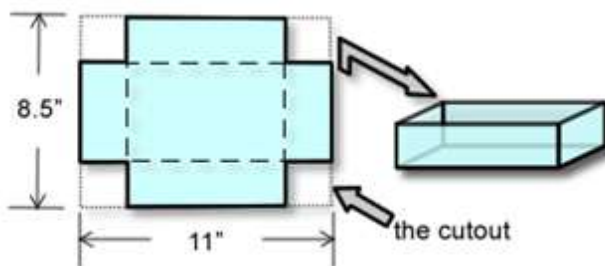


This investigation helps students continue to practice conceptualizing quantities in context for the purpose of modeling relationships between covarying quantities (how they are related and how their values change together). We utilize an extended example (modeling the volume of an open-topped box in terms of the length of the side of a square cutout) to help students recognize how taking time to understand a situation and explore relationships within the context helps them develop meaningful representations of those relationships such as formulas and graphs. As students consider where to cut the paper to remove the squares from the corners, encourage them to explore the maximum length of the side of the square cutout and to verbalize why the side of the square cutout cannot exceed $\frac{1}{2}$ the length of the shortest side of the paper. Students should also realize that when the sides of the box are folded up, the length of the side of the square cutout is the same length as the box's height.

- *1. Consider what is involved in building a box (without a top) from an 8.5" by 11" sheet of paper by cutting squares from each corner and folding up the sides.



Pass out sheets of paper, scissors, and tape and ask students to create an open-top box by cutting equal-sized squares from the corners of an 8.5-inch by 11-inch sheet of paper.

- To understand how the quantities in the situation are related, it is important to first model the situation by doing the following.
 - Cut four equal-sized squares from the corners of an 8.5-inch by 11-inch sheet of paper.
 - Fold up the sides and tape them together at the edges.
- Do the cutouts have to be square? Explain.

Yes, the cutout must be a square. If it is rectangular, for example, the resulting box will have two different heights (the sides folded up to create the sides of the box will not match up).
- Describe each quantity with a varying value. Then describe each quantity with a fixed value.

Use the animations linked in the PowerPoint to help students explore how the quantities in the situation vary.

Students should notice that the following quantities each vary in value:

- the length of the side of the cutout,
- the length of the base of the box,
- the width of the base of the box,
- the box's volume,
- the box's height, and
- the box's surface area.

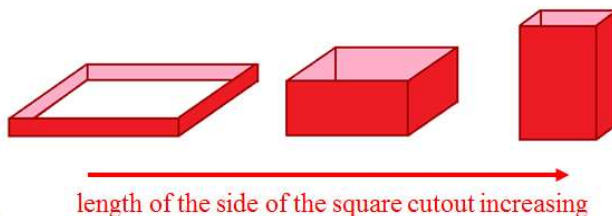
Also, consider helping students conceptualize the difference between linear, square, and cubic units (compare 1 inch, 1 square inch, and 1 cubic inch, and contrast what they are used to measure and why).

After posing the above questions while holding a conversation using visuals and animations, ask students to identify each quantity in the situation with a fixed value, including the length of the sheet of paper before the squares were cut, the width of the sheet of paper before the squares were cut, and the surface area of the sheet of paper before the squares were cut. Help students see that the expressions that represent the box's width and length emerge from conceptualizing the relationship between the fixed side lengths and the independent variable (e.g., $11 - 2x$).

- d. Describe how the configuration of the box changes as the length of the side of the square cutout increases? What is the largest value of x , the square cutout side length (in inches), that makes sense?

You could have four or five boxes displayed on a table in the front of the room and ask students to vote on which box they believe has the largest volume. Number the boxes on the bottom (1-5), record and display the class vote, and then return to the results after graphing the volume vs. cutout length to match each box to its associated point on the graph.

Students should notice that the box starts as a flat and short box, transforms to a box with a larger volume, and ends as a tall, skinny box. The boxes with bases that are very large or very small will not have large volumes.



The largest value that makes sense for the cutout length is 4.25 inches (half the length of the shortest side of the paper, 8.5 inches).

