights into the motivation for the instructors' interactions with students, thus providing additional data relative to our research question, to understand GSIs' mental actions when interacting with students. During the 75–90 minute-long interview sessions, we viewed and discussed video clips from the GSI's classroom teaching that we conjectured would provide additional insights about choices and utterances made by the instructor when interacting with students, or provide triangulating data to confirm or refute our video analysis. During the audiotaped interview we watched the video episode with the GSI and prompted her to explain her motive for specific comments, questions, and actions.

Data analysis

All observation and interview data were transcribed. The transcriptions of the classroom videos were first studied to identify instances in which the instructor was interacting with students.[4] Similar to Jacobs and Empson's (2016) approach, we determined a teacher–student interaction occurring in the context of a conceptually focused question from the Pathways curriculum as the unit of analysis since it has "coherence with respect to a purpose" (p. 188). These interactions were coded according to the five dimensions identified in prior research (Carlson et al., 2007; Teuscher et al., 2016), including whether: (a) the instructor was interested in the student's thinking, (b) the instructor attempted to understand the student's thinking, (c) the instructor attempted to create a model of the student's thinking, (d) the instructor's model of the student's thinking appeared to inform her instructional actions, and (e) the instructor used her model of the student's thinking to anticipate how she might be interpreted by the student. We were also open to identifying novel GSIs' decentering actions related to both their mental actions and behaviors during interactions with students. The coding list is presented in Table A1. To enhance reliability of coding, the researchers first independently coded randomly selected interactions. Percent agreement was 90% and discrepancies were discussed and resolved by reaching a consensus. The coding process led to the five categories of interactions, each of which included diverse levels of evidence of GSIs' decentering actions, ranging from very strong evidence to very weak evidence (see Table A2).

To answer the research question, we grouped the interactions with the same level of evidence of GSIs' decentering actions and looked for common trends in these interactions. This allowed us to identify general properties of instructor mental actions and behaviors for each group (i.e., each level). The products of this process are illustrated in Table A3. As an example, see the three interactions that display very weak evidence of GSIs' decentering actions for our Level 0 decentering descriptors. This analysis led to our associating specific instructor behaviors with each of the five categories of decentering levels and mental actions that emerged from our data.

Results

Corresponding to our research question, the primary results of this study are captured in our framework of five levels of teacher–student interactions, relative to: (a) the instructor's attention (or lack of attention) to a student's thinking, (b) the instructor's construction of a model of the student's thinking, and (c) the instructor's use of her model of the student's thinking to inform her instructional actions (see Table 1). The higher levels of our framework correspond to more advanced decentering behaviors; however, since the excerpts used in our illustrations were from three different GSIs' classrooms, we make no claims about the developmental nature of the levels in our framework.

A continuum of GSIs' decentering actions

The framework entries illustrate how a graduate student instructor's decentering actions (column 2 in Table 1) correspond with associated instructor behaviors (column 3 in Table 1) and mental actions that supported these decentering actions (column 1 in Table 1). By looking across a row of the framework one can extract the link between an instructor's response and the model the instructor has constructed and used during interaction.

Column 2 of the framework provides a brief description of the five decentering levels. Our general descriptions of each decentering level illustrate subtle differences in instructors' actions relative to their focus on and use of student thinking when interacting with a student. The mental actions described in column 1 ranged from an instructor using her first-order model in an unreflective way (Level 0), to the instructor creating, adjusting, and operating from her second-order model of the student's thinking and her anticipation of how she might be interpreted by the student (Level 4). The framework also illustrates how an instructor's questioning patterns differ for five decentering levels (column 3 in Table 1). In particular, an instructor's questioning ranges from (1) probing a student's answer to (2) probing a student's thinking to (3) posing question to advance a student's thinking toward the instructor's perspective to (4) posing question aimed at revealing a student's unique way(s) of thinking to (5) posing question that is based on a student's unique way(s) of thinking and is asked for the purpose of supporting the student in making new connections. This progression might be useful for supporting instructors in shifting their instructional perspective to be more responsive to specific ways of thinking that students present.

Levels of GSIs' actions to understand and act on students' thinking

In this section we provide our characterization of each decentering level with an excerpt from a classroom discussion. To characterize each level, we used excerpts from three GSIs' classes that we identified to be most representative of the five decentering levels. The first two excerpts were from Karen's and Dave's classes, respectively, while the remaining three ones were from Greg's class. In our presentation of results we first introduce the mathematical goals of the lesson and describe the task under discussion. We then provide the excerpt and follow with a discussion of our analysis and interpretation of the teacher–student interaction.

Level 0: No interest

In Excerpt 1, Karen was leading a Pathways lesson that focused on foundational knowledge for understanding exponential functions. The students were engaged with a task that prompted them to describe the relative size of two quantities (see Figure 2).

Students were expected to perform a multiplicative comparison of the lengths of two segments. The curriculum prompted the student to determine the length of one segment using the second segment's length as the unit of measurement. Since the measure of both segments was given in a common unit (in this case inches) it is possible to determine the relative size of quantity A in units of quantity B, by calculating quantity

Table 1. A framework that characterizes a continuum of graduate student instructors' decentering actions.

Mental Actions	Decentering Levels	Description of the Behaviors

• Does not reflect on aspects of the interaction that are contributed by the student
• Assumes that the student's thinking is identical to her own
• Uses first-order model during interaction to organize and guide her actions
	Level 0: No Interest
	Shows no interest in the student's thinking, but shows interest in the student's answer. Makes no attempt to make sense of the student's thinking, but takes actions to get the student to say the correct answer
		• Asks question to elicit the student's answer
		• Listens to the student's answer
		• Does not pose question aimed at revealing the student's thinking
		  ◦ Poses question that focuses on a procedure [e.g., *What do you do next?]*
		  ◦ Poses question to have the student echo key phrase or complete steps to get the correct answer