- e. Determine the values of each of the following for your box.
 Box's height: _____ Length of the box's base: _____
 Width of the box's base: _____ Volume of the box: _____

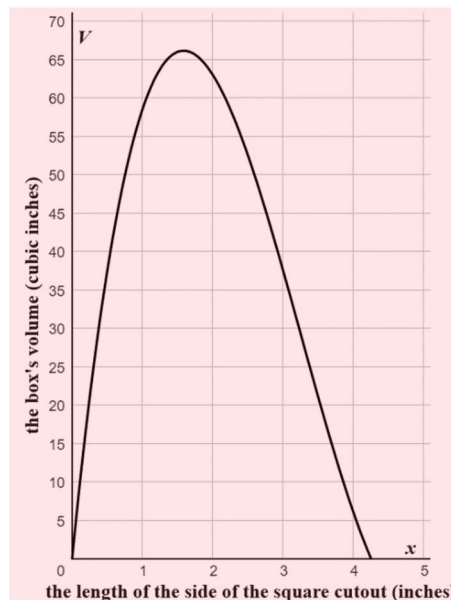
Answers will vary; make sure the units are mentioned. Make a table on the board to illustrate the dimensions of 5-8 different-shaped boxes built by the class. List the length of the side of the square cutout, height of the box, width of the box, and volume of the box in a table. If you are teaching remotely, you should have 3 or 4 boxes made in advance so you can illustrate the dimensions of the boxes in the table. Encourage students to discuss patterns in the table—e.g., that the length of the base of the box is 11 inches minus 2 times the cutout side length.

- f. Describe how the volume of the box varies as the length of the side of the cutout varies from 0 to 4.25 inches.

You can ground this discussion in the work from parts (d) and (e), with applets and animations, by creating a table of values, or using any other approach that emerges from your students' insights.

As the length of the side of the cutout varies from 0 to a little less than 2 inches, the box's volume increases. Then as the length of the side of the cutout varies from a little less than 2 inches to 4.25 inches, the volume of the box decreases.

- g. Based on your responses to previous parts, draw a rough sketch of a graph of the volume of the box (in cubic inches) in terms of the length of the side of the square cutout (in inches). *Be sure to scale and label your axes. Students' sketches do not need to have the same level of accuracy as the solution we provide.*
- h. Use a ruler to measure the length of the side of the square cutout (in inches) of your box.
 Answer: _____
Answers will vary; make sure the units are mentioned.



- *2. Let x represent the length of the side of the square cutout in inches. Let w represent the width of the box's base in inches. Let l represent the length of the box's base in inches. Let V represent the volume of the box in cubic inches.

- a. Complete the table of values.

x	w	l	V
0	8.5	11	0
1	6.5	9	58.5
2.5	3.5	6	52.5
4.25	0	2.5	0
	Impossible	2	

- b. Explain how the length of the base of the box is related to the length of the side of the square cutout and how the width of the base of the box is related to the length of the side of the square cutout.

(a) It is useful for students to develop this table themselves. This will allow them to see how various quantities (e.g., the length of the base of the box, the width of the base of the box, and the volume of the box) are related to the length of the side of the cutout. Encourage them to generalize the arithmetic approach to computing the values in the table. After students build the table, ask them to discuss any patterns they noticed. *It will be useful to have the completed table displayed on an overhead or on the board for this discussion.*

Discuss issues of domain and range by pointing out that it is impossible to have a box length of 2 inches because this would imply that 9 inches had been cut off from the length of the paper, meaning the square cutout was 4.5 inches long; however, it is impossible to remove more than $2(4.25) = 8.5$ inches from the width of the paper. If you want to discuss other impossible cases, consider a cutout length of 5 inches (impossible because then the width of the box would be negative) and a box length of 15 inches (impossible because that is greater than the original length of the paper).

(b) Extend the table in part (a) if necessary to help students see the pattern for determining the length of the base of the box—that the base's length is 11 inches minus 2 times the length of the side of the cutout. Some students will benefit by doing the computations for 3 or 4 values of the length of the side of the square cutout. It can also be helpful to graph these relationships. However, before showing the linear graph, ask students to tell you what they think the graph will look like. Many students won't predict that the relationship is linear with a vertical intercept of (0, 11) and a rate of change of -2. *Use applets linked in the PowerPoints to help facilitate these conversations.*

The length of the box is found by subtracting twice the length of the side of the cutout from the original length of the paper.

The width of the box is found by subtracting twice the length of the side of the cutout from the original width of the paper.