• Does not reflect on aspects of the interaction that are contributed by the student
• Recognizes that the student's thinking is different than her own but does not intentionally analyze the student's mental actions
• Uses first-order model during interaction to organize and guide her actions
	Level 1: Interest
	Shows interest in the student's thinking, but makes no attempt to make sense of the student's thinking. Attempts to move the student to her way of thinking without trying to understand or build on the expressed thinking and perspective of the student
		• Poses question to reveal the student's thinking [e.g., *Why*? *What does that mean*? *What does that term represent?*], but does not attempt to understand the student's thinking
		• Guides the student toward her way of thinking
		  ◦ Poses question for the purpose of getting the student to adopt her way of thinking or has the student echo key phrase
		  ◦ Gives explanation aimed at getting the student to adopt her way of thinking
		  ◦ Evaluates how the student's response compares to her way of thinking or approach

• Reflects on aspects of the interaction that are contributed by the student
• Recognizes that the student's thinking is different than her own and attempts to analyze the student's mental actions
• Builds the second-order model of the student's thinking
• Uses first-order model during interaction to organize and guide her actions
	Level 2: Make Sense
	Makes an effort to make sense of the student's thinking and perspective but does not use this knowledge in communication
		• Poses question to reveal the student's thinking and attempts to adopt the student's perspective
		• Does not use the student's thinking for the purpose of advancing the student's current way of thinking
		  ◦ Does not pose question aimed at prompting the student to reflect on or advance her current way of thinking
		  ◦ Redirects the conversation toward her way of thinking or approach

• Reflects on aspects of the interaction that are contributed by the student
• Assumes that the student has unique ways of thinking
• Builds and continually adjusts the second-order model of the student's thinking
• Operates from the second-order model of the student's thinking to make decisions on how to act
	Level 3: Use
	Makes sense of the student's thinking and perspective and makes general moves to use the student's thinking when interacting with the student
		• Prompts the student to explain the student's way of thinking
		  ◦ Poses question to gain insights into the student's way of thinking
		• Leverages the student's expressed thinking to advance the student's understanding of key ideas in the lesson
		  ◦ Poses question or gives explanation that is attentive to the student's thinking and aimed at advancing the student's understanding of the key idea
		  ◦ Poses question or gives explanation to support the student in making connections between different viable ways of thinking of the key idea

(*Continued*)

Table 1. (Continued).

• Reflects on aspects of the interaction that are contributed by the student
• Assumes that the student has unique ways of thinking
• Builds and continually adjusts the second-order model of both the student's thinking and how the student might be interpreting her utterance
• Operates from the second-order model of both the student's thinking and how the student might be interpreting her utterance
	Level 4: Use and Adjust
	Constructs an image of the student's thinking and perspective and then adjusts her action to take into account both the student's thinking and how the student might be interpreting her utterance
		• Poses question to reveal and understand the student's thinking
		• Follows up on the student's response in order to perturb the student in a way that extends the student's current way of thinking
		    • Poses question informed by the student's current way of thinking and meaning of the key idea
		    • Gives explanation informed by the student's current way of thinking and meaning of the key idea

> Suppose you have two segments A and B. Suppose segment A has a length of 25 inches and segment B has a length of 60 inches.
> a. How many times as long is the length of segment A as the length of segment B?
> b. How many times as long is the length of segment B as the length of segment A?

Figure 2. The relative size of two quantities (Carlson et al., 2016).

A/quantity B. If quantity A/quantity B is equal to k, then it is said that quantity A is k times as large as quantity B (Carlson et al., 2016).

Excerpt 1

(1) Karen: If we want to see how big A is in terms of B . . . Remember, so quantity A over quantity B, what is that?
(2) Student 1: Twenty-five over sixty?
(3) Karen: Twenty-five over sixty, good. So, if we write twenty-five over sixty, what does that represent, again?
(4) Student 2: A over B.
(5) Karen: A over B, what does that mean like in English sentence?
(6) Student 2: How big A is compared to B.
(7) Karen: Exactly, how big A is compared to B. So, twenty-five over sixty, we would say A is equal to twenty-five over sixty whatever B's length is, okay? If we write out twenty-five over sixty in a calculator we will get .416 about. So we could say A is .416 times B.

During the exchange in Excerpt 1, Karen prompted her students to echo the correct answer, that determining how big A is in terms of B involves evaluating the quotient, quantity A over quantity B. Although she posed questions and listened to students' answers, at no point during the exchange did she ask students to say how they were thinking; rather, she appeared to have a single goal of getting students to say what she deemed to be the correct answer. At the end of the exchange, she appeared satisfied that Student 2 had finally repeated what she had initially suggested.

Since the questions posed by Karen did not inquire about a student's thinking, we conclude that she did not put aside her way of thinking (first-order model); nor did she attempt to understand or build a mental model of a student's thinking. Her questions and responses to students were based on her first-order model (i.e., her own meaning of the idea). We classify Karen's interactions in this excerpt at Level 0, indicating that she did not show interest in her students' thinking or engage in mental actions aimed at understanding her students' thinking (see the first row of Table 1).

In contrast, when a GSI shows some interest in the student's thinking, the excerpt is classified at Level 1, provided the instructor makes no attempt to make sense of the student's thinking.

Level 1: Interest

Similar to Excerpt 1, the teacher–student interaction in Excerpt 2 occurred in the context of a lesson focused on introducing exponential functions in the Pathways curriculum. This lesson extended multiplicative comparisons to represent the relative size of two quantities with percentages. The materials supported students in understanding the idea of percentage as a type of measurement where the unit of measure is 1/100 of the value of the reference quantity (i.e., 1%).

Prior to the exchange in Excerpt 2 students were asked to discuss the idea of percent in small groups. After a few minutes Dave asked the class, "What is a percent?" (Statement 1).

Excerpt 2

(1) Dave: So group 1, what did you come up with? What is a percent? There is no right or wrong answer here. I am just interested to hear what you thought about.
(2) Student 1: It's the number less than 1.
(3) Dave: It's the number less than 1? Okay. That is interesting in some sense and gets kind of what I am getting at. What about you guys, what is one percent of something?
(4) Student 2: One of something for every hundred.
(5) Dave: One of something for every hundred? Okay that is getting a lot closer to what I am interested in. How about the group 3 back there?
(6) Student 3: We just say a tenth of a value.
(7) Dave: So if I say here is one percent of something that means cut that thing up in ten parts and take one of them and that is one percent, right? That is probably the closest to what the actual definition is but one tenth is not the right fraction. What about you guys?
(8) Student 4: We said the same thing as group 2.
(9) Dave: It is one out of one hundred? Oh, Okay. So maybe instead of one tenth of something, it is one hundredth of something; do you guys see the difference there? So we will get to this a little more precisely in a minute but a percent … the intuitive idea is it is one hundredth of anything, right? But in practice we actually use it as a unit of measurement.