- c. Construct an expression to represent the width of the box in terms of the length of the side of the square cutout, x .

$$8.5 - 2x$$

$$\underbrace{\overbrace{8.5}^{\substack{\text{the width of the} \\ \text{original paper} \\ \text{(inches)}}} - \overbrace{2x}^{\substack{\text{the width (in inches) removed from} \\ \text{the paper to form the box's width} \\ \text{when the cutout length is } x \text{ inches}}}}_{\text{the box's width (inches)}}$$

- d. Construct an expression to represent the length of the box in terms of the length of the side of the square cutout, x .

$$11 - 2x$$

$$\underbrace{\overbrace{11}^{\substack{\text{the length of the} \\ \text{original paper} \\ \text{(inches)}}} - \overbrace{2x}^{\substack{\text{the length (in inches) removed from} \\ \text{the paper to form the box's length} \\ \text{when the cutout length is } x \text{ inches}}}}_{\text{the box's length (inches)}}$$

- e. Define a formula to determine the volume of the box in terms of the length of the side of the square cutout.

Let V represent the volume of the box (measured in cubic inches).

$$V = (8.5 - 2x)(11 - 2x)x$$

$$\underbrace{V}_{\substack{\text{the box's volume} \\ \text{(cubic inches)}}} = \underbrace{\overbrace{(8.5 - 2x)}^{\substack{\text{the box's width} \\ \text{(inches)}}} \overbrace{(11 - 2x)}^{\substack{\text{the box's length} \\ \text{(inches)}}} \overbrace{x}^{\substack{\text{the box's height} \\ \text{(inches)}}}}_{\text{the box's volume (cubic inches)}}$$

Parts f and g present students with the issue that the only way to refer to a box that has a specific cutout length – for example, 0.5 inches or 3 inches – is to write out the formula. The intent for these representations is to motivate the need for function notation.

- f. Use your formula from part (e) to represent the volume of the box when x , the length of the side of the cutout, is 0.5 inches.

$$V = (8.5 - 2(0.5))(11 - 2(0.5))(0.5)$$

- g. Use your formula from part (e) to represent the volume of the box when x , the length of the side of the cutout, is 3 inches.

$$V = (8.5 - 2(3))(11 - 2(3))(3)$$

Hold a discussion about how tedious it is to have to write out this formula every time we want to refer to the volume of a box with some cutout length. You may briefly introduce function notation as a way to specify the input value and output value together – $V = f(0.5)$ and $V = f(3)$ – without actually computing the volume.

Exercise #3 supports students in understanding and expressing the meanings that are conveyed by the graphical representation of the box's volume in terms of x . Encourage students to articulate the meaning of a point on the graph and to recognize (a) that the horizontal distance of a point from the vertical axis represents the cutout length and (b) that the vertical distance from the horizontal axis represents the value of the volume of the box that corresponds to this cutout length. They should see points on the graph as the simultaneous coupling of the two quantities' values as they change together.

3. a. Create a graph that represents the volume of the box, V (measured in cubic inches), in terms of the length of the side of the square cutout, x (measured in inches).

(When determining the window setting on your calculator, consider the possible values of x and the possible values of V that make sense.)

Construct the graph and label points on the graph that correspond to boxes you (or the class) built. State what two of these points convey about the box.

Students can use their graphing calculator, online programs such as Desmos, applets linked in the PowerPoint, and/or their prior work in this investigation to help them complete this part.

Be sure to discuss how the domain (set of possible independent variable values) and range (set of possible dependent variable values) are displayed in the function's graph. We have provided two examples.

Point 1: (0.75, 49.875) When the length of the side of the square cutout is 0.75 inch, the volume of the box is 49.875 cubic inches.

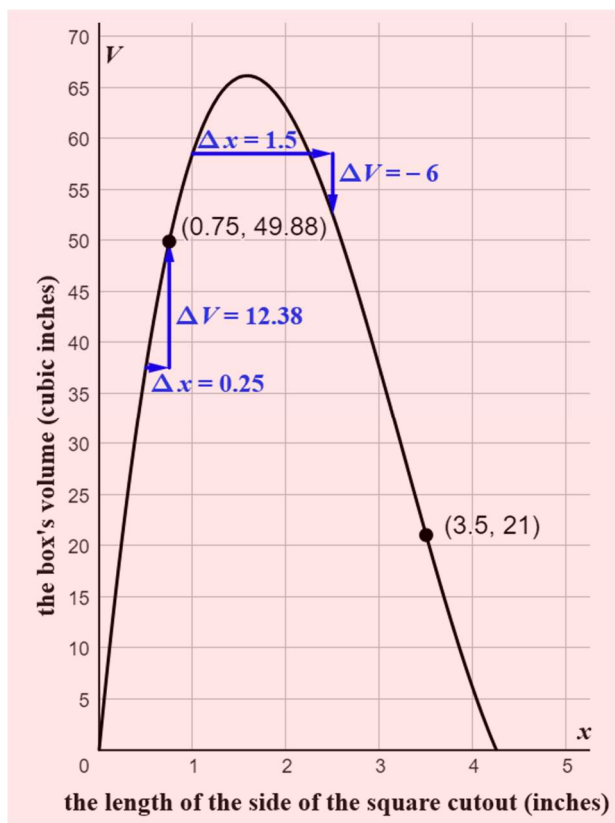
Point 2: (3.5, 21) When the length of the side of the square cutout is 3.5 inches, the volume of the box is 21 cubic inches.