In Statement 1, Dave's expressions suggest that he was open to diverse responses and interested in how his students were thinking about the idea of percent. However, as we see later, when students did provide diverse responses he failed to comment on and make moves to make sense of what the students conveyed. For instance, following Student 1's response, Dave took no action to clarify or understand how the student was thinking (Statement 3). Although he appeared to recognize that the student's thinking differed from his own, he did not probe this student's thinking; instead, he posed the same question to another group (Statement 3). He also appeared to consider if this student's response was in alignment with his way of thinking by saying, "[this] gets kind of what I am getting at." This pattern of interaction is observed throughout the rest of the exchange (Statements 4–9). Dave ended the discussion by restating his understanding of the idea of percent (Statement 9).

Dave's rationale for these actions expressed in the follow-up interview provided further evidence of him not attempting to make sense of students' thinking and operating from his first-order model (i.e., his meaning of a percent) during the exchange. He first expressed that he was expecting all students to say that a percent is "1 one hundredth of something." This expectation is aligned with his first-order model as he expressed in Statement 9 in Excerpt 2. He also expressed that he did not know what else to do except to continue posing the same question to other students until Student 3 finally expressed an answer that was somewhat aligned with his own (Statement 6). In response to Student 3's idea, he initially posed a question to elaborate on the idea (Statement 7). In the interview, Dave explained his rationale for the question in a way that, "this group has the right intuition, they are thinking about it [a percent] as a fraction, but just as the wrong fraction. Asking them a question like this usually helps them clear it up for themselves." This statement illustrated that he posed this question for the purpose of getting students to adopt his way of thinking about the idea.

Like the GSI exhibiting a Level 0 decentering behavior,[5] the GSI exhibiting Level 1 decentering did not build a mental model of a student's thinking and his actions were still informed by his first-order model. However, his explicit request for students to share their thinking about what a percent represents and his appearing to be interested in student thinking are what distinguish Level 1 decentering behavior from Level 0 decentering behavior (see the second row of Table 1).

In contrast to Level 1 decentering behavior, a GSI exhibiting Level 2 decentering behavior takes actions to make sense of student thinking.

Level 2: Make sense

The teacher–student interaction in Excerpt 3 is from Greg's lesson focused on the behavior of exponential functions; that is, how the values of the independent quantity and values of the dependent quantity of an exponential function change together. Students were prompted to define a function that results in the value of the dependent variable tripling every time the value of the independent variable is increased by 1. After defining this function Greg asked students to explain how the value of the dependent variable, $f(x)$, of the exponential function changes when increasing the value of the independent variable by 3 (See Figure 3).

After students were given time in their small groups to discuss this question, Greg faced the class and asked for volunteers to share their thinking. Student 1 responded to his prompt by saying that she changed the function to 3 to the 3*x* power (Statement 2). The student followed by explaining that 3^{3x} represents a three-unit growth change (Statement 4).

Excerpt 3

(1) Greg: So, what happens to the value of $f(x)$ whenever x increases not by 1 but by 3. Student 1, what did you guys come up with?
(2) Student 1: Umm... how I solved I just ended up changing the function just to display how I was trying to explain it was 3 to the 3*x* power.
(3) Greg: Okay.
(4) Student 1: Just because that is showing us a three-unit growth change instead of being a one-unit growth change.
(5) Greg: So you did. Wait, so you redefined the function or just wrote kind of three to the . . . ?
(6) Student 1: Three to the 3*x* power.
(7) Greg: So like raise to three times x? [*He is writing on the board* 3^{3x}]
(8) Student 1: Yeah.
(9) Greg: Okay. So why did you write that?
(10) Student 1: Because umm . . . Because when x increases by three that's how that's the growth factor is changing to three rather than one unit it's changing from one unit to three units so I just . . .
(11) Greg: Ok so instead of . . . So this 3 is representing up here [*He is underlining 3 in the exponent*] change in input of 3 instead of change in input of 1?
(12) Student 1: Yeah.

Greg responded by posing a question to verify the student's intent to change the function to $f(x) = 3^{3x}$ (Statement 5). In the interview, he explained the reason for asking that question in a way that he attempted to gain more insight about the student's way of thinking: "I try to figure out what she was thinking. I wasn't sure she did that [redefined the function] or if she was just writing the expression 3^{3x} as a way to get started with solving the problem." He also added, "creating a new function is not quite like how you would solve the task," which shows that the student's idea of redefining the function was not aligned with the instructor's way of thinking. As a response to Greg's questions in Statements 5 and 7, the student expressed that she wrote the expression 3^{3x}, even though she had previously expressed that she redefined the function to represent a three-unit growth change (Statements 2 and 4). Thus, Greg's questions in Statements 5 and 7 appear to redirect the conversation toward what he expected the student to conclude (i.e., the output becomes 3^3 times as large).

1. Define a function f that has the following property; every time x increases by 1, $f(x)$ triples.
2. What happens to the value of $f(x)$ whenever x increases by 3?

Figure 3. Covarying quantities in an exponential growth context (Carlson et al., 2016).

Greg continued to pose questions that probed Student 1's thinking and in particular her meaning for the 3 in the exponent of the expression, 3^{3x} (Statements 9 and 11). When asked during the interview to describe his motivation for posing the questions, he responded by saying, "She says something about changing from one-unit to three-unit, so I was trying to figure out if this 3 represents the change in *x*. I mean if that is how she was thinking about it." His response further supports that he was trying to understand how she was conceptualizing the 3 in the exponent.

However, it is also noteworthy that Greg did not try to understand why the student placed an *x* in the exponent. When asked about this during the interview he responded by saying, "I was trying to manage the conversation so I can move to the fact that the answer is essentially 3^3. I am guessing I subtlety ignored x in the exponent, I just moved on." We claim that he thought that initially focusing on the meaning of 3^3 would lead to the correct answer of how the value of $f(x)$ would be impacted by increasing *x* by 3 instead of 1. He chose to ignore the thinking that led to the student incorrectly including *x* in the expression, 3^{3x}, when explaining that when *x* increases by 3 instead of 1, $f(x)$ increases by a factor of 3^3 (or 27), instead of increasing by a factor of 3. Greg's choice to not address this meaning suggests that he may not have possessed the knowledge for spontaneously addressing the student's weak meaning during this interaction.

The GSI's decentering actions in this interaction (Excerpt 3) are classified at Level 2. Unlike Level 0 and Level 1, in Level 2 decentering, some of the instructor's questions appeared to be motivated by his desire to understand the reasoning that led to the student's responses. We note that he made some progress in understanding the rationale for the student's answer and building a second-order model of the thinking associated with the student's answer; however, he elected to focus on an aspect of the student's response that he conjectured would move the student toward his way of thinking (i.e., his first-order model) (see the third row of Table 1).

A GSI exhibiting Level 3 decentering behavior is distinct from a GSI exhibiting Level 2 decentering behavior in that she uses her image of the student's thinking when interacting with the student, while at Level 2 the GSI does not.

Level 3: Use

In Excerpt 4, the mathematical context of the interactions from Greg's lesson was focused on the ideas of multiplicative comparison, percent comparison, percent change, and growth/decay factors; and how these ideas are connected. He posed the task shown in Figure 4 that had this focus.

In what follows we describe Greg's conceptions (first-order model) of what it means to understand the ideas of multiplicative comparison, percent comparison, percent change, growth factor, and his connections among these ideas. We then discuss the teacher–student interactions and the thinking Greg engaged in to model a student's thinking.

Greg's first-order model. At the beginning of the interview Greg expressed that the four questions in the task (Figure 4) probed four different ways of comparing the initial value ($x_i = 5$) and final value ($x_f = 9$) of the variable x. He explained that he viewed the multiplicative comparison required for answering part (a) and the use of percent comparison to answer part (b) in terms of the idea of measurement. Accordingly, when determining how many times as large 9 is than 5, he indicated that measuring 9 in units of 5 resulted in a measurement of 1.8. When determining what percent 9 is of 5 he explained that measuring 9 in units of 1/100 of 5 would give a measurement of 180. Greg also

Suppose the value of x increases from x_i=5 to x_f=9

a. x_f is how many times as large as x_i?

b. x_f is what percent of x_i?

c. What is the change in x?

d. By what percent does x increase?

Figure 4. Comparing two values of a quantity (Carlson et al., 2016).

conceptualized the idea of percent change (see part (d)) in terms of measurement as indicated by his statement that, "it [percent change] represents the difference between 9 and 5 measured in units of 1/100 of 5."

Our analysis of data from the interview with Greg further revealed that he viewed a percent change as the difference between the "final percent" and the "initial percent." This thinking was revealed in his statement that 180% is the final percent that represents the measure of 9 in units of 1/100 of 5, and the initial percent of 100% represents the measure of 5 in units of 1/100 of 5, and that 80% is the percent change from the initial to the final percent. He also expressed that 80% represents the measure of 4 (i.e., 9–5) in units of 1/100 of 5, indicating that his first-order model of the meaning of percent change included at least two distinct ways of thinking.

The teaching episode. During a discussion that preceded the Excerpt 4 exchange Greg asked the students to share how they solved each question in the task (Figure 4). When responding to part (d), Student 1 explained that he first determined the amount of change as the input value increased from 5 to 9. He followed by expressing the change of 4 as a percent of 5 by determining the measure of 4 relative to 5 and arriving at the fraction 4/5 in percent form (i.e., 80%). Subsequent to Student 1's explanation Greg directed his students to compare how questions in part (b) and part (d) differ (Statement 1).

Excerpt 4

(1) Greg: Does everybody see the difference between part d and part b? Student 2, what do you think the difference between part b and d is?
(2) Student 2: So like I looked at it when I am solving d, I said 5 out of 5 is 100%, 10 out of 5 is 200%, so 9/5 would be 180 [percent], so kind of subtracting that I said the difference from 180 [percent] to 100 [percent] is 80 percent – that is how I solved d. I didn't answer your question.
(3) Greg: Okay, no, actually that's good. I like that. Let me … so you essentially did this. [*He is writing on the board 180%–100% = 80%*]
(4) Student 2: That is how I solved d.
(5) Greg: Okay. Anybody else think about this that way? We got the same answer both ways, right? Why is it that we got the same answer, like conceptually?
(6) Greg: Student 2, can you say one more time why you thought to do this 180% – 100%?
(7) Student 2: Because 5 out of 5 is 100%, so that'd be your second part, and 10 divided by 5 would be 2 or 200% so taking that and solving it, your change is 80% …
(8) Greg: Okay, so this 180% was like …
(9) Student 2: Was the 9/5 representation in percent form.
(10) Greg: And then this [100%] was kind of like a 5/5 but in percent form. When we subtract those, what are we computing?
(11) Student 2: We're computing the percent that x increases by.
(12) Greg: Okay. Yeah. It's like … we essentially just computed the change, but we did it in percent form right? Does everybody see that? Do you take … like the total percent, 180% and you take away 100% of the reference value, and you should just end up with whatever we've changed by, right? Does that make sense? Like if we take away 100% from our new percentage, we should end up with how much we've changed.

In response to the question (Statement 1), Student 2 began by explaining that she determined the percent change from 5 to 9 and subtracted 100% from 180% when determining the percent change in x as its value increases from 5 to 9 (Statement 2). When asked about the student's approach during the interview, Greg explained that he expected students to approach the problem by adding 100% to the given percent change. He conveyed that this way of thinking enables students to easily interpret the base of an exponential function (i.e., growth factor) in terms of the percent change for contiguous values of the dependent quantity that are associated with incremental equal increases in the values of the function's independent quantity. This view was consistent with the instructional goals and learning trajectory embedded in the Pathways curriculum. Greg also conveyed that he did not expect Student 2 to express this way of thinking because he viewed the student as a procedural thinker.