Be sure to have students explain what the coordinates represent in the context of the problem. Students should be supported in verbalizing that the point (3.5, 21) also conveys that the box's volume is 21 cubic inches when the square cutout has a side-length of 3.5 inches. (*Remind students to include units when describing what each point conveys.*)

- b. As x (the length of the side of the cutout) increases from 0.5 inch to 0.75 inch, how does the volume of the box change? Indicate these changes on your graph from part (a).

The volume of the box increases from 37.5 cubic inches to 49.875 cubic inches (a change of about 12.38 cubic inches).

Draw arrows on the graph that represent these changes. Practice asking other similar questions to students until they are fluent in describing these covarying values—this is an important ability for interpreting the meaning conveyed by graphs. As you select different intervals for x , ask students if the box's volume is always increasing/decreasing on the interval.

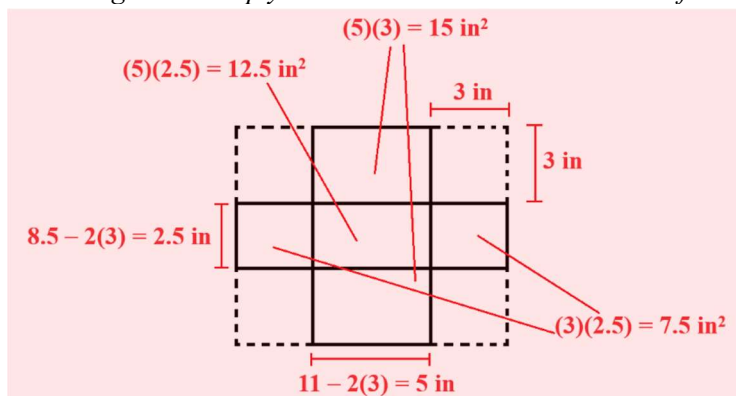


- c. As x (the length of the side of the cutout) increases from 1 inch to 3 inches, how does the volume of the box change?
The volume of the box is initially 58.5 cubic inches and increases; then decreases to a volume of 37.5 cubic inches.
Ask students to determine if the volume of the box is always decreasing on this interval. Students should recognize that if we imagine x continuously varying from 1 inch to 3 inches (despite the physical limitations of cutting and constructing a box), then the volume of the box increases on the interval from $x = 1$ to $x \approx 1.585$ and then decreases on the interval from $x \approx 1.585$ to $x = 3$, for a total change in volume of -21 cubic inches.
- d. As x (the length of the side of the cutout) increases from 2.1 inches to 2.7 inches, how does the volume of the box change?
The volume of the box decreases from 61.404 cubic inches to 46.872 cubic inches. Ask students to determine if the volume of the box is always decreasing on this interval.
- e. Indicate on the graph i) a change of cutout length from 1 inch to 2.5 inches and ii) the corresponding change in the box's volume.
You should use the notation $\Delta x = 2.5 - 1 = 1.5$ and $\Delta V = 52.5 - 58.5 = -6$ to reinforce how a change is referenced symbolically. Make sure students point to the horizontal and vertical arrows that correspond to these changes.
- f. Estimate the interval(s) of values for the length of the side of the cutout x for which the volume of the box decreases as x increases.
[1.585, 4.25]

While discussing this next task (Exercise #4), encourage students to build a table or explore on the graph (by zooming in) to develop better and better approximations for the maximum volume of a box. Probe students to discuss whether it is possible to locate the exact volume that corresponds to the box's maximum volume. (Some students may suggest using the 'calculate max' command on their graphing calculator. You could have a class discussion about how the calculator determines this value.)

- *4. a. Using your graphing calculator, solve for the possible lengths of the side of the square cutout if the volume of the box is 62.5 cubic inches.
When the volume of the box is 62.5 cubic inches, the length of the side of the square cutout is either 1.1746 inches or 2.0333 inches.
Remind students that "solving for" the input variable is a reversal of evaluating a formula at a value of the independent variable. More specifically, we explain "solving for" as determining any value(s) of the independent variable associated with a particular value of the dependent variable.
- b. Using your graphing calculator, solve for the possible lengths of the side of the square cutout if the volume of the box is 23 cubic inches.
When the volume of the box is 23 cubic inches, the length of the side of the square cutout is either 0.277 inches or 3.439 inches.
- c. Using your graphing calculator, find an approximate maximum value for the box's volume.
The approximate value for the maximum volume of the box is 66.148 cubic inches.
- d. Using your graphing calculator, solve for the length of the side of the square cutout that produces the maximum volume you found in part (c).
 $x \approx 1.585$ inches (Encourage students to identify this location on the graph they drew previously.)

5. a. If the length of the side of the square cutout is 3 inches, what is the box's exterior surface area (in square inches)? *Draw a diagram to help you visualize the box's exterior surface area.*



Students might think about summing the areas of the five rectangles ($15(2) + 7.5(2) + 12.5 = 57.5$ square inches), or they might think about finding the area of the original rectangle and removing the areas of the four cutouts ($(11)(8.5) - 4(3)^2 = 57.5$ square inches).

- b. How does the box's exterior surface area vary with the length of the side of the square cutout?
 As the length of the side of the square cutout increases from 0 to 4.25, the box's surface area decreases from 93.5 to 21.25 square inches.
Students might provide more general descriptions without the actual measurements. It would be useful to hold up an 8.5-inch by 11-inch sheet of paper (with equal sized squares marked on each corner) and ask students to imagine how the exterior surface area varies with the length of the side of the square cut from the four corners. You can also use applets linked in the PowerPoint.
- c. Define a formula to determine the box's exterior surface area, S (in square inches), in terms of x , the length of the side of the square that is cut from each of the four corners.

If the student visualizes adding up the exterior surface areas of the box's base and four sides, then the formula for determining the exterior surface area in terms of x is:

$$S = (11 - 2x)(8.5 - 2x) + 2((x)(11 - 2x)) + 2((x)(8.5 - 2x))$$

exterior area of the box's base (square inches)
exterior area of one of the longer folded-up sides (square inches)
exterior area of one of the shorter folded-up sides (square inches)

$$\underbrace{S}_{\substack{\text{the box's exterior surface} \\ \text{area (square inches)}}} = \underbrace{(11 - 2x)(8.5 - 2x)}_{\substack{\text{exterior area of the box's base} \\ \text{(square inches)}}} + \underbrace{2((x)(11 - 2x))}_{\substack{\text{there are two} \\ \text{identical sides}}} + \underbrace{2((x)(8.5 - 2x))}_{\substack{\text{there are two} \\ \text{identical sides}}}$$

the box's exterior surface area (square inches)

If the student visualizes the exterior surface area as being determined by removing the four squares from each corner of the paper, then they would construct the formula

$$S = ((11)(8.5)) - 4(x^2)$$

exterior area of the original paper (square inches)
exterior area of one removed cutout (square inches)

$$\underbrace{S}_{\substack{\text{the box's exterior surface} \\ \text{area (square inches)}}} = \underbrace{((11)(8.5))}_{\substack{\text{exterior area of the original} \\ \text{paper (square inches)}}} - \underbrace{4(x^2)}_{\substack{\text{there are four} \\ \text{identical cutouts}}}$$

the box's exterior surface area (square inches)

It would be useful to simplify the first formula by multiplying and combining like terms and discuss why the simplified version of the first formula is equivalent to the second formula, $S = 93.5 - 4x^2$.

- d. What is the exterior surface area of the box with the maximum volume? Explain the thinking you used to determine your answer.

Since the box's maximum volume occurs when the length of the side of the square cutout x is approximately 1.585 inches, the box's exterior surface area for this value of x is approximately 83.5 square inches.

$$\begin{aligned} S &= 93.5 - 4x^2 \\ &\approx 93.5 - 4(1.585)^2 \\ &\approx 83.5 \text{ square inches} \end{aligned}$$

- e. What is the volume of the box that has an exterior surface area of 25 square inches?

Using a graphing calculator, we determine that $x \approx 4$ when the box's exterior surface area is 25 square inches. We evaluate the box's volume using this value of x .

$$\begin{aligned} V &= x(11 - 2x)(8.5 - 2x) \\ &\approx 4(11 - 2(4))(8.5 - 2(4)) \\ &\approx 6 \text{ cubic inches} \end{aligned}$$