During the interaction Greg prompted Student 2 to explain her reasoning to gain further insights into her thinking (Statements 3, 6, 8, and 10). He explained that he believed the student thought that, "100% is the initial value, 180% is the final value; final minus initial gives us the change." He then asked the class to explain why the approaches expressed by two different students (i.e., Student 1 and Student 2) produced the same answer to the question in part (d) (Statement 5). During the interview he expressed his rationale for asking this general question by saying,

> With that question, I was hoping to relate this 180% minus 100% back to the other way of thinking of 4 is what percent of 5. In both cases we were dealing with the change, we were measuring change in percentage, but we took two different approaches.

These statements provide evidence that Greg was aware that the students' thinking differed from what he expected. We further claim that he built a model of both Student 1's and Student 2's ways of reasoning about the task (i.e., second-order model). He then used his second-order model in his subsequent interaction with the class when he prompted them to consider how the two ways of thinking compare (Statement 5). He also continued by asking Student 2 more specific questions (Statements 8 and 10) to compare her approach and Student 1's approach to expressing a change as a percentage. During the interview he conveyed that the purpose of these questions was to prompt Student 2 to think of her initial value (i.e., 100%) as 5 measured as a percent of 5, and her final value (i.e., 180%) as 9 measured as a percent of 5. We claim that Greg's model of Student 1's and Student 2's distinct ways of thinking provided the basis for his questioning.

We characterize Greg's actions as Level 3 decentering. In contrast to Level 2 decentering, this GSI exhibited Level 3 decentering; he was operating from his model of the student's thinking (second-order model) to make decisions on how to act (see the fourth row of Table 1).

Like a GSI exhibiting a Level 3 decentering behavior, a GSI exhibiting Level 4 decentering behavior also builds and operates from her model of a student's thinking when determining how to act. What distinguishes a GSI exhibiting a Level 4 decentering behavior is the instructor's anticipation of how her actions might be interpreted by the student.

Level 4: Use and adjust

The interactions in Excerpt 5 occurred in Greg's same lesson, which was described in Excerpt 4, and in the context of the same task (see Figure 4). The discussion was focused on comparing the initial ($x_i = 5$) and the final ($x_f = 9$) value of x and expressing this comparison as a percent (i.e., part (b)). Descriptions of Greg's mathematical meanings related to the key ideas of the task were included in the previous section. Briefly, he thought that measuring 9 in units of 5 involves determining how many times as large 9 is than 5, and that a percent comparison of 9 and 5 involves conceptualizing 9 as being measured in units of 1/100 of 5, leading to his result that 9 is 180% of 5. As a percent comparison, the unit of measure (i.e., 5/100) is 1/100 of the size of the previous unit of measure (i.e., 5); the measurement itself (the percentage) is then 100 times as large. This conception held by this GSI was consistent with the conception expressed in the Pathways instructional materials. This conception characterizes the mental model he held (his first-order model) of what was involved in understanding how to respond to the questions he posed in Statement 1 of Excerpt 5.

Greg expressed interest in understanding the reasoning students used when responding to the question "x_f is what percent of x_i?" One student responded by saying, "9 divided by 5 equals 1.8. Then I moved the decimal two places and got 180." Greg responded by asking the student to explain why he moved the decimal point two places. Another student conveyed that, "You are going to need to multiply by 100 because you are looking at percentage." After hearing these students' procedural explanations Greg prompted students to reflect on their reason for multiplying the decimal 1.8 by 100 (Excerpt 5, Statement 1).

Excerpt 5

(1) Greg: How do we go from having 1.8 to 180? Why is it that we multiply by 100?
(2) Student 1: A percentage is always . . . I'm going to call it base, but like a 100 percent is always a whole.
(3) Greg: Yeah. In some regard a 100% is like one full thing, right? Somehow you have this feel that a 100 percent is like one full base value. So, let's take a second to think about this. What is it that we mean when we say 1% of 5? What do we mean by 1 percent?
(4) Student 2: It's 1/100 of whatever value you're having as your reference.
(5) Greg: Yeah, 1% of 5 . . . Does everybody agree? This is 1/100 of 5? How much is 1/100 of 5? How might we actually find out how much that is?
(6) Student 3: 5 divided by 100.
(7) Greg: 5 divided by 100? So from here, let's try this again. What is it about these percentages that forces us to essentially multiply by 100 to get into percent form?
(8) Student 3: If you take 100 off the bottom . . .
(9) Greg: Okay, can you say more?
(10) Student 3: If you multiply 5 out of 100 by 100 you get the original 5 back.
(11) Greg: Okay. If we multiply by 100 we get essentially our reference value back, right? So, how might we describe 180 % of 5? If 1% of 5 is 1/100 of 5, what do you think about 180 % of 5?
(12) Student 3: 180 times 5/100?
(13) Greg: Can you say more? How did you come up with that?
(14) Student 3: I thought that 180% of 5 should be 180 times as large as 1% of 5. 1% of 5 equals to 5/100. So 180 % of 5 can be described as 180 times 5/100.
(15) Greg: Good. So, we might say this 180% of 5 is essentially 180 times 1/100 of 5, right? So now I mean essentially this is why we end up multiplying by 100. Because at some point we took 1/100 of 5, or 1/100 of our reference value.

During the interview Greg conveyed that his instructional goal at that time was to support his students in thinking of a percentage as a type of measurement where the measurement unit is 1/100 of the value of the reference quantity. Specifically, he expressed that he wanted his students to see that determining what percent 9 is of 5 involves measuring 9 in units of 1/100 of 5 (i.e., 1% of 5). He elaborated by saying that he also wanted them "to conceptually understand the reason why the measurement (the percentage) is 100 times as large."

In response to Greg's question (Statement 1), Student 1 expressed that 100% represented "the whole" (Statement 2). Greg then asked the student to think about the meaning of 1%. During the interview Greg expressed that he had thought that the student thought of 100% as "the whole" and 1% as 1/100 of that whole. He also expressed that 1% of 5 is 1/100 of 5. At this point in the conversation with Student 1 (Statement 3), we claim that Greg compared the student's way of thinking with what he thought was a more productive way of thinking and then posed the question, "What is it that we mean when we say 1% of 5?" When asked during the interview to describe his rationale for posing this question, he responded by saying, "Since the student put on the table that 100% is one full thing, I essentially leveraged her intuition so that this idea of moving toward 1/100 of our base value doesn't seem so foreign." This response provided further evidence that the second-order model Greg had built of Student 1's thinking informed his questions in Statement 3.

After Greg again asked his students to explain why they multiply by 100 (Statement 7), Student 3 conveyed that multiplying 5/100 by 100 would result in going back to the reference value of 5 (Statements 8 and 10). Greg then prompted the student to think about the process of determining 180% of 5 (Statement 11). During the interview Greg expressed that he imaged Student 3's idea in a way that the student thought of some percent of 5 in terms of units of 5/100. Based on this student's expressed meaning, he conjectured that the student must have thought that [180% of 5] could be expressed as 5/100s. Furthermore, the idea that he intended to convey at that time was that [180% of 5] is 180 times as large as [1% of 5], which was another way of expressing the idea that measuring 9 in units of 1/100 of 5 results in a measurement of 180.

Student 3's answer that 180% of 5 is 180 times 5/100 (Statements 12 and 14) indicates that Greg's instructional choice resulted in the student using what Greg viewed as a productive way of thinking. At the end of the exchange Greg summarized the ideas in terms of the thinking just expressed by Student 3; the idea that the measurement unit for measuring 9 relative to 5 in percent form (i.e., 5/100) is 1/100 times as large as the measurement unit when measuring 9 relative to 5 (i.e., 5), resulting in the measurement in percent (i.e., 180%) to be 100 times as large as 1.8 (Statement 15).

We claim that this GSI adapted his questioning and interactions with each of the three students based on his second-order model of each student's thinking and his anticipation of how he might be interpreted by the student with whom he was interacting. This GSI's choice to adjust his actions to take into account the student's thinking and how the student might interpret him is what distinguishes decentering at Level 4 from decentering at Level 3 (see the fifth row of Table 1).

Discussion and conclusions

This study used decentering as an explanatory construct for making inferences about graduate student instructors' actions when enacting responsive teaching. Consistent with prior studies (Carlson et al., 2007; Teuscher et al., 2016; Thompson, 2013; Ulrich et al., 2014), our data support that Piaget's (1955) construct of decentering is useful for explaining and characterizing instructors' responses to and interactions with students when teaching.

Our study extends prior research on characterizing responsive teaching in several ways. First, our analysis of GSIs' decentering actions characterizes and contrasts an instructor acting for the sole purpose of getting her students to adopt her way of thinking, with instructor behaviors associated with constructing a model of a student's thinking and then using that model when interacting with the student. These characterizations may provide useful insights for understanding how to support teachers in engaging students in what others have framed as responsive teaching (Staples, 2007; Staples & King, 2017; Stein et al., 2008). As one example, in Staples' (2007) framework, a teacher's question posing into students' understandings was characterized as a strategy to scaffold the production of students' ideas. A shift to focus on how the teacher interprets and uses the student's thinking provides insights into the rationale for the teacher's actions, potentially providing insights for supporting other teachers in adopting the perspective of students during teaching. Our analysis of the GSI's interactions in Levels 3 and 4 revealed that an instructor was able to pose questions that were attentive to students' ideas when the instructor reflected on his interactions with students while attempting to understand and model students' thinking.

Second, our study is one response to Jacobs et al.'s (2010) call for researchers to investigate the link between teachers' professional noticing and their execution of on-the-spot decisions in their classrooms. Although Jacobs and Empson (2016) extended this research by describing a teacher's execution of on-the-spot decisions that "build on children's thinking" (p. 195), our use of the construct of decentering allowed us to make inferences about GSIs' ways of thinking (mental actions) that might illustrate the process of an instructor building a model of a student's thinking and how this model enabled the instructor to make informed instructional decisions in the moment of teaching. This perspective helps to explain why a teacher might respond to a student in a particular way, with the framework levels 3 and 4 characterizing a teacher's model of a student's thinking that enabled him to interact productively with the student.

Our participants were graduate student instructors who had completed at least 2 years of graduate-level mathematics, well beyond what is required of a secondary teacher. They attended a weekly 90-minute professional development seminar concurrent with their teaching, where they were encouraged to consider student thinking when making instructional decisions during class. Despite efforts of the professional development leader to support the GSIs in focusing on student thinking, our observations of the GSIs, both during the seminar and when teaching, suggest that more research is needed to understand the mechanisms for supporting GSIs to enact decentering behaviors. This finding is consistent with findings reported by Speer and Wagner (2009) that one cannot assume that an instructor with strong mathematical content knowledge will necessarily be effective in scaffolding questions that are rooted in knowledge of students' thinking. We further highlight our contributions to undergraduate mathematics teaching, in light of the scarce amount of research in this area (Speer, Smith, & Horvath, 2010). We join Speer et al.'s (2010) call for future studies to investigate factors that influence undergraduate instructors' moment to moment instructional decisions and suggest that studies examine how a teacher's meaning for a specific mathematical idea influences her decentering actions.

A framework that characterizes a continuum of GSI decentering actions

This study produced a framework of five levels of decentering actions that GSIs in our study exhibited when responding to students' utterances during teaching (see Table 1). We do not claim that the levels of the framework characterize a developmental path for a GSI's development of decentering behaviors. In fact, within a single class period, it is possible for a GSI to exhibit all five decentering behaviors, dependent on the goals of the instructor during specific moments of instruction and teacher–student interaction. An instructor who has the capacity to decenter at Level 4 may elect to not continue probing the student's thinking during moments of teaching. Some reasons include the time it takes to understand what a particular student is thinking or the instructor does not feel comfortable in handling how she imagines the mathematics conversation unfolding. In Excerpt 3, the GSI acknowledged that he chose to focus on the aspect of the student's answer that aligned with his idea of a correct answer and that he elected to move on because of limited time. However, our observations suggest that an instructor who has only engaged in Level 0 and Level 1 decentering behaviors may need support in shifting her teaching orientation toward understanding and supporting student thinking. We believe it would be useful for future studies to investigate this observation and explore mechanisms for improving instructors' decentering abilities.

We see every teacher action to understand a student's thinking as potentially advancing teacher's knowledge of how the student might think about a particular idea. We further conjecture that decentering is a mechanism by which a teacher acquires greater insights into how students can be supported in learning an idea. We believe that teachers who attempt to understand students' thinking when interacting with them as they are learning an idea, may become more and more anticipatory of how students might be thinking in future situations when interacting with students around that same idea. We call for more research to explore decentering as a mechanism for advancing teachers' images of how students develop their understandings of an idea.

Our decentering framework should be useful for researchers who are interested in characterizing a teacher's actions to understand and act on student thinking when attempting to engage students in other mathematical contexts. In designing interventions, teacher educators and professional development leaders could use the framework as an analytical tool to scaffold pre-service and in-service teachers' learning to place more focus on student thinking as a basis for their instructional decisions and actions. Similarly, professional development leaders could use the framework to support GSIs in making sense of and using student thinking. In a video-based intervention, for instance, GSIs could watch videos of an instructional episode and rate each episode using our framework levels, then discuss the rationale for their ratings.

Limitations of this study and recommendations

The framework emerged from our analysis of the three GSIs in the context of their having attended Pathways professional development and using Pathways curriculum, both of which supported advancements in GSIs' mathematical meanings of precalculus mathematics. Thus, we suggest caution in generalizing our descriptions of these GSIs' mathematical goals and behaviors to other teachers in other settings.

Our future study will include tracking instructors' decentering behaviors over multiple years in the context of professional development that is continually refined to support instructors in reflecting on their teaching in relation to its impact on student thinking and learning. Our data for this study suggest that future studies into the influence(s) of an instructor's decentering actions might benefit from characterizing how the instructor's meanings for specific mathematical ideas influence the model the instructor builds of a student's thinking and subsequent instructional actions.

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Notes

1. To avoid using "his/her" repeatedly, we will use feminine pronoun when referring to third-person singular throughout the article .
2. The Pathways project was also being implemented at 6 other universities, but the data collection for this study was at one site only, the project leader's research and development site.
3. Pseudonyms were used.
4. Despite the fact that teachers' decentering actions can be observed through their written questions, prompts, or feedbacks (Thompson, 2000), this study focused only on oral teacher–student interactions.
5. We used "Level 0 decentering behavior" to refer to the lowest level of the continuum of decentering actions even though both Level 0 and Level 1 describe an absence of decentering actions.

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References

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*, *93*(4), 373–397. doi:10.1086/461730

Carlson, M. P., Oehrtman, M., & Moore, K. (2016). *Precalculus: Pathways to calculus* (6th ed.). Phoenix, AZ: Rational Reasoning.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education. III* (Vol. 7, pp. 114–162). Providence, RI: American Mathematical Society.

Carlson, M. P., Bowling, S., Moore, K., & Ortiz, A. (2007). The role of the facilitator in promoting meaningful discourse among professional learning communities of secondary mathematics and science teachers. In T. Lamberg & L. R. Wiest (Eds.), *Proceedings of the 29th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 841–848). Stateline (Lake Tahoe), NV: University of Nevada, Reno. Retrieved from http://www.pmena.org/pmenaproceedings/PMENA%2029%202007%20Proceedings.pdf

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378. doi:10.2307/4149958

Franke, M. L., Webb, N. M., Chan, A. G., Ing, M., Freund, D., & Battey, D. (2009). Teacher questioning to elicit students' mathematical thinking in elementary school classrooms. *Journal of Teacher Education*, *60*(4), 380–392. doi:10.1177/0022487109339906

Hackenberg, A. (2005). A model of mathematical learning and caring relations. *For the Learning of Mathematics*, *25*(1), 45–51. Retrieved from http://www.jstor.org/stable/40248486

Jacobs, V. R., & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. *ZDM Mathematics Education*, *48*(1-2), 185–197. doi:10.1007/s11858-015-0717-0

Jacobs, V. R., Lamb, L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202. Retrieved from https://www.jstor.org/stable/20720130

Johnson, E. M. S., & Larsen, S. P. (2012). Teacher listening: The role of knowledge of content and students. *The Journal of Mathematical Behavior*, *31*(1), 117–129. doi:10.1016/j.jmathb.2011.07.003

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior*, *31*(1), 48–59. doi:10.1016/j.jmathb.2011.09.001

Musgrave, S., & Carlson, M. P. (2017). Understanding and advancing graduate teaching assistants' mathematical knowledge for teaching. *The Journal of Mathematical Behavior, 45*, 137–149. doi:10.1016/j.jmathb.2016.12.011

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematics success for all*. Reston, VA: Author.

Piaget, J. (1955). *The language and thought of the child*. Cleveland: Meridian Books.

Sherin, M. G. (2002). When teaching becomes learning. *Cognition and Instruction*, *20*(2), 119–150. doi:10.1207/S1532690XCI2002_1

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education*, *60*(1), 20–37. doi:10.1177/0022487108328155

Speer, N. M., Smith III, J. P., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior*, *29*(2), 99–114. doi:10.1016/j.jmathb.2010.02.001

Speer, N. M., & Wagner, J. F. (2009). Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. *Journal for Research in Mathematics Education*, *40*(5), 530–562. doi:10.5951/jresematheduc.40.5.0530

Staples, M. (2007). Supporting whole-class collaborative inquiry in a secondary mathematics classroom. *Cognition and Instruction*, *25*(2), 161–217. doi:10.1080/07370000701301125

Staples, M., & King, S. (2017). Eliciting, supporting and guiding the math: Three key functions of the teacher's role in facilitating meaningful mathematical discourse. In D. A. Spangler & J. J. Wanko (Eds.), *Enhancing classroom practice with research behind principles to actions* (pp. 25–36). Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P., & Thompson, P. W. (2000). Interaction or intersubjectivity? A reply to Lerman. *Journal for Research in Mathematics Education*, *31*(2), 191–209. doi:10.2307/749751

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York, NY: Praeger Scientific.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313–340. doi:10.1080/10986060802229675

Teuscher, D., Moore, K. C., & Carlson, M. P. (2016). Decentering: A construct to analyze and explain teacher actions as they relate to student thinking. *Journal of Mathematics Teacher Education*, *19*(5), 433–456. doi:10.1007/s10857-015-9304-0

Thompson, P. W. (2000). Radical constructivism: Reflections and directions. In L. P. Steffe & P. W. Thompson (Eds.), *Radical constructivism in action: Building on the pioneering work of Ernst von Glaserfeld* (pp. 412–448). London: Falmer Press.

Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital directions for research in mathematics education* (pp. 57–93). New York, NY: Springer.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing,* WISDOMe Monographs (Vol. 4, pp. 1–24). Laramie, WY: University of Wyoming.

Ulrich, C., Tillema, E. S., Hackenberg, A. J., & Norton, A. (2014). Constructivist model building: Empirical examples from mathematics education. *Constructivist Foundations*, *9*(3), 328–339. Retrieved from http://constructivist.info/9/3/328

Table A1. The coding list and descriptions of the codes.

Codes	**Description of Codes and Sub-codes:**
IUR	Interacting with the student unreflectively
Sub-codes [mental actions]	
IUR: UFOM	Using first-order model during interaction to organize and guide her actions
IUR: ASTI	Assuming that the student's thinking is identical to her way of thinking
Sub-codes [behaviors]	
IUR: AQCA	Asking question to reveal the correct answer
IUR: AQFP	Asking question that focuses on a procedure or calculation
IUR: HSKP	Having the student echo key phrase
IUR: LSA	Listening to the student's answer
IUR: GSAT	Getting the student to adopt her way of thinking
IUR: ESR	Evaluating how the student's response compares to her way of thinking
IR	Interacting with the student reflectively
Sub-codes [mental actions]	
IR: CSOM	Creating second-order model of the student's thinking
IR: OSOM	Operating from second-order model of the student's thinking
IR: RSDT	Recognizing that the student's thinking is different than her way of thinking
IR: AMA	Adjusting response according to how the student might be interpreting her utterance
Sub-codes [behaviors]	
IR: AQST	Asking question to reveal the student's thinking
IR: FMST	Following up on the student's response to make sense of the student's thinking
IR: LST	Leveraging the student's thinking to advance the student's understanding
IR: MCST	Making connections among different ways of student thinking
IR: PS	Perturbing the student in a way that extends the student's current way of thinking

Table A2. The categories generated when using the codes during the data analysis.

Categories	**Codes Referring to Mental Actions**	**Codes Referring to Behaviors**
Very strong evidence	IR, IR: RSDT, IR: CSOM, IR: OSOM, IR: AMA	IR: AQST, IR: FMST, IR: LST, IR: MCST, IR: PS
Strong evidence	IR, IR: RSDT, IR: CSOM, IR: OSOM	IR: AQST, IR: FMST, IR: LST, IR: MCST
Acceptable evidence	IR, IR: RSDT, IR: CSOM, IUR: UFOM	IR: AQST, IR: FMST, IUR: GSAT, IUR: ESR
Weak evidence	IUR, IR: RSDT, IUR: UFOM	IR: AQST, IUR: GSAT, IUR: ESR, IUR: HSKP
Very weak evidence	IUR, IUR: ASTI, IUR: UFOM	IUR: AQCA, IUR: AQFP, IUR: HSKP, IUR: LSA

Table A3. Example of looking for common trends among the three samples of interactions with very weak evidence of gsis' decentering actions.

Excerpts	*Codes (Used for Excerpts)*	*Commonalities*
Excerpt 1 **GSI**: If we want to see how big A is in terms of B . . . Remember, so quantity A over quantity B, what is that? **S**: Twenty-five over sixty? **GSI**: Twenty-five over sixty, good. So, if we write twenty-five over sixty, what does that represent, again? **S**: A over B. **GSI**: A over B, what does that mean like in English sentence? **S**: How big A is compared to B. **GSI**: Exactly, how big A is compared to B. So, twenty-five over sixty, we would say A is equal to twenty-five over sixty whatever B's length is, okay? If we write out twenty-five over sixty in a calculator we will get .416 about. So we could say A is .416 times B.	IUR IUR: ASTI IUR: UFOM IUR: AQCA IUR: AQFP IUR: HSKP IUR: LSA	• Interacting with the student unreflectively (**Excerpts1& 2& 3**) • Assuming that the student's thinking is identical to her way of thinking (**Excerpts1& 2& 3**) • Using first-order model during interaction to organize and guide her actions (**Excerpts1 & 2 & 3**) • Asking question to elicit the student's answer (**Excerpts1& 2 & 3**) • Listening to the student's answer (**Excerpts1 & 2 & 3**) • Asking question that focuses on a procedure or calculation (**Excerpts1 & 2 & 3**) • *Posing question to have the student echo key phrase* (**Excerpt1**)
Excerpt 2 **GSI**: So the original price is 52 dollars and then there is an increase in 10%. What is our percent change here? **S**: Ten (*several students*). **GSI**: Ten percent. Positive or negative? **S**: Positive (*several students*). **GSI**: Positive, ok? So percent change, it is just 10%. So we are paying the full 52 dollars, right and then an additional ten percent. So we want to pay 52 dollars, what is the ten percent of 52? **S**: 5.20. **GSI**: 5.20. How did you get 5.20? What did you multiply it by? **S**: I moved the decimal place. **GSI**: Yep, just move the decimal place, get ten percent, ok? Or, you could multiply it by 0.1, alright? So, our total price would be 52 dollars and then an additional ten percent of 52 dollars (*She is writing on the board 52(1)+52(0.1)*). So, we could write this as 52 times 1, representing our original 52, plus 0.1, ok? (*She is writing on the board 52(1 + 0.1) = 52 (1.1) = 57.20*). So when you have an increase in price, you are goanna have your original price and then it is goanna be 1 plus your increase [0.1], ok? So you are goanna pay 110% of the original price.	IUR IUR: ASTI IUR: UFOM IUR: AQCA IUR: AQFP IUR: LSA	
Excerpt 3 **GSI**: The original price is 13 dollars and then we have a 50% increase. So if we have our original 13 dollars that we have to pay and we have to pay 50%, what do we want to multiply 13 by to get our new price? **S**: 1.5 (*several students*). **GSI**: 1.5 exactly! Our new price is what percent of our old price? **S**: 150 (*several students*). **GSI**: 150%, ok? If we don't really see that obviously right away, what could we do: Our new price 13 times 1.5 divided by 13, that is just 1.5. Then a percentage of that is 1.5 times 100, 150%. Our new price is 150% as big as our old price. Question about this? Ok.	IUR IUR: ASTI IUR: UFOM IUR: AQCA IUR: AQFP IUR: LSA